ROMANIAN MATHEMATICAL MAGAZINE

**JP.635** If $a, b > 0; ab = 1$ then:

$$\frac{a}{3a^2+b} + \frac{b}{3b^2+a} \leq \frac{1}{2}$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution 1 by Daniel Sitaru – Romania*

$$\frac{a}{3a^2+b} + \frac{b}{3b^2+a} \leq \frac{1}{2}$$

$$2a(3b^2+a) + 2b(3a^2+b) \leq (3a^2+b)(3b^2+a)$$

$$6ab^2 + 2a^2 + 6a^2b + 2b^2 \leq 9a^2b^2 + 3a^3 + 3b^3 + ab$$

$$6b \cdot ab + 2(a^2+b^2) + 6a \cdot ab \leq 9 \cdot 1 + 3(a^3+b^3) + 1$$

$$6(a+b) + 2(a^2+b^2) \leq 3(a^3+b^3) + 10$$

Denote: $\begin{cases} S = a+b \\ P = ab \end{cases} \Rightarrow P = 1,\ 6S + 2(S^2-2) \leq 3(S^3-3S) + 10$

$$6S + 2S^2 - 4 \leq 3S^3 - 9S + 10, \qquad 3S^3 - 2S^2 - 15S + 14 \geq 0$$

$$S = a+b \geq 2\sqrt{ab} = 2\sqrt{1} = 2 \Rightarrow S - 2 \geq 0 \quad (1)$$

$$3S^3 - 6S^2 + 4S^2 - 8S - 7S + 14 \geq 0$$

$$3S^2(S-2) + 4S(S-2) - 7(S-2) \geq 0$$

$$(S-2)(3S^2+4S-7) \geq 0, \qquad 3S^2 + 4S - 7 \geq 0$$

$$3S^2 - 3S + 7S - 7 \geq 0, \qquad 3S(S-1) + 7(S-1) \geq 0$$

$$(S-1)(3S+7) \geq 0 \quad \text{(true!)}$$

Equality holds for $a = b = 1$.

*Solution 2 by Marin Chirciu– Romania*

$$\frac{a}{3a^2+b} + \frac{b}{3b^2+a} \leq \frac{1}{2}, ab = 1 \Leftrightarrow 3(a^3+b^3) + 10 \geq 2(a^2+b^2) + 6(a+b), ab = 1$$

$$\Leftrightarrow 3(t^3-3t) + 10 \geq 2(t^2-2) + 6t, t = a+b \Leftrightarrow 3t^3 - 2t^2 - 15t + 14 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (t-1)(t-2)(3t+4) \geq 0, \text{see } t \geq 2, t = a+b \geq 2\sqrt{ab} = 2\sqrt{1} = 2.$$

Equality holds if and only if $a = b = 1$.

**Remark.**

If $a, b > 0, ab = 1$ and $1 \leq \lambda \leq 12$ then

$$\frac{a}{\lambda a^2+b}+\frac{b}{\lambda b^2+a}\leq\frac{2}{\lambda+1}.$$

**Solution.**

$$\frac{a}{\lambda a^2+b}+\frac{b}{\lambda b^2+a}\leq\frac{2}{\lambda+1},ab=1\Leftrightarrow$$

$$2\lambda(a^3+b^3)+2\lambda^2+2\geq(\lambda+1)(a^2+b^2)+\lambda(\lambda+1)(a+b),ab=1\Leftrightarrow$$

$$\Leftrightarrow 2\lambda(t^3-3t)+2\lambda^2+2\geq(\lambda+1)(t^2-2)+\lambda(\lambda+1)t,t=a+b\Leftrightarrow$$

$$2\lambda t^3-(\lambda+1)t^2-\lambda(\lambda+7)t+2(\lambda^2+\lambda+2)\geq 0\Leftrightarrow$$

$$\Leftrightarrow(t-2)(2\lambda t^2+(3\lambda-1)t-\lambda^2-\lambda-2)\geq 0,\text{ see } t\geq 2,$$

$$t=a+b\geq 2\sqrt{ab}=2\sqrt{1}=2.$$

**Equality holds if and only if $a=b=1$.**

**Note: For $\lambda=3$ we obtain Problem JP.635 from RMM 44, Winter 2026.**

## *Solution 3 by Jose Luis Diaz Barrero-Spain*

**Since $ab=1$, then $b=\frac{1}{a}$, and we have**

$$\frac{a}{3a^2+b}=\frac{a^2}{3a^3+1},\qquad\frac{b}{3b^2+a}=\frac{a}{a^3+3}.$$

**So the inequality is equivalent to**

$$2a^2(a^3+3)+2a(3a^3+1)\leq(3a^3+1)(a^3+3)$$

**Rearranging terms and calling $P(a)$ to the LHS of the resulting calculations, we have to prove that it holds**

$$P(a)=3a^6-2a^5-6a^4+10a^3-6a^2-2a+3\geq 0.$$

**Polynomial $P$ is palindromic, so divide by $a^3>0$ and set $t=a+\frac{1}{a}\geq 2$. Using that $a^2+\frac{1}{a^2}=t^2-2$ and $a^3+\frac{1}{a^3}=t^3-3t$, we get**

$$\frac{P(a)}{a^3}=3t^3-2t^2-15t+14=(t-2)(t-1)(3t+7)\geq 0,$$

**since $t\geq 2$. Equality holds if and only if $t=2$, that is, $a=b=1$.**