ROMANIAN MATHEMATICAL MAGAZINE

JP.633 If $a, b, c \geq 1; a^2 + b^2 + c^2 = 3$ then:

$$(a^5 + 10a^3 + 5a - 1)(b^5 + 10b^3 + 5b - 1)(c^5 + 10c^3 - 5c - 1) \geq 3375$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(a-1)^5 \geq 0$$

$$a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1 \geq 0$$

$$a^5 + 10a^3 + 5a - 1 \geq 5a^4 + 10a^2 = 5a^2(a^2 + 2)$$

$$(a^5 + 10a^3 + 5a - 1)(b^5 + 10b^3 + 5b - 1)(c^5 + 10c^3 + 5c - 1) \geq$$

$$\geq 5a^2(a^2+2) \cdot 5b^2(b^2+2) \cdot 5c^2(c^2+2) =$$

$$= 125a^2b^2c^2(a^2+2)(b^2+2)(c^2+2) \overset{AM-GM}{\geq}$$

$$\geq 125\left(\frac{a^2+b^2+c^2}{3}\right)^3 \cdot \left(\frac{a^2+2+b^2+2+c^2+2}{3}\right)^3 =$$

$$= 125 \cdot \left(\frac{3}{3}\right)^3 \cdot \left(\frac{3+6}{3}\right)^3 = 125 \cdot 27 = 3375$$

Equality holds for $a = b = c = 1$.

Solution 2 by Marin Chirciu-Romania

Lemma.

If $a \geq 1$, then:

$$a^5 + 10a^3 + 5a - 1 \geq 15$$

Proof.

$$a \geq 1 \Rightarrow (a-1)^5 \geq 0 \Leftrightarrow a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1 \geq 0 \Rightarrow$$

$$a^5 + 10a^3 + 5a - 1 \geq 5a^4 + 10a^2 \geq 5 \cdot 1^4 + 10 \cdot 1^2 = 15.$$

Let's get to the main problem.

Using the Lemma, we obtain:

$$LHS = \prod\left(a^5 + 10a^3 + 5a - 1\right) \overset{Lemma}{\geq} \prod 15 = 15^3 = 3375 = RHS$$

Equality holds if and only if $a = b = c = 1$.

Remark.

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If $a,b,c,d \geq 1$, then:

$$\prod (a^3+3a-1) \geq 81$$

Marin Chirciu

Solution

Lemma.

If $a \geq 1$, then:

$$a^3+3a-1 \geq 3.$$

Proof.

$$a \geq 1 \Rightarrow (a-1)^3 \geq 0 \Leftrightarrow a^3-3a^2+3a-1 \geq 0 \Rightarrow a^3+3a-1 \geq 3a^2 \geq 3 \cdot 1^2 = 3.$$

Let's get back to the main problem.

Using the Lemma, we obtain:

$$LHS = \prod (a^3+3a-1) \overset{Lemma}{\geq} \prod 3 = 3^4 = 81 = RHS$$

Equality holds if and only if $a=b=c=1$.

Solution 3 by Jose Luis Diaz Barrero-Spain

Since $a,b,c \geq 1$, we have $a^2,b^2,c^2 \geq 1$, and therefore

$$3 = a^2+b^2+c^2 \geq 3,$$

with equality if and only if $a^2=b^2=c^2=1$, that is $a=b=c=1$. Let

$f(x)=x^5+10x^3+5x-1$. Then $f(1)=15$ and

$$f(a)f(b)f(c) = 15^3 = 3375,$$

so the inequality holds, with equality.

Note. Since $f'(x)=5x^4+30x^2+5>0$, then f is increasing, so the condition $a,b,c \geq 1$ alone, without the constrain $a^2+b^2+c^2=3$, already gives $f(a)f(b)f(c) \geq 3375$.