ROMANIAN MATHEMATICAL MAGAZINE

JP.631 Let a, b, c, d be nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq \frac{1}{2} \text{ and } ab + bc + cd + da = 4. \text{ Prove that:}$$

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} \leq \frac{64}{63}$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

Let $x = \frac{a+c}{2}$ and $y = \frac{b+d}{2}$. We have $xy = 1$, and from $(2a-1)(2c-1) \geq 0$ and $(2b-1)(2d-1) \geq 0$, we get $4ac \geq 4x - 1, \quad 4bd \geq 4y - 1$.

Write the required inequality as follows:

$$\frac{a+c+6}{ac+3(a+c)+9} + \frac{b+d+6}{bd+3(b+d)+9} \leq \frac{64}{63}$$

$$\frac{x+3}{ac+6x+9} + \frac{y+3}{bd+6y+9} \leq \frac{32}{63}.$$

It is true if

$$\frac{4(x+3)}{4x-1+4(6x+9)} + \frac{4(y+3)}{4y-1+4(6y+9)} \leq \frac{32}{63},$$

that is

$$\frac{x+3}{4x+5} + \frac{y+3}{4y+5} \leq \frac{8}{9}, \qquad \frac{x+3}{4x+5} + \frac{3x+1}{5x+4} \leq \frac{8}{9},$$

$$(x-1)^2 \geq 0.$$

So, the proof is finished. Equality occurs for $a = b = \frac{3}{2}$ and $c = d = \frac{1}{2}$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\textit{Lemma: For any } u, v \geq \frac{1}{2}, \qquad \frac{2}{7} + \frac{2}{2u+2v+5} \geq \frac{1}{u+3} + \frac{1}{v+3}.$$

$$\textit{Proof: } \frac{2}{7} + \frac{2}{2u+2v+5} - \frac{1}{u+3} - \frac{1}{v+3} = \frac{4(u+v+6)\left(u-\frac{1}{2}\right)\left(v-\frac{1}{2}\right)}{7(2u+2v+5)(uv+3u+3v+9)} \geq 0$$

$$\textit{Applying the lemma to } (a,c) \textit{ and } (b,d), \textit{we get:}$$

$$\frac{1}{a+3} + \frac{1}{c+3} + \frac{1}{b+3} + \frac{1}{d+3} \leq \frac{4}{7} + \frac{2}{2a+2c+5} + \frac{2}{2b+2d+5}$$

$$\textit{Let } x = a + c \textit{ and } y = b + d, \textit{then } xy = ab + bc + cd + da = 4.$$

So, $x+y\geq 2\sqrt{xy}=4$. *We have*:

$$\frac{4}{9}-\frac{2}{2x+5}-\frac{2}{2y+5}==\frac{4(4xy+10x+10y+25)-36(x+y)-180}{9(2x+5)(2y+5)}\overset{xy=4}{=}$$

$$=\frac{4(x+y-4)}{9(2x+5)(2y+5)}\overset{x+y\geq 4}{\geq}0$$

Thus, $$\frac{2}{2a+2c+5}+\frac{2}{2b+2d+5}\leq\frac{4}{9}$$

Hence, $$\frac{1}{a+3}+\frac{1}{c+3}+\frac{1}{b+3}+\frac{1}{d+3}\leq\frac{4}{7}+\frac{4}{9}=\frac{64}{63}$$

Equality holds if and only if $(a,b,c,d)=\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)$.