ROMANIAN MATHEMATICAL MAGAZINE

JP.632 Let a, b, c, d, e be nonnegative real numbers such that $a \geq b \geq c \geq d \geq e$ and $(a+b)(c+d+e) = 6$. Prove that:

$$\frac{1}{2a+3} + \frac{1}{2b+3} + \frac{1}{2c+3} + \frac{1}{2d+3} + \frac{1}{2e+3} \geq 1$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

Denoting

$$x = \frac{a+b}{2}, \quad y = \frac{c+d+e}{3},$$

we have $xy = 1$ and $x \geq 1 \geq y$. By the AM-HM inequality, we have

$$\frac{1}{2a+3} + \frac{1}{2b+3} \geq \frac{4}{(2a+3)+(2b+3)} = \frac{2}{2x+3}$$

and

$$\frac{1}{2c+3} + \frac{1}{2d+3} + \frac{1}{2e+3} \geq \frac{9}{(2c+3)+(2d+3)+(2e+3)} = \frac{3}{2y+3}.$$

So, it suffices to show that

$$\frac{2}{2x+3} + \frac{3}{2y+3} \geq 1.$$

Indeed,

$$\frac{2}{2x+3} + \frac{3}{2y+3} - 1 = \frac{2}{2x+3} + \frac{3x}{3x+2} - 1 = \frac{2(x-1)}{(2x+3)(3x+2)} \geq 0.$$

The equality occurs for $a = b = c = d = e = 1$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

Let $x = a+b$ and $y = c+d+e$ so that $xy = 6$. Here,

$$\frac{1}{2a+3} + \frac{1}{2b+3} + \frac{1}{2c+3} + \frac{1}{2d+3} + \frac{1}{2e+3} \overset{\text{Bergström}}{\geq} \frac{(1+1)^2}{2(a+b)+6} + \frac{(1+1+1)^2}{2(c+d+e)+9} =$$

$$= \frac{4}{2x+6} + \frac{9}{2y+9} = \frac{2}{x+3} + \frac{9}{2y+9}$$

We have, $x = a+b \geq 2c \geq \frac{2(c+d+e)}{3} = \frac{2y}{3}$. So, $xy \geq \frac{2y^2}{3}$, i.e., $\frac{2y^2}{3} \leq 6$. $\therefore y \leq 3$

Now,

$$\frac{2}{x+3}+\frac{9}{2y+9}\overset{xy=6}{=}\frac{2y}{3y+6}+\frac{9}{2y+9}=\frac{2y}{2y+y+6}+\frac{9}{2y+9}\overset{y\leq 3}{\geq}\frac{2y}{2y+3+6}+\frac{9}{2y+9}=1$$

Therefore,

$$\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}+\frac{1}{2d+3}+\frac{1}{2e+3}\geq 1$$

Equality holds if and only if $a=b=c=d=e=1$.