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JP.634 Find $x, y \in (0, \pi)$ such that:

$$\frac{\sin^2 x}{1+\sin^2 x} + \frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)} + \frac{1}{4\sin x \sin y} = 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sin^2 x\,(1+\sin^2 y)\cdot 4\sin x \sin y + \sin^2 y \cdot 4\sin x \sin y + (1+\sin^2 x)(1+\sin^2 y) =$$

$$= 4\sin x \sin x\,(1+\sin^2 x)(1+\sin^2 y)$$

$$4\sin x \sin y\,(\sin^2 x + \sin^2 x \sin^2 y + \sin^2 y - 1 - \sin^2 x - \sin^2 y - \sin^2 x \sin^2 y) +$$

$$+(1+\sin^2 x)(1+\sin^2 y) = 0$$

$$4\sin x \sin y \cdot (-1) + 1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y = 0$$

$$1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y = 4\sin x \sin y$$

$$1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y \geq$$

$$\overset{AM-GM}{\geq} 4\sqrt[4]{1 \cdot \sin^2 x \cdot \sin^2 y \cdot \sin^2 x \sin^2 y} =$$

$$= 4\sqrt[4]{\sin^4 x \sin^4 y} = 4\sin x \sin y$$

Equality holds for:

$$\sin x = \sin y = 1 \Rightarrow x = y = \frac{\pi}{2}.$$

Solution 2 by Marin Chirciu-Romania

$$\frac{\sin^2 x}{1+\sin^2 x} + \frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)} + \frac{1}{4\sin x \sin y} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{\sin^2 x + \sin^2 y + \sin^2 x \sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)} + \frac{1}{4\sin x \sin y} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{(1+\sin^2 x)(1+\sin^2 y) - 1}{(1+\sin^2 x)(1+\sin^2 y)} + \frac{1}{4\sin x \sin y} = 1 \Leftrightarrow$$

$$\Leftrightarrow 1 - \frac{1}{(1+\sin^2 x)(1+\sin^2 y)} + \frac{1}{4\sin x \sin y} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{(1+\sin^2 x)(1+\sin^2 y)} = \frac{1}{4\sin x \sin y} \Leftrightarrow (1+\sin^2 x)(1+\sin^2 y) = 4\sin x \sin y$$

$$\Leftrightarrow 1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y = 4\sin x \sin y \Leftrightarrow$$

$$\Leftrightarrow (\sin x-\sin y)^2+(\sin x\sin y-1)^2=0\Leftrightarrow$$

$$\Leftrightarrow \begin{cases}\sin x-\sin y=0\\ \sin x\sin y-1=0\end{cases}\Leftrightarrow \begin{cases}\sin x=\sin y\\ \sin x\sin y=1\end{cases}\Leftrightarrow \begin{cases}\sin x=\sin y\\ \sin^2 x=1\end{cases}\Leftrightarrow \sin x=\sin y=1\Leftrightarrow$$

$$\Leftrightarrow x=y=\frac{\pi}{2}.$$

We deduce that $(x,y)=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ is the unique solution of the equation.

Remark: If $x,y\in(0,\pi)$ and $\lambda\geq 0$ then

$$\frac{\sin^2 x}{\lambda+\sin^2 x}+\frac{\lambda\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+\sin^2 x)(\lambda+\sin^2 y)=(\lambda+1)^2\sin x\sin y$$

Marin Chirciu

Solution

$$\frac{\sin^2 x}{\lambda+\sin^2 x}+\frac{\lambda\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{\lambda\sin^2 x+\lambda\sin^2 y+\sin^2 x\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{(\lambda+\sin^2 x)(\lambda+\sin^2 y)-\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow 1-\frac{\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}=\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}\Leftrightarrow$$

$$\Leftrightarrow(\lambda+\sin^2 x)(\lambda+\sin^2 y)=(\lambda+1)^2\sin x\sin y$$

Solution 3 by Jose Luis Diaz Barrero-Spain

Since $x,y\in(0,\pi)$, we have $\sin x,\sin y>0$. First we write the equation as

$$1-\frac{\sin^2 x}{1+\sin^2 x}-\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}=\frac{1}{4\sin x\sin y},$$

and we observe that

$$1-\frac{\sin^2 x}{1+\sin^2 x}-\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}=\frac{1}{(1+\sin^2 x)(1+\sin^2 y)},$$

So the given equation is equivalent to

$$(1+\sin^2 x)(1+\sin^2 y)=4\sin x\sin y.$$

By the AM-GM inequality, $1+\sin^2 x\geq 2\sin x$ and $1+\sin^2 y\geq 2\sin y$, hence

$$(1+\sin^2 x)(1+\sin^2 y)\geq 4\sin x\sin y,$$

with equality if and only if $\sin x=\sin y=1$. Therefore the unique solution is $x=y=\frac{\pi}{2}$.

Solution 4 by Hikmat Mammadov-Azerbaijan

$$u=\sin x,\ \ v=\sin y$$

$$x,y\in(0,\pi)\Rightarrow u,v\in(0,1]$$

$$\int_0^u \frac{2t}{(1+t^2)^2}dt=\left[-\frac{1}{1+t^2}\right]_0^u=\left(-\frac{1}{1+u^2}\right)-\left(-\frac{1}{1+0^2}\right)=1-\frac{1}{1+u^2}=\frac{u^2}{1+u^2}$$

$$\int_0^v \frac{2z}{(1+z^2)^2}dz=\left[-\frac{1}{1+z^2}\right]_0^v=\left(-\frac{1}{1+v^2}\right)-\left(-\frac{1}{1+0^2}\right)=1-\frac{1}{1+v^2}=\frac{v^2}{1+v^2}$$

$$\frac{u^2}{1+u^2}+\frac{v^2}{(1+u^2)(1+v^2)}=\int_0^u \frac{2t}{(1+t^2)^2}dt+\frac{1}{1+u^2}\int_0^v \frac{2z}{(1+z^2)^2}dz$$

$$=\left(1-\frac{1}{1+u^2}\right)+\frac{1}{1+u^2}\left(1-\frac{1}{1+v^2}\right)$$

$$=1-\frac{1}{1+u^2}+\frac{1}{1+u^2}-\frac{1}{(1+u^2)(1+v^2)}=1-\frac{1}{(1+u^2)(1+v^2)}$$

$$\frac{1}{1+u^2}=\int_0^\infty e^{-s(1+u^2)}ds,\qquad \frac{1}{1+v^2}=\int_0^\infty e^{-t(1+v^2)}dt$$

$$\frac{1}{(1+u^2)(1+v^2)}\int_0^\infty\int_0^\infty e^{-s(1+u^2)-t(1+v^2)}ds\,dt$$

$$1-\frac{1}{(1+u^2)(1+v^2)}+\frac{1}{4uv}=1\Rightarrow 1-1+\frac{1}{4uv}=\frac{1}{(1+u^2)(1+v^2)}$$

$$\Rightarrow\frac{1}{4uv}=\frac{1}{(1+u^2)(1+v^2)}\Rightarrow 4uv=(1+u^2)(1+v^2)$$

$$\Rightarrow 4uv=1+u^2+v^2+u^2v^2\Rightarrow 1+u^2+v^2+u^2v^2-4uv=0$$

$$\Rightarrow 1-2uv+u^2v^2+u^2-2uv+v^2=0\Rightarrow(1-uv)^2+(u-v)^2=0$$

$$\Rightarrow\begin{cases}(1-uv)^2=0\\(u-v)^2=0\end{cases}\Rightarrow\begin{cases}1-uv=0\\u-v=0\end{cases}\Rightarrow\begin{cases}u=v\\u^2=1\end{cases}$$

$$u,v\in(0,1]\Rightarrow u=1,v=1$$

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$$\begin{cases}\sin x = 1\\ \sin y = 1\end{cases} \Rightarrow \begin{cases}x = \frac{\pi}{2}\\ y = \frac{\pi}{2}\end{cases}$$

$$(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

