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SOLUTIONS

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PROBLEMS FOR JUNIORS

JP.631 Let a, b, c, d be nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq \frac{1}{2} \text{ and } ab + bc + cd + da = 4. \text{ Prove that:}$$

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} \leq \frac{64}{63}$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

Let $x = \frac{a+c}{2}$ and $y = \frac{b+d}{2}$. We have $xy = 1$, and from $(2a-1)(2c-1) \geq 0$ and $(2b-1)(2d-1) \geq 0$, we get $4ac \geq 4x - 1, \quad 4bd \geq 4y - 1$.

Write the required inequality as follows:

$$\frac{a+c+6}{ac+3(a+c)+9} + \frac{b+d+6}{bd+3(b+d)+9} \leq \frac{64}{63}$$

$$\frac{x+3}{ac+6x+9} + \frac{y+3}{bd+6y+9} \leq \frac{32}{63}.$$

It is true if

$$\frac{4(x+3)}{4x-1+4(6x+9)} + \frac{4(y+3)}{4y-1+4(6y+9)} \leq \frac{32}{63},$$

that is

$$\frac{x+3}{4x+5} + \frac{y+3}{4y+5} \leq \frac{8}{9}, \qquad \frac{x+3}{4x+5} + \frac{3x+1}{5x+4} \leq \frac{8}{9},$$

$$(x-1)^2 \geq 0.$$

So, the proof is finished. Equality occurs for $a = b = \frac{3}{2}$ and $c = d = \frac{1}{2}$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\textit{Lemma: For any } u, v \geq \frac{1}{2}, \qquad \frac{2}{7} + \frac{2}{2u+2v+5} \geq \frac{1}{u+3} + \frac{1}{v+3}.$$

$$\textit{Proof: } \frac{2}{7} + \frac{2}{2u+2v+5} - \frac{1}{u+3} - \frac{1}{v+3} = \frac{4(u+v+6)\left(u-\frac{1}{2}\right)\left(v-\frac{1}{2}\right)}{7(2u+2v+5)(uv+3u+3v+9)} \geq 0$$

Applying the lemma to (a,c) *and* (b,d), *we get*:

$$\frac{1}{a+3}+\frac{1}{c+3}+\frac{1}{b+3}+\frac{1}{d+3}\le\frac{4}{7}+\frac{2}{2a+2c+5}+\frac{2}{2b+2d+5}$$

Let $x=a+c$ *and* $y=b+d$, *then* $xy=ab+bc+cd+da=4$.

So, $x+y\ge 2\sqrt{xy}=4$. *We have*:

$$\frac{4}{9}-\frac{2}{2x+5}-\frac{2}{2y+5}==\frac{4(4xy+10x+10y+25)-36(x+y)-180}{9(2x+5)(2y+5)}\overset{xy=4}{=}$$

$$=\frac{4(x+y-4)}{9(2x+5)(2y+5)}\overset{x+y\ge 4}{\geqslant}0$$

Thus, $$\frac{2}{2a+2c+5}+\frac{2}{2b+2d+5}\le\frac{4}{9}$$

Hence, $$\frac{1}{a+3}+\frac{1}{c+3}+\frac{1}{b+3}+\frac{1}{d+3}\le\frac{4}{7}+\frac{4}{9}=\frac{64}{63}$$

Equality holds if and only if $(a,b,c,d)=\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)$.

JP.632 Let a,b,c,d,e be nonnegative real numbers such that $a\ge b\ge c\ge d\ge e$ and $(a+b)(c+d+e)=6$. Prove that:

$$\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}+\frac{1}{2d+3}+\frac{1}{2e+3}\ge 1$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

Denoting

$$x=\frac{a+b}{2},\quad y=\frac{c+d+e}{3},$$

we have $xy=1$ and $x\ge 1\ge y$. By the AM-HM inequality, we have

$$\frac{1}{2a+3}+\frac{1}{2b+3}\ge\frac{4}{(2a+3)+(2b+3)}=\frac{2}{2x+3}$$

and

$$\frac{1}{2c+3}+\frac{1}{2d+3}+\frac{1}{2e+3}\geq\frac{9}{(2c+3)+(2d+3)+(2e+3)}=\frac{3}{2y+3}.$$

So, it suffices to show that

$$\frac{2}{2x+3}+\frac{3}{2y+3}\geq 1.$$

Indeed,

$$\frac{2}{2x+3}+\frac{3}{2y+3}-1=\frac{2}{2x+3}+\frac{3x}{3x+2}-1=\frac{2(x-1)}{(2x+3)(3x+2)}\geq 0.$$

The equality occurs for $a=b=c=d=e=1$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

Let $x=a+b$ **and** $y=c+d+e$ **so that** $xy=6$. **Here,**

$$\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}+\frac{1}{2d+3}+\frac{1}{2e+3}\overset{\text{Bergström}}{\geq}\frac{(1+1)^2}{2(a+b)+6}+\frac{(1+1+1)^2}{2(c+d+e)+9}=$$

$$=\frac{4}{2x+6}+\frac{9}{2y+9}=\frac{2}{x+3}+\frac{9}{2y+9}$$

We have, $x=a+b\geq 2c\geq\frac{2(c+d+e)}{3}=\frac{2y}{3}$ **. So,** $xy\geq\frac{2y^2}{3}$ **, i. e.,** $\frac{2y^2}{3}\leq 6$ **.** $\therefore y\leq 3$

Now,

$$\frac{2}{x+3}+\frac{9}{2y+9}\overset{xy=6}{=}\frac{2y}{3y+6}+\frac{9}{2y+9}=\frac{2y}{2y+y+6}+\frac{9}{2y+9}\overset{y\leq 3}{\geq}\frac{2y}{2y+3+6}+\frac{9}{2y+9}=1$$

Therefore,

$$\frac{1}{2a+3}+\frac{1}{2b+3}+\frac{1}{2c+3}+\frac{1}{2d+3}+\frac{1}{2e+3}\geq 1$$

Equality holds if and only if $a=b=c=d=e=1$.

JP.633 **If** $a,b,c\geq 1; a^2+b^2+c^2=3$ **then:**

$$\left(a^5+10a^3+5a-1\right)\left(b^5+10b^3+5b-1\right)\left(c^5+10c^3-5c-1\right)\geq 3375$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer

$$(a-1)^5\geq 0$$

$$a^5-5a^4+10a^3-10a^2+5a-1\geq 0$$

$$a^5+10a^3+5a-1\geq 5a^4+10a^2=5a^2(a^2+2)$$

$$(a^5+10a^3+5a-1)(b^5+10b^3+5b-1)(c^5+10c^3+5c-1)\geq$$

$$\geq 5a^2(a^2+2)\cdot 5b^2(b^2+2)\cdot 5c^2(c^2+2)=$$

$$=125a^2b^2c^2(a^2+2)(b^2+2)(c^2+2)\overset{AM-GM}{\geq}$$

$$\geq 125\left(\frac{a^2+b^2+c^2}{3}\right)^3\cdot\left(\frac{a^2+2+b^2+2+c^2+2}{3}\right)^3=$$

$$=125\cdot\left(\frac{3}{3}\right)^3\cdot\left(\frac{3+6}{3}\right)^3=125\cdot 27=3375$$

Equality holds for $a=b=c=1$.

Solution 2 by Marin Chirciu-Romania

Lemma.

If $a\geq 1$, then:

$$a^5+10a^3+5a-1\geq 15$$

Proof.

$$a\geq 1\Rightarrow (a-1)^5\geq 0\Leftrightarrow a^5-5a^4+10a^3-10a^2+5a-1\geq 0\Rightarrow$$

$$a^5+10a^3+5a-1\geq 5a^4+10a^2\geq 5\cdot 1^4+10\cdot 1^2=15.$$

Let's get to the main problem.

Using the Lemma, we obtain:

$$LHS=\prod\left(a^5+10a^3+5a-1\right)\overset{Lemma}{\geq}\prod 15=15^3=3375=RHS$$

Equality holds if and only if $a=b=c=1$.

Remark.

If $a,b,c,d\geq 1$, then:

$$\prod(a^3+3a-1)\geq 81$$

Marin Chirciu

Solution: **Lemma.**

If $a\geq 1$, then: $a^3+3a-1\geq 3$.

Proof.

$$a \geq 1 \Rightarrow (a-1)^3 \geq 0 \Leftrightarrow a^3 - 3a^2 + 3a - 1 \geq 0 \Rightarrow a^3 + 3a - 1 \geq 3a^2 \geq 3 \cdot 1^2 = 3.$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS = \prod (a^3 + 3a - 1) \overset{Lemma}{\geq} \prod 3 = 3^4 = 81 = RHS$$

Equality holds if and only if $a = b = c = 1$.

Solution 3 by Jose Luis Diaz Barrero-Spain

Since $a, b, c \geq 1$, we have $a^2, b^2, c^2 \geq 1$, and therefore $3 = a^2 + b^2 + c^2 \geq 3$,

with equality if and only if $a^2 = b^2 = c^2 = 1$, that is $a = b = c = 1$. Let

$f(x) = x^5 + 10x^3 + 5x - 1$. Then $f(1) = 15$ and $f(a)f(b)f(c) = 15^3 = 3375$,

so the inequality holds, with equality.

Note. Since $f'(x) = 5x^4 + 30x^2 + 5 > 0$, then f is increasing, so the condition $a, b, c \geq 1$

alone, without the constrain $a^2 + b^2 + c^2 = 3$, already gives $f(a)f(b)f(c) \geq 3375$.

JP.634 Find $x, y \in (0, \pi)$ such that:

$$\frac{\sin^2 x}{1 + \sin^2 x} + \frac{\sin^2 y}{(1 + \sin^2 x)(1 + \sin^2 y)} + \frac{1}{4 \sin x \sin y} = 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sin^2 x\,(1 + \sin^2 y) \cdot 4 \sin x \sin y + \sin^2 y \cdot 4 \sin x \sin y + (1 + \sin^2 x)(1 + \sin^2 y) =$$

$$= 4 \sin x \sin x\,(1 + \sin^2 x)(1 + \sin^2 y)$$

$$4 \sin x \sin y\,(\sin^2 x + \sin^2 x \sin^2 y + \sin^2 y - 1 - \sin^2 x - \sin^2 y - \sin^2 x \sin^2 y) +$$

$$+ (1 + \sin^2 x)(1 + \sin^2 y) = 0$$

$$4 \sin x \sin y \cdot (-1) + 1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y = 0$$

$$1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y = 4 \sin x \sin y$$

$$1 + \sin^2 x + \sin^2 y + \sin^2 x \sin^2 y \geq$$

$$\overset{AM-GM}{\geq} 4\sqrt[4]{1 \cdot \sin^2 x \cdot \sin^2 y \cdot \sin^2 x \sin^2 y} = 4\sqrt[4]{\sin^4 x \sin^4 y} = 4 \sin x \sin y$$

Equality holds for: $\sin x = \sin y = 1 \Rightarrow x = y = \frac{\pi}{2}$.

Solution 2 by Marin Chirciu-Romania

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{4\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{\sin^2 x+\sin^2 y+\sin^2 x\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{4\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{(1+\sin^2 x)(1+\sin^2 y)-1}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{4\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow 1-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{4\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}=\frac{1}{4\sin x\sin y}\Leftrightarrow(1+\sin^2 x)(1+\sin^2 y)=4\sin x\sin y$$

$$\Leftrightarrow 1+\sin^2 x+\sin^2 y+\sin^2 x\sin^2 y=4\sin x\sin y\Leftrightarrow$$

$$\Leftrightarrow(\sin x-\sin y)^2+(\sin x\sin y-1)^2=0\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\sin x-\sin y=0\\ \sin x\sin y-1=0\end{cases}\Leftrightarrow\begin{cases}\sin x=\sin y\\ \sin x\sin y=1\end{cases}\Leftrightarrow\begin{cases}\sin x=\sin y\\ \sin^2 x=1\end{cases}\Leftrightarrow\sin x=\sin y=1\Leftrightarrow$$

$$\Leftrightarrow x=y=\frac{\pi}{2}.$$

We deduce that $(x,y)=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ is the unique solution of the equation.

Remark: If $x,y\in(0,\pi)$ and $\lambda\geq 0$ then

$$\frac{\sin^2 x}{\lambda+\sin^2 x}+\frac{\lambda\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+\sin^2 x)(\lambda+\sin^2 y)=(\lambda+1)^2\sin x\sin y$$

Marin Chirciu

Solution

$$\frac{\sin^2 x}{\lambda+\sin^2 x}+\frac{\lambda\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{\lambda\sin^2 x+\lambda\sin^2 y+\sin^2 x\sin^2 y}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow\frac{(\lambda+\sin^2 x)(\lambda+\sin^2 y)-\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow 1-\frac{\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}+\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}=1\Leftrightarrow$$

$$\Leftrightarrow \frac{\lambda^2}{(\lambda+\sin^2 x)(\lambda+\sin^2 y)}=\frac{\lambda^2}{(\lambda+1)^2\sin x\sin y}\Leftrightarrow$$

$$\Leftrightarrow (\lambda+\sin^2 x)(\lambda+\sin^2 y)=(\lambda+1)^2\sin x\sin y$$

Solution 3 by Jose Luis Diaz Barrero-Spain

Since $x,y\in(0,\pi)$, we have $\sin x,\sin y>0$. First we write the equation as

$$1-\frac{\sin^2 x}{1+\sin^2 x}-\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}=\frac{1}{4\sin x\sin y},$$

and we observe that

$$1-\frac{\sin^2 x}{1+\sin^2 x}-\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}=\frac{1}{(1+\sin^2 x)(1+\sin^2 y)},$$

So the given equation is equivalent to

$$(1+\sin^2 x)(1+\sin^2 y)=4\sin x\sin y.$$

By the AM-GM inequality, $1+\sin^2 x\geq 2\sin x$ and $1+\sin^2 y\geq 2\sin y$, hence

$$(1+\sin^2 x)(1+\sin^2 y)\geq 4\sin x\sin y,$$

with equality if and only if $\sin x=\sin y=1$. Therefore the unique solution is $x=y=\frac{\pi}{2}$.

Solution 4 by Hikmat Mammadov-Azerbaijan

$$u=\sin x,\ \ v=\sin y$$

$$x,y\in(0,\pi)\Rightarrow u,v\in(0,1]$$

$$\int_0^u\frac{2t}{(1+t^2)^2}dt=\left[-\frac{1}{1+t^2}\right]_0^u=\left(-\frac{1}{1+u^2}\right)-\left(-\frac{1}{1+0^2}\right)=1-\frac{1}{1+u^2}=\frac{u^2}{1+u^2}$$

$$\int_0^v\frac{2z}{(1+z^2)^2}dz=\left[-\frac{1}{1+z^2}\right]_0^v=\left(-\frac{1}{1+v^2}\right)-\left(-\frac{1}{1+0^2}\right)=1-\frac{1}{1+v^2}=\frac{v^2}{1+v^2}$$

$$\frac{u^2}{1+u^2}+\frac{v^2}{(1+u^2)(1+v^2)}=\int_0^u\frac{2t}{(1+t^2)^2}dt+\frac{1}{1+u^2}\int_0^v\frac{2z}{(1+z^2)^2}dz$$

$$=\left(1-\frac{1}{1+u^2}\right)+\frac{1}{1+u^2}\left(1-\frac{1}{1+v^2}\right)$$

$$=1-\frac{1}{1+u^2}+\frac{1}{1+u^2}-\frac{1}{(1+u^2)(1+v^2)}=1-\frac{1}{(1+u^2)(1+v^2)}$$

$$\frac{1}{1+u^2}=\int_0^\infty e^{-s(1+u^2)}\,ds,\qquad \frac{1}{1+v^2}=\int_0^\infty e^{-t(1+v^2)}\,dt$$

$$\frac{1}{(1+u^2)(1+v^2)}\int_0^\infty\int_0^\infty e^{-s(1+u^2)-t(1+v^2)}\,ds\,dt$$

$$1-\frac{1}{(1+u^2)(1+v^2)}+\frac{1}{4uv}=1\Rightarrow 1-1+\frac{1}{4uv}=\frac{1}{(1+u^2)(1+v^2)}$$

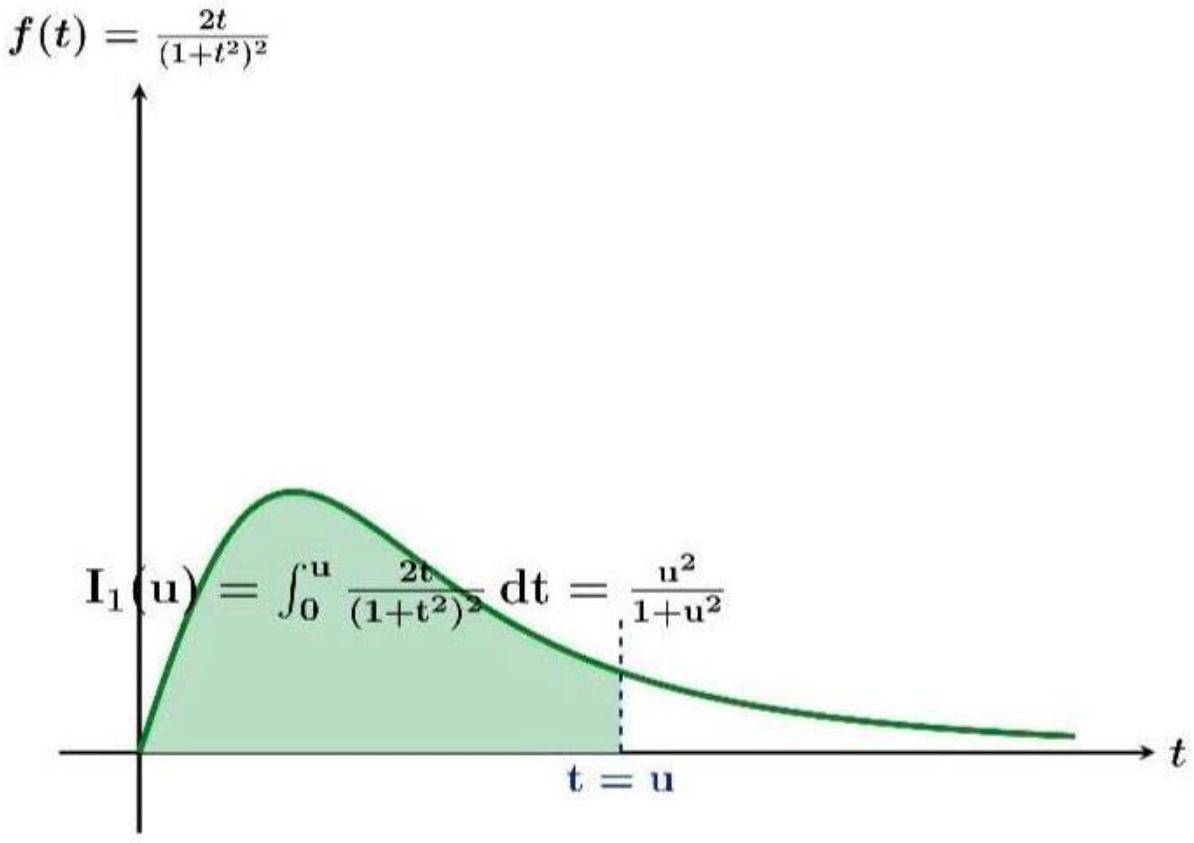
$$\Rightarrow\frac{1}{4uv}=\frac{1}{(1+u^2)(1+v^2)}\Rightarrow 4uv=(1+u^2)(1+v^2)$$

$$\Rightarrow 4uv=1+u^2+v^2+u^2v^2\Rightarrow 1+u^2+v^2+u^2v^2-4uv=0$$

$$\Rightarrow 1-2uv+u^2v^2+u^2-2uv+v^2=0\Rightarrow(1-uv)^2+(u-v)^2=0$$

$$\Rightarrow\begin{cases}(1-uv)^2=0\\(u-v)^2=0\end{cases}\Rightarrow\begin{cases}1-uv=0\\u-v=0\end{cases}\Rightarrow\begin{cases}u=v\\u^2=1\end{cases}$$

$$u,v\in(0,1]\Rightarrow u=1,v=1,\begin{cases}\sin x=1\\\sin y=1\end{cases}\Rightarrow\begin{cases}x=\frac{\pi}{2}\\y=\frac{\pi}{2}\end{cases},\ (x,y)=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$$



JP.635 If $a,b>0;ab=1$ then:

$$\frac{a}{3a^2+b}+\frac{b}{3b^2+a}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a}{3a^2+b}+\frac{b}{3b^2+a}\leq\frac{1}{2}$$

$$2a(3b^2+a)+2b(3a^2+b)\leq(3a^2+b)(3b^2+a)$$

$$6ab^2+2a^2+6a^2b+2b^2\leq 9a^2b^2+3a^3+3b^3+ab$$

$$6b\cdot ab+2(a^2+b^2)+6a\cdot ab\leq 9\cdot 1+3(a^3+b^3)+1$$

$$6(a+b)+2(a^2+b^2)\leq 3(a^3+b^3)+10$$

Denote: $\begin{cases}S=a+b\\P=ab\end{cases}\Rightarrow P=1,\ 6S+2(S^2-2)\leq 3(S^3-3S)+10$

$$6S+2S^2-4\leq 3S^3-9S+10,\qquad 3S^3-2S^2-15S+14\geq 0$$

$$S=a+b\geq 2\sqrt{ab}=2\sqrt{1}=2\Rightarrow S-2\geq 0\quad (1)$$

$$3S^3-6S^2+4S^2-8S-7S+14\geq 0$$

$$3S^2(S-2)+4S(S-2)-7(S-2)\geq 0$$

$$(S-2)(3S^2+4S-7)\geq 0,\qquad 3S^2+4S-7\geq 0$$

$$3S^2-3S+7S-7\geq 0,\qquad 3S(S-1)+7(S-1)\geq 0$$

$$(S-1)(3S+7)\geq 0\quad \text{(true!)}$$

Equality holds for $a=b=1$.

Solution 2 by Marin Chirciu– Romania

$$\frac{a}{3a^2+b}+\frac{b}{3b^2+a}\leq\frac{1}{2},ab=1\Leftrightarrow 3(a^3+b^3)+10\geq 2(a^2+b^2)+6(a+b),ab=1$$

$$\Leftrightarrow 3(t^3-3t)+10\geq 2(t^2-2)+6t,t=a+b\Leftrightarrow 3t^3-2t^2-15t+14\geq 0\Leftrightarrow$$

$$\Leftrightarrow (t-1)(t-2)(3t+4)\geq 0,\text{ see } t\geq 2,t=a+b\geq 2\sqrt{ab}=2\sqrt{1}=2.$$

Equality holds if and only if $a=b=1$.

Remark.

If $a,b>0,ab=1$ and $1\leq\lambda\leq 12$ then

$$\frac{a}{\lambda a^2+b}+\frac{b}{\lambda b^2+a}\leq\frac{2}{\lambda+1}.$$

Solution.

$$\frac{a}{\lambda a^2+b}+\frac{b}{\lambda b^2+a}\leq\frac{2}{\lambda+1}, ab=1\Leftrightarrow$$

$$2\lambda(a^3+b^3)+2\lambda^2+2\geq(\lambda+1)(a^2+b^2)+\lambda(\lambda+1)(a+b), ab=1\Leftrightarrow$$

$$\Leftrightarrow 2\lambda(t^3-3t)+2\lambda^2+2\geq(\lambda+1)(t^2-2)+\lambda(\lambda+1)t, t=a+b\Leftrightarrow$$

$$2\lambda t^3-(\lambda+1)t^2-\lambda(\lambda+7)t+2(\lambda^2+\lambda+2)\geq 0\Leftrightarrow$$

$$\Leftrightarrow (t-2)(2\lambda t^2+(3\lambda-1)t-\lambda^2-\lambda-2)\geq 0, \text{see } t\geq 2,$$

$$t=a+b\geq 2\sqrt{ab}=2\sqrt{1}=2.$$

Equality holds if and only if $a=b=1$.

Note: For $\lambda=3$ we obtain Problem JP.635 from RMM 44, Winter 2026.

Solution 3 by Jose Luis Diaz Barrero-Spain

Since $ab=1$, then $b=\frac{1}{a}$, and we have

$$\frac{a}{3a^2+b}=\frac{a^2}{3a^3+1},\qquad \frac{b}{3b^2+a}=\frac{a}{a^3+3}.$$

So the inequality is equivalent to

$$2a^2(a^3+3)+2a(3a^3+1)\leq(3a^3+1)(a^3+3)$$

Rearranging terms and calling $P(a)$ to the LHS of the resulting calculations, we have to prove that it holds

$$P(a)=3a^6-2a^5-6a^4+10a^3-6a^2-2a+3\geq 0.$$

Polynomial P is palindromic, so divide by $a^3>0$ and set $t=a+\frac{1}{a}\geq 2$. Using that $a^2+\frac{1}{a^2}=t^2-2$ and $a^3+\frac{1}{a^3}=t^3-3t$, we get

$$\frac{P(a)}{a^3}=3t^3-2t^2-15t+14=(t-2)(t-1)(3t+7)\geq 0,$$

since $t\geq 2$. Equality holds if and only if $t=2$, that is, $a=b=1$.

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JP.636 If $a, b > 0$ and $ab = 1$ then:

$$\frac{a}{\sqrt{a^2+1}} + \frac{b}{\sqrt{b^2+1}} \leq \sqrt{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$ab = 1 \Rightarrow b = \frac{1}{a}$$

$$\frac{a}{\sqrt{a^2+1}} + \frac{b}{\sqrt{b^2+1}} = \frac{a}{\sqrt{a^2+1}} + \frac{\frac{1}{a}}{\sqrt{\frac{1}{a^2}+1}} = \frac{a}{\sqrt{a^2+1}} + \frac{1}{\sqrt{a^2+1}} = \frac{a+1}{\sqrt{a^2+1}} \leq \sqrt{2} \quad \text{(to prove)}$$

$$a+1 \leq \sqrt{2}\cdot\sqrt{a^2+1}, \qquad (a+1)^2 \leq 2(a^2+1), \qquad a^2+2a+1 \leq 2a^2+2$$

$$2a^2+2-a^2-2a-1 \geq 0,\ a^2-2a+1 \geq 0,\ (a-1)^2 \geq 0 \quad \text{(True!)}$$

Equality holds for $a = b = 1$.

Solution 2 by Marin Chirciu– Romania

$$\frac{a}{\sqrt{a^2+2}} + \frac{b}{\sqrt{b^2+1}} \leq \sqrt{2} \Leftrightarrow a\sqrt{b^2+1} + b\sqrt{a^2+1} \leq \sqrt{2}\sqrt{a^2+1}\sqrt{b^2+1} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{a^2b^2+a^2} + \sqrt{a^2b^2+b^2} \leq \sqrt{2}\sqrt{a^2b^2+a^2+b^2+1}, ab = 1 \Leftrightarrow$$

$$\sqrt{1+a^2} + \sqrt{1+b^2} \leq \sqrt{2}\sqrt{2+a^2+b^2} \Leftrightarrow \left(\sqrt{1+a^2} + \sqrt{1+b^2}\right)^2 \leq 2(2+a^2+b^2) \Leftrightarrow$$

$$\Leftrightarrow (1+a^2) + (1+b^2) + 2\sqrt{1+a^2}\sqrt{1+b^2} \leq 2(2+a^2+b^2) \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{1+a^2}\sqrt{1+b^2} \leq 2+a^2+b^2, ab = 1 \Leftrightarrow 2\sqrt{2+a^2+b^2} \leq 2+a^2+b^2 \Leftrightarrow$$

$$\Leftrightarrow 2 \leq \sqrt{2+a^2+b^2} \Leftrightarrow 4 \leq 2+a^2+b^2 \Leftrightarrow a^2+b^2 \geq 2, \text{ see } a^2+b^2 \geq 2ab = 2\cdot 1 = 2$$

Equality holds if and only if $a = b = 1$. Remark:

If $a, b > 0, ab = 1$ and $0 \leq \lambda \leq \frac{5}{2}$ then:

$$\frac{a}{\sqrt{a^2+\lambda}} + \frac{b}{\sqrt{b^2+\lambda}} \leq \frac{2}{\sqrt{\lambda+1}}$$

Marin Chirciu

Solution

$$\frac{a}{\sqrt{a^2+\lambda}}+\frac{b}{\sqrt{b^2+\lambda}}\le\frac{2}{\sqrt{\lambda+1}}\Leftrightarrow\sqrt{\lambda+1}\left(a\sqrt{b^2+\lambda}+b\sqrt{a^2+\lambda}\right)\le 2\sqrt{a^2+\lambda}\sqrt{b^2+\lambda}$$

$$\Leftrightarrow\sqrt{\lambda+1}\left(\sqrt{a^2b^2+\lambda a^2}+\sqrt{a^2b^2+\lambda b^2}\right)\le 2\sqrt{a^2b^2+\lambda a^2+\lambda b^2+\lambda^2},ab=1\Leftrightarrow$$

$$\Leftrightarrow\sqrt{\lambda+1}\left(\sqrt{1+\lambda a^2}+\sqrt{1+\lambda b^2}\right)\le 2\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2},ab=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\left(\sqrt{1+\lambda a^2}+\sqrt{1+\lambda b^2}\right)^2\le 4(1+\lambda a^2+\lambda b^2+\lambda^2),ab=1\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\left((1+\lambda a^2)+(1+\lambda b^2)+2\sqrt{1+\lambda a^2}\sqrt{1+\lambda b^2}\right)\le 4(1+\lambda a^2+\lambda b^2+\lambda^2)$$

$$\Leftrightarrow(\lambda+1)\left(2+\lambda a^2+\lambda b^2+2\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}\right)\le 4(1+\lambda a^2+\lambda b^2+\lambda^2).$$

Denoting $t=\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}$, the inequality can be written:

$$(\lambda+1)(1+t^2-\lambda^2+2t)\le 4t^2\Leftrightarrow(3-\lambda)t^2-2(\lambda+1)t+(\lambda+1)^2(\lambda-1)\ge 0\Leftrightarrow$$

$$\Leftrightarrow(t-\lambda-1)\left(t-\tfrac{(\lambda+1)(\lambda-2)}{3-\lambda}\right)\ge 0,\text{ see } t\ge\lambda+1\ge\tfrac{(\lambda+1)(\lambda-2)}{3-\lambda},0\le\lambda\le\tfrac{5}{2}.$$

We have used $t\ge\lambda+1\Leftrightarrow t=\sqrt{1+\lambda a^2+\lambda b^2+\lambda^2}\ge\lambda+1\Leftrightarrow$

$$\Leftrightarrow 1+\lambda a^2+\lambda b^2+\lambda^2\ge(\lambda+1)^2\Leftrightarrow$$

$$\Leftrightarrow a^2+b^2\ge 2,\text{ see } a^2+b^2\ge 2ab=2\cdot 1=2.$$

Equality holds if and only if $a=b=1$. Note: For $\lambda=1$ we obtain Problem JP.636 from RMM 43, Winter 2026.

Solution 3 by Jose Luis Diaz Barrero-Spain

On account of the inequality $(x+y)^2\le 2(x^2+y^2)$, we have

$$\left(\frac{a}{\sqrt{a^2+1}}+\frac{b}{\sqrt{b^2+1}}\right)^2\le 2\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}\right)$$

Since $ab=1$, we get $b=\frac{1}{a}$, and so

$$\frac{b^2}{b^2+1}=\frac{\frac{1}{a^2}}{\frac{1}{a^2}+1}=\frac{1}{1+a^2}$$

Hence

$$\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}=\frac{a^2}{a^2+1}+\frac{1}{a^2+1}=1,$$

and the square of the left-hand side is at most 2. Because the left-hand side is positive, it is at most $\sqrt{2}$. Equality holds if and only if $a = b = 1$.

Alternative solution. Since $a > 0$, there exists a unique $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $a = \tan\alpha$.

From $ab = 1$ we get $b = \frac{1}{\tan\alpha} = \cot\alpha$. As $\sin\alpha > 0$ and $\cos\alpha > 0$,

$$\sqrt{a^2+1} = \sqrt{\tan^2\alpha + 1} = \frac{1}{\cos\alpha}, \quad \sqrt{b^2+1} = \sqrt{\cot^2\alpha + 1} = \frac{1}{\sin\alpha}.$$

Therefore

$$\frac{a}{\sqrt{a^2+1}} = \tan\alpha\cos\alpha = \sin\alpha, \qquad \frac{b}{\sqrt{b^2+1}} = \cot\alpha\sin\alpha = \cos\alpha$$

and the left-hand side of the inequality equals

$$\sin\alpha + \cos\alpha = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \frac{\sqrt{2}}{2}\cos\alpha\right) = \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right) \le \sqrt{2}.$$

Equality holds if and only if $\alpha + \frac{\pi}{4} = \frac{\pi}{2}$, that is, $\alpha = \frac{\pi}{4}$, which corresponds to $a = b = 1$.

JP.637 **Solve for positive real numbers the following system of equations:**

$$x^2 + y^2 = 1,$$

$$1 + 2\cdot(3x - 4x^3)\cdot(3y - 4y^3) = 0.$$

Proposed by Neculai Stanciu – Romania

Solution 1 by proposer

We have $1 + 2(3x - 4x^3)(3y - 4y^3) = 0 \Leftrightarrow 9xy + 16x^3y^3 - 12xy^3 - 12x^3y = -\frac{1}{2} \Leftrightarrow$

$$\Leftrightarrow 9xy + 16(xy)^3 - 12xy(x^2+y^2) = -\frac{1}{2} \overset{x^2+y^2=1}{\Leftrightarrow} -3(xy) + 16(xy)^3 = -\frac{1}{2} \Rightarrow$$

$\Rightarrow xy \in \left\{-\frac{1}{2}; \frac{1}{4}\right\} \overset{x,y>0}{\Rightarrow} xy = \frac{1}{4}$. So, $x + y = \sqrt{x^2+y^2+2xy} = \frac{\sqrt{6}}{2}$ and $x^2 - \frac{\sqrt{6}}{2}x + \frac{1}{4} = 0$, we get that

$$(x, y) \in \left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4}, \frac{\sqrt{6}+\sqrt{2}}{4}\right), \left(\frac{\sqrt{6}+\sqrt{2}}{4}, \frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}.$$

Solution 2 by proposer

$$x = \cos\alpha, y = \sin\alpha; 3y - 4y^3 = \sin 3\alpha, 3x - 4x^3 = -\cos 3\alpha.$$

From $1+2(3x-4x^3)(3y-4y^3)=0$, we deduce that

$$\sin 3\alpha\cos 3\alpha=\frac{1}{2}\Rightarrow \sin 6\alpha=1\Rightarrow \alpha=\frac{1}{6}\arcsin 1=\frac{\pi}{12}.$$

Then $x+y=\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{6}}{2}$, and as in solution 1 we obtain

$$(x,y)=\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$x^2+y^2=1,\ \ x>0,y>0\Rightarrow\ \ \exists t\in\left(0,\frac{\pi}{2}\right):\begin{cases}x=\sin t\\ y=\cos t\end{cases}$$

$$\begin{cases}3x-4x^3=3\sin t-4\sin^3 t=\sin(3t)\\ 3y-4y^3=3\cos t-4\cos^3 t=-(4\cos^3 t-3\cos t)=-\cos(3t)\end{cases}$$

$$1+2(3x-4x^3)(3y-4y^3)=0\Rightarrow 1+2\cdot\sin(3t)\cdot(-\cos(3t))=0$$

$$1-2\sin(3t)\cos(3t)=0\Rightarrow 1-\sin(2\cdot 3t)=0\Rightarrow \sin(6t)=1$$

$$t\in\left(0,\frac{\pi}{2}\right)\Rightarrow 6t\in(0,3\pi)$$

$$\sin(6t)=1, 6t\in(0,3\pi)\Rightarrow 6t\in\left\{\frac{\pi}{2},\frac{5\pi}{2}\right\}\Rightarrow\begin{cases}t_1=\frac{\pi}{12}\\ t_2=\frac{5\pi}{12}\end{cases}$$

(i) $t=\frac{\pi}{12}$

$$x=\sin\left(\frac{\pi}{12}\right)=\sin(15^\circ)=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$y=\cos\left(\frac{\pi}{12}\right)=\cos(15^\circ)=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $t=\frac{5\pi}{12}$

$$x=\sin\left(\frac{5\pi}{12}\right)=\sin(75^\circ)=\sin(45^\circ+30^\circ)=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$y=\cos\left(\frac{5\pi}{12}\right)=\cos(75^\circ)=\cos(45^\circ+30^\circ)=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ=$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$(x,y)\in\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}$$

Solution 4 by Marin Chirciu-Romania

We make the substitution $(x,y)=(\sin t,\cos t), t\in\left(0,\frac{\pi}{2}\right)$ and

$\sin 3t=3\sin t-4\sin^3 t,$

$\cos 3t=4\cos^3 t-3\cos t\Rightarrow \sin 3t=3x-4x^3,\cos 3t=4y^3-3y.$

We obtain $1-2\sin 3t\cos 3t=0\Leftrightarrow 1-\sin 6t=0\Leftrightarrow \sin 6t=1\Leftrightarrow 6t=\frac{\pi}{2}+2k\pi$

$k=0,1\Leftrightarrow t=\frac{\pi}{12}+\frac{k\pi}{3}, k=0,1.$

1. $k=0\Rightarrow t=\frac{\pi}{12}\Rightarrow (x,y)=\left(\sin\frac{\pi}{12},\cos\frac{\pi}{12}\right)\Leftrightarrow (x,y)=\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right).$

1. $k=1\Rightarrow t=\frac{\pi}{12}\Rightarrow (x,y)=\left(\sin\frac{5\pi}{12},\cos\frac{5\pi}{12}\right)\Leftrightarrow (x,y)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

The set of the solutions of the system is $S=\left\{\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right),\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right\}.$

Solution 5 by Jose Luis Diaz Barrero-Spain

Since $x,y>0$ and $x^2+y^2=1$, we can write $x=\sin\alpha, y=\cos\alpha, 0<\alpha<\frac{\pi}{2}$. The triple-angle formulas give

$$3x-4x^3=3\sin\alpha-4\sin^3\alpha=\sin 3\alpha,$$

$$3y-4y^3=3\cos\alpha-4\cos^3\alpha=-\cos 3\alpha.$$

Substituting into the second equation, we get

$$1-2\sin 3\alpha\cos 3\alpha=0\Leftrightarrow 1-\sin 6\alpha=0\Leftrightarrow \sin 6\alpha=1.$$

Hence $6\alpha=\frac{\pi}{2}+2k\pi$, that is, $\alpha=\frac{\pi}{12}+\frac{k\pi}{3}$ with $k\in\mathbb{Z}$. Since $\alpha\in\left(0,\frac{\pi}{2}\right)$, the only admissible values are

$$\alpha=\frac{\pi}{12}\text{ or }\alpha=\frac{5\pi}{12}.$$

Using $\sin\frac{\pi}{12}=\cos\frac{5\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\frac{\pi}{12}=\sin\frac{5\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$, the solutions of the system are

$$(x,y)=\left(\frac{\sqrt{6}-\sqrt{2}}{4},\frac{\sqrt{6}+\sqrt{2}}{4}\right) \text{ and } (x,y)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right).$$

JP.638 If $a,b,c>0$, prove that :

$$a+b+c\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

Proposed by Neculai Stanciu – Romania

Solution 1 by proposer

By AM-GM $a^4+b^4+c^4=\frac{a^4+b^4}{2}+\frac{b^4+c^4}{2}+\frac{c^4+a^4}{2}\ge a^2b^2+b^2c^2+c^2a^2=$

$$=a^2\left(\frac{b^2+c^2}{2}\right)+b^2\left(\frac{c^2+a^2}{2}\right)+c^2\left(\frac{a^2+b^2}{2}\right)\ge a^2bc+b^2ca+c^2ab=$$

$=abc(a+b+c)$, q.e.d. Equality iff $a=b=c$.

Solution 2 by Marin Chirciu-Romania

$$RHS=\sum\frac{a^3}{bc}\overset{Holder}{\ge}\frac{(\sum a)^3}{3\sum bc}\overset{SOS}{\ge}\frac{(\sum a)^3}{(\sum a)^2}=\sum a=LHS$$

Equality holds if and only if $a=b=c$. Remark: If $a,b,c>0$ and $n\in\mathbb{N}^*$ then

$$3\left(\frac{\sum a}{3}\right)^{2n-1}\le\sum\frac{a^{2n+1}}{bc}.$$

Marin Chirciu

Solution.

$$RHS=\sum\frac{a^{2n+1}}{bc}\overset{Holder}{\ge}\frac{(\sum a)^{2n+1}}{3^{2n-1}\sum bc}\overset{SOS}{\ge}\frac{(\sum a)^{2n-1}}{3^{2n-1}\frac{1}{3}(\sum a)^2}=\frac{(\sum a)^{2n-1}}{3^{2n-2}}=3\left(\frac{\sum a}{3}\right)^{2n-1}=LHS$$

Equality holds if and only if $a=b=c$. Note: For $n=1$ we obtain Problem JP.638 from RMM 43.

Solution 3 by Jose Luis Diaz Barrero-Spain

Since

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}=\frac{a^4+b^4+c^4}{abc}$$

and $abc > 0$, the inequality is equivalent to $a^4 + b^4 + c^4 \geq abc(a + b + c)$

Recall that for all real numbers u, v, w, it holds

$$u^2 + v^2 + w^2 - uv - vw - wu = \frac{1}{2}[(u - v)^2 + (v - w)^2 + (w - u)^2] \geq 0$$

Applying this with $(u, v, w) = (a^2, b^2, c^2)$ and then with $(u, v, w) = (ab, bc, ca)$, we obtain $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$,

as desired. Equality holds if and only if $a = b = c$.

Solution 4 by Himat Mammadov-Azerbaijan

$$\int_y^x (t - y)\, dt = \left[\frac{t^2}{2} - yt\right]_y^x = \left(\frac{x^2}{2} - yx\right) - \left(\frac{y^2}{2} - y^2\right) = \frac{x^2 - 2xy + y^2}{2} = \frac{(x - y)^2}{2} \geq 0$$

$$\int_y^x (t - y)\, dt \geq 0 \Rightarrow \frac{x^2 + y^2}{2} \geq xy \Rightarrow x^2 + y^2 \geq 2xy$$

$$\int_{bc}^{a^2} (t - bc)\, dt + \int_{ca}^{b^2} (t - ca)\, dt + \int_{ab}^{c^2} (t - ab)\, dt \geq 0$$

$$\frac{(a^2 - bc)^2 + (b^2 - ca)^2 + (c^2 - ab)^2}{2} \geq 0$$

$$(a^4 - 2a^2bc + b^2c^2) + (b^4 - 2ab^2c + c^2a^2) + (c^4 - 2abc^2 + a^2b^2) \geq 0$$

$$a^4 + b^4 + c^4 + a^2b^2 + b^2c^2 + c^2a^2 \geq 2abc(a + b + c)$$

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$$

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$$

$$a^4 + b^4 + c^4 + a^4 + b^4 + c^4 \geq 2 \cdot (a^2b^2 + b^2c^2 + c^2a^2) \geq 2abc(a + b + c)$$

$$a^4 + b^4 + c^4 \geq abc(a + b + c), \qquad \frac{a^4 + b^4 + c^4}{abc} \geq \frac{abc(a + b + c)}{abc}$$

$$\frac{a^4}{abc} + \frac{b^4}{abc} + \frac{c^4}{abc} \geq a + b + c, \qquad \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \Rightarrow Q.E.D.$$

Solution 5 by Himat Mammadov-Azerbaijan

$$\bar{u}=\left(\sqrt{\frac{a^3}{bc}},\sqrt{\frac{b^3}{ca}},\sqrt{\frac{c^3}{ab}}\right),\bar{v}=\left(\sqrt{abc},\sqrt{abc},\sqrt{abc}\right)$$

$$(\bar{u}\cdot\bar{v})^2\le|\bar{u}|^2\cdot|\bar{v}|^2$$

$$\bar{u}\cdot\bar{v}=\sqrt{\frac{a^3}{bc}}\cdot\sqrt{abc}+\sqrt{\frac{b^3}{ca}}\cdot\sqrt{abc}+\sqrt{\frac{c^3}{ab}}\cdot\sqrt{abc}$$

$$\bar{u}\cdot\bar{v}=\sqrt{\frac{a^3\cdot abc}{bc}}+\sqrt{\frac{b^3\cdot abc}{ca}}+\sqrt{\frac{c^3\cdot abc}{ab}}=\sqrt{a^4}+\sqrt{b^4}+\sqrt{c^4}=a^2+b^2+c^2$$

$$|\bar{u}|^2=\left(\sqrt{\frac{a^3}{bc}}\right)^2+\left(\sqrt{\frac{b^3}{ca}}\right)^2+\left(\sqrt{\frac{c^3}{ab}}\right)^2=\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$|\bar{v}|^2=\left(\sqrt{abc}\right)^2+\left(\sqrt{abc}\right)^2+\left(\sqrt{abc}\right)^2=abc+abc+abc=3abc$$

$$(a^2+b^2+c^2)^2\le\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right)\cdot(3abc)\Rightarrow\frac{(a^2+b^2+c^2)^2}{3abc}\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2,\ \ a^2b^2+b^2c^2+c^2a^2\ge abc(a+b+c)$$

$$a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)\ge a^2b^2+b^2c^2+c^2a^2+2abc(a+bc)\ge$$

$$\ge 3abc(a+b+c)$$

$$(a^2+b^2+c^2)^2\ge 3abc(a+b+c)$$

$$\frac{(a^2+b^2+c^2)^2}{3abc}\ge\frac{3abc(a+b+c)}{3abc}=a+b+c$$

$$a+b+c\le\frac{(a^2+b^2+c^2)^2}{3abc}\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$a+b+c\le\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\Rightarrow Q.E.D.$$

Solution 6 by Himat Mammadov-Azerbaijan

$$z_1=\sqrt{a},z_2=\sqrt{b},z_3=\sqrt{c}\ \ (z_1,z_2,z_3\in\mathbb{R}^+\subset\mathbb{C})$$

$$|z_1|^2=a,|z_2|^2=b,|z_3|^2=c$$

$$w_1=\frac{z_1^3}{z_2z_3},\ \ w_2=\frac{z_2^3}{z_3z_1},\qquad w_3=\frac{z_3^3}{z_1z_2}$$

$$|w_1|=\left|\frac{z_1^3}{z_2z_3}\right|=\frac{|z_1|^3}{|z_2||z_3|}=\frac{a^{\frac{3}{2}}}{\sqrt{b}\sqrt{c}}=\frac{a^{\frac{3}{2}}}{\sqrt{bc}}$$

$$|w_1|^2=\frac{a^3}{bc},\ \ |w_2|^2=\frac{b^3}{ca},\ \ |w_3|^2=\frac{c^3}{ab}$$

$$|w_1|^2+|w_2|^2+|w_3|^2=\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}$$

$$w_1w_2=\frac{z_1^3}{z_2z_3}\cdot\frac{z_2^3}{z_3z_1}=\frac{z_1^2z_2^2}{z_3^2}=\frac{a\cdot b}{c}$$

$$|w_1w_2|=\frac{ab}{c},|w_2w_3|=\frac{bc}{a},\qquad |w_3w_1|=\frac{ca}{b}$$

$$|w_1|^2+|w_2|^2+\left|w_3^2\right|^2\geq|w_1w_2|+|w_2w_3|+|w_3w_1|$$

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\geq\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}$$

$$\frac{ab}{c}+\frac{bc}{a}\geq 2\sqrt{\frac{ab}{c}\cdot\frac{bc}{a}}=2b,\qquad \frac{bc}{a}+\frac{ca}{b}\geq 2c,\qquad \frac{ca}{b}+\frac{ab}{c}\geq 2a$$

$$2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\geq 2(a+b+c)\Rightarrow\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq a+b+c$$

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\geq\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq a+b+c$$

$$a+b+c\leq\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\Rightarrow Q.E.D.$$

JP.639 If $x,y,z>0,\sum x=1$ and $\lambda\leq 4$ then:

$$\lambda\sum x^3+3(\lambda+1)xyz\geq\frac{2\lambda+1}{9}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using pqr – Method

We denote $p=\sum x=1,q=\sum xy,r=xyz.$

We have $\sum x^3 = p^3 - 3pq + 3r, p = 1 \Leftrightarrow \sum x^3 = 1 - 3q + 3r$.

We write the inequality:

$$\lambda(1 - 3q + 3r) + 3(\lambda + 1)r \geq \frac{2\lambda + 1}{9} \Leftrightarrow \frac{7\lambda - 1}{9} - 3\lambda q + 3(2\lambda + 1)r \geq 0,$$

which follows from Schur's inequality

$$p^3 - 4pq + 9r \geq 0, p = 1 \Leftrightarrow 1 - 4q + 9r \geq 0 \Leftrightarrow r \geq \frac{4q - 1}{9}$$

It suffices to show that:

$$\frac{7\lambda - 1}{9} - 3\lambda q + 3(2\lambda + 1) \cdot \frac{4q - 1}{9} \geq 0 \Leftrightarrow 3(\lambda - 4)q \leq 1 \Leftrightarrow 3q \leq 1,$$

see $1 = p^2 = (x + y + z)^2 \geq 3(xy + yz + zx) = 3q$.

Equality holds if and only if $x = y = z = \frac{1}{3}$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

Let $p = x + y + z = 1, \quad q = xy + yz + zx, \quad r = xyz.$

$p^3 + 9r \overset{\text{Schur's}}{\geq} 4pq \overset{p=1}{\Leftrightarrow} r \geq \frac{4q - 1}{9}$. Here:

$$x^3 + y^3 + z^3 = p^3 - 3pq + 3r = 1 - 3q + 3r.$$

Now,

$$\lambda(x^3 + y^3 + z^3) + 3(\lambda + 1)r = \lambda - 3\lambda q + (6\lambda + 3)r \geq$$

$$\geq \lambda - 3\lambda q + (6\lambda + 3)\frac{4q - 1}{9} = \frac{2\lambda + 1}{9} + \frac{(\lambda - 4)(1 - 3q)}{9} \geq \frac{2\lambda + 1}{9}$$

where the last inequality follows from

$$3q = 3(xy + yz + zx) \leq (x + y + z)^2 = 1 \quad \text{and} \quad \lambda \geq 4.$$

Remark: The condition $\lambda \geq 4$ is both necessary and sufficient (the printed condition $\lambda \leq 4$ is incorrect.) Indeed, take $x = y = \frac{1-t}{2}, z = t$ with $0 < t < 1$. As $t \to 0^+$, we obtain

$$\lambda(x^3 + y^3 + z^3) + 3(\lambda + 1)xyz \to \frac{\lambda}{4}.$$

$$\text{Thus, if } \lambda < 4, \text{ then } \frac{\lambda}{4} < \frac{2\lambda + 1}{9}$$

,so the proposed inequality fails for sufficiently small positive t.

Therefore, the exact condition is $\boxed{\lambda \geq 4.}$

JP.640 If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then:

$$\sum \left(\frac{\cos(y-z)}{\sin x}\right)^n \geq 3 \cdot 2^n$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

For $n = 0$ we obtain equality $3 = 3$. For $n = 1$ see Lemma.

For $n \geq 2$ we use Holder's inequality.

$$LHS = \sum \left(\frac{\cos(y-z)}{\sin x}\right)^n \overset{Holder}{\geq} \frac{\left(\sum \frac{\cos(y-z)}{\sin x}\right)^n}{3^{n-1}} \overset{Lemma}{\geq} \frac{6^n}{3^{n-1}} = 3 \cdot 2^n = RHS.$$

Lemma: If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then: $\sum \frac{\cos(y-z)}{\sin x} \geq 6$.

Proof: If $x, y, z > 0, x + y + z = \frac{\pi}{2}$ then: $\sum \tan x \tan y = 1$.

$$x + y + z = \frac{\pi}{2} \Leftrightarrow x + y = \frac{\pi}{2} - z \Leftrightarrow \tan(x+y) = \tan\left(\frac{\pi}{2} - z\right) \Leftrightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \cot z$$

$$\Leftrightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1}{\tan z} \Leftrightarrow \tan z (\tan x + \tan y) = 1 - \tan x \tan y$$

$$\Leftrightarrow \sum \tan x \tan y = 1$$

$$\frac{\cos(y-z)}{\sin x} = \frac{\cos(y-z)}{\sin\left(\frac{\pi}{2} - y - z\right)} = \frac{\cos(y-z)}{\cos(y+z)} = \frac{\cos y \cos z + \sin y \sin z}{\cos y \cos z - \sin y \sin z} =$$

$$= \frac{1 + \tan y \tan z}{1 - \tan y \tan z} = 1 + \frac{2 \tan y \tan z}{1 - \tan y \tan z} = 1 + \frac{2 \tan y \tan z}{\tan x \tan y + \tan x \tan z}.$$

$$LHS = \sum \frac{\cos(y-z)}{\sin x} = \sum \left(1 + \frac{2 \tan y \tan z}{\tan x \tan y + \tan x \tan z}\right) =$$

$$= 3 + 2 \sum \frac{\tan y \tan z}{\tan x \tan y + \tan x \tan z} \overset{Nesbitt}{\geq} 3 + 2 \cdot \frac{3}{2} = 6 = RHS.$$

Equality holds if and only if $x = y = z = \frac{\pi}{6}$.

Solution 2 by Jose Luis Diaz Barrero-Spain

Since $x = \frac{\pi}{2} - (y+z)$, then we have $\sin x = \cos(y+z) > 0$, because $0 < y+z < \frac{\pi}{2}$.

Hence

$$\frac{\cos(y-z)}{\sin x} = \frac{\cos(y-z)}{\cos(y+z)} = \frac{\cos y \cos z + \sin y \sin z}{\cos y \cos z - \sin y \sin z} = \frac{1+\tan y \tan z}{1-\tan y \tan z}$$

Put $u = \tan y \tan z$, $v = \tan z \tan x$ and $w = \tan x \tan y$. These are positive numbers and, since $x + y + z = \frac{\pi}{2}$, it is well known that

$$\tan x \tan y + \tan y \tan z + \tan z \tan x = 1,$$

that is, $u + v + w = 1$. In particular $0 < u, v, w < 1$. Setting

$$A = \frac{1+u}{1-u}, \quad B = \frac{1+v}{1-v}, \quad C = \frac{1+w}{1-w},$$

we must prove that $A^n + B^n + C^n \geq 3 \cdot 2^n$. Indeed, since $\frac{1+u}{1-u} = \frac{2}{1-u} - 1$, the AM-HM inequality yields

$$A + B + C = 2\left(\frac{1}{1-u} + \frac{1}{1-v} + \frac{1}{1-w}\right) - 3 \geq 2 \cdot \frac{9}{(1-u)+(1-v)+(1-w)} - 3$$

$$= 2 \cdot \frac{9}{2} - 3 = 6.$$

Finally, by the power mean inequality, we have

$$A^n + B^n + C^n \geq 3\left(\frac{A+B+C}{3}\right)^n \geq 3 \cdot 2^n,$$

as desired. Equality holds if and only if $u = v = w = \frac{1}{3}$, and substituting in the preceding we get equality when $x = y = z = \frac{\pi}{6}$.

Solution 3 by Jenish Rijal-Bardiya-Nepal

For $n = 0$, the inequality is trivial and we get equality.

Let $a = \tan y \tan z$, $b = \tan z \tan x$, $c = \tan x \tan y$. Since $x + y + z = \frac{\pi}{2}$, we have: $a + b + c = 1$.

Here, $\frac{\cos(y-z)}{\sin x}=\frac{\cos(y-z)}{\cos(y+z)}=\frac{1+\tan y\tan z}{1-\tan y\tan z}=\frac{1+a}{1-a}=\frac{1+a}{b+c}$ (and analogs)

$$\sum_{cyc}\frac{\cos(y-z)}{\sin x}=\sum_{cyc}\frac{1+a}{b+c}=\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}\frac{1}{b+c}\overset{\substack{\text{Nesbitt}\\+\\\text{Bergström}}}{\geq}$$

$$\geq\frac{3}{2}+\frac{(1+1+1)^2}{2(a+b+c)}=\frac{3}{2}+\frac{9}{2}=6$$

Finally for $n>0$: $\sum_{cyc}\left(\frac{\cos(y-z)}{\sin x}\right)^n\overset{\text{Power Mean}}{\geq}3\cdot\left(\frac{1}{3}\cdot\sum_{cyc}\frac{\cos(y-z)}{\sin x}\right)^n\geq 3\cdot\left(\frac{6}{3}\right)^n=3\cdot 2^n$

Equality for $n>0$ holds if and only if $x=y=z=\frac{\pi}{6}$.

Remark: For $0<n<1$, Power Mean is not valid. But for every $0<n<1$,

$f(t)=\left(\frac{1+t}{1-t}\right)^n$; $n>0$ is strictly convex. Hence the desired inequality

can be proven also (for $0<n<1$)via *Jensen's* inequality.

JP.641 Solve for real numbers:

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}=x+y+z$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}=$$

$$=2\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)=$$

$$=2\left(\sqrt{4(x-4)}+\sqrt{25(y-25)}+\sqrt{49(z-49)}\right)\overset{AM-GM}{\leq}$$

$$\leq 2\cdot\left(\frac{4+x-4}{2}+\frac{25+y-25}{2}+\frac{49+z-49}{2}\right)=$$

$$=2\frac{x+y+z}{2}=x+y+z$$

Equality holds for:

$$\begin{cases}4=x-4\\25=y-25\\49=z-49\end{cases}\Leftrightarrow\begin{cases}x=8\\y=50\\z=98\end{cases}$$

Solution 2 by Marin Chirciu-Romania

$$LHS = 4\sqrt{x-4} + 10\sqrt{y-25} + 14\sqrt{z-49} \overset{CBS}{\leq}$$

$$\leq \sqrt{(4^2+10^2+14^2)(x-4+y-25+z-49)} =$$

$$= \sqrt{312(x+y+z-78)} \overset{SOS}{\leq} x+y+z = RHS,$$

where $\sqrt{312(x+y+z-78)} \overset{SOS}{\leq} x+y+z \Leftrightarrow (x+y+z-156)^2 \geq 0$, with equal for

$\frac{\sqrt{x-4}}{4^2} = \frac{\sqrt{y-25}}{10^2} = \frac{\sqrt{z-49}}{14^2}$ and $x+y+z = 156 \Leftrightarrow (x,y,z) = (8,50,98)$.

We deduce that $(x,y,z) = (8,50,98)$ is the unique solution of the equation.

Remark.

Let be $a, b, c > 0$ fixed. Solve for real numbers:

$$2a\sqrt{x-a^2} + 2b\sqrt{y-b^2} + 2c\sqrt{z-c^2} = x+y+z.$$

Marin Chirciu

Solution

$$LHS = 2a\sqrt{x-a^2} + 2b\sqrt{y-b^2} + 2c\sqrt{z-c^2} \overset{CBS}{\leq}$$

$$\leq \sqrt{(4a^2+4b^2+4c^2)(x-a^2+y-b^2+z-c^2)} =$$

$$= \sqrt{(4a^2+4b^2+4c^2)(x+y+z-a^2-b^2-c^2)} \overset{SOS}{\leq} x+y+z = RHS,$$

where $\sqrt{(4a^2+4b^2+4c^2)(x+y+z-a^2-b^2-c^2)} \overset{SOS}{\leq} x+y+z \Leftrightarrow$

$$\Leftrightarrow \left(x+y+z-2(a^2+b^2+c^2)\right)^2 \geq 0, \text{ with equal for:}$$

$\frac{\sqrt{x-a^2}}{4a^2} = \frac{\sqrt{y-b^2}}{4b^2} = \frac{\sqrt{z-c^2}}{4c}$ and $x+y+z = 2(a^2+b^2+c^2) \Leftrightarrow (x,y,z) = (2a^2, 2b^2, 2c^2)$.

We deduce that $(x,y,z) = (2a^2, 2b^2, 2c^2)$ is the unique solution of the equation.

Note: For $(a,b,c) = (2,5,7)$ we obtain Problem JP.641 from RMM 43, Winter 2026.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x-4 \geq 0 \\ y-25 \geq 0 \\ z-49 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq 4 \\ y \geq 25 \\ z \geq 49 \end{cases}$$

$$\forall t>0,\int_0^{\sqrt{a-t^2}}2(t-u)\,du=2\left[tu-\frac{u^2}{2}\right]_0^{\sqrt{a-t^2}}=[2tu-u^2]_0^{\sqrt{a-t^2}}$$

$$=2t\sqrt{a-t^2}-\left(\sqrt{a-t^2}\right)^2-(0-0)=2t\sqrt{a-t^2}-(a-t^2)=2t\sqrt{a-t^2}-a+t^2$$

$$\left(t-\sqrt{a-t^2}\right)^2\geq 0\Leftrightarrow t^2-2t\sqrt{a-t^2}+\left(\sqrt{a-t^2}\right)^2\geq 0$$

$$\Leftrightarrow t^2-2t\sqrt{a-t^2}+a-t^2\geq 0\Leftrightarrow a-2t\sqrt{a-t^2}\geq 0\Leftrightarrow 2t\sqrt{a-t^2}\leq a$$

$$\int_0^{\sqrt{a-t^2}}2(t-u)\,du=2t\sqrt{a-t^2}-a+t^2\leq a-a+t^2=t^2$$

$$t_1=2\left(t_1^2=4\right)\Rightarrow\int_0^{\sqrt{x-4}}2(2-u)du=4\sqrt{x-4}-(x-4)\leq 4$$

$$\Rightarrow 4\sqrt{x-4}-x+4\leq 4\Rightarrow 4\sqrt{x-4}\leq x$$

$$\text{Equality}\Leftrightarrow 2-\sqrt{x-4}=0\Leftrightarrow\sqrt{x-4}=2\Leftrightarrow x-4=4\Leftrightarrow x=8$$

$$t_2=5\left(t_2^2=25\right)\Rightarrow\int_0^{\sqrt{y-25}}2(5-u)\,du=10\sqrt{y-25}-(y-25)\leq 25$$

$$\Rightarrow 10\sqrt{y-25}-y+25\leq 25\Rightarrow 10\sqrt{y-25}\leq y$$

$$\text{Equality}\Leftrightarrow 5-\sqrt{y-25}=0\Leftrightarrow\sqrt{y-25}=5\Leftrightarrow y-25=25\Leftrightarrow y=50$$

$$t_3=7\left(t_3^2=49\right)\Rightarrow\int_0^{\sqrt{z-49}}2(7-u)\,du=14\sqrt{z-49}-(z-49)\leq 49$$

$$\Rightarrow 14\sqrt{z-49}-z+49\leq 49\Rightarrow 14\sqrt{z-49}\leq z$$

$$\text{Equality}\Leftrightarrow 7-\sqrt{z-49}=0\Leftrightarrow\sqrt{z-49}=7\Leftrightarrow z-49=49\Leftrightarrow z=98$$

$$\sum\Rightarrow 4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\leq x+y+z$$

$$\begin{cases}4\sqrt{x-4}=x\\10\sqrt{y-25}=y\\14\sqrt{z-49}=z\end{cases}\Leftrightarrow\begin{cases}\sqrt{x-4}=2\\\sqrt{y-25}=5\\\sqrt{z-49}=7\end{cases}\Leftrightarrow\begin{cases}x-4=4\\y-25=25\\z-49=49\end{cases}\Leftrightarrow\begin{cases}x=8\\y=50\\z=98\end{cases}$$

$$(x,y,z)=(8,50,98)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x-4\geq 0 \\ y-25\geq 0 \\ z-49\geq 0 \end{cases} \Leftrightarrow \begin{cases} x\geq 4 \\ y\geq 25 \\ z\geq 49 \end{cases}$$

$$f(t)=\sqrt{t},\ \ t>0$$

$$f'(t)=\frac{1}{2\sqrt{t}},\ \ f''(t)=-\frac{1}{4t^{\frac{3}{2}}}<0.\ (\forall t>0)\Rightarrow f(t) \text{ is strictly concave}$$

Jensen's Inequality: $\lambda_1 f(t_1)+\lambda_2 f(t_2)+\lambda_3 f(t_3)\leq f(\lambda_1 t_1+\lambda_2 t_2+\lambda_3 t_3)$

$$\lambda_1=\frac{4}{78},\lambda_2=\frac{25}{78},\lambda_3=\frac{49}{78}\ \ \left(\lambda_1+\lambda_2+\lambda_3=\frac{4+25+49}{78}=1\right)$$

$$t_1=\frac{x-4}{4},t_2=\frac{y-25}{25},t_3=\frac{z-49}{49}$$

$$\frac{4}{78}\sqrt{\frac{x-4}{4}}+\frac{25}{78}\sqrt{\frac{y-25}{25}}+\frac{49}{78}\sqrt{\frac{z-49}{49}}\leq$$

$$\leq\sqrt{\frac{4}{78}\cdot\frac{x-4}{4}+\frac{25}{78}\cdot\frac{y-25}{25}+\frac{49}{78}\cdot\frac{z-49}{49}}$$

$$\frac{4}{78}\cdot\frac{\sqrt{x-4}}{2}+\frac{25}{78}\cdot\frac{\sqrt{y-25}}{5}+\frac{49}{78}\cdot\frac{\sqrt{z-49}}{7}\leq\sqrt{\frac{(x-4)+(y-25)+(z-49)}{78}}$$

$$\frac{2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}}{78}\leq\sqrt{\frac{x+y+z-78}{78}}$$

$$2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\leq 78\sqrt{\frac{x+y+z-78}{78}}$$

$$2\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)\leq 2\cdot 78\sqrt{\frac{x+y+z-78}{78}}$$

$$4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\leq 156\sqrt{\frac{x+y+z-78}{78}}$$

$$x+y+z\leq 156\sqrt{\frac{x+y+z-78}{78}}\Rightarrow(x+y+z)^2\leq 156^2\cdot\frac{x+y+z-78}{78}$$

$$(x+y+z)^2 \leq 312(x+y+z-78) \Leftrightarrow (x+y+z-156)^2 \leq 0 \Rightarrow$$

$$\Rightarrow x+y+z=156$$

Equality Condition $\Leftrightarrow t_1=t_2=t_3$

$$\Leftrightarrow \frac{x-4}{4}=\frac{y-25}{25}=\frac{z-49}{49}=k$$

$$\Leftrightarrow \begin{cases} x-4=4k \\ y-25=25k \\ z-49=49k \end{cases} \Leftrightarrow \begin{cases} x=4k+4 \\ y=25k+25 \\ z=49k+49 \end{cases}$$

$$x+y+z=(4k+4)+(25k+25)+(49k+49)=78k+78=156 \Leftrightarrow$$

$$\Leftrightarrow 78k=78 \Leftrightarrow k=1$$

$$\Rightarrow \begin{cases} x=4(1)+4 \\ y=25(1)+25 \\ z=49(1)+49 \end{cases} \Leftrightarrow \begin{cases} x=4+4 \\ y=25+25 \\ z=49+49 \end{cases} \Leftrightarrow \begin{cases} x=8 \\ y=50 \\ z=98 \end{cases}$$

$$(x,y,z)=(8,50,98)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x-4\geq 0 \\ y-25\geq 0 \\ z-49\geq 0 \end{cases} \Leftrightarrow \begin{cases} x\geq 4 \\ y\geq 25 \\ z\geq 49 \end{cases}$$

$$u=\left(\sqrt{x-4},\sqrt{y-25},\sqrt{z-49}\right),\ \ v=(2,5,7)$$

$$u\cdot v=2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}$$

$$\|u\|^2=\left(\sqrt{x-4}\right)^2+\left(\sqrt{y-25}\right)^2+\left(\sqrt{z-49}\right)^2=(x-4)+(y-25)+(z-49)=$$

$$=x+y+z-78$$

$$\|v\|^2=2^2+5^2+7^2=4+25+49=78$$

$$(u\cdot v)^2\leq \|u\|^2\cdot\|v\|^2$$

$$\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)^2\leq (x+y+z-78)\cdot 78$$

$$4\left(2\sqrt{x-4}+5\sqrt{y-25}+7\sqrt{z-49}\right)^2\leq 4\cdot 78\cdot(x+y+z-78)$$

$$\left(4\sqrt{x-4}+10\sqrt{y-25}+14\sqrt{z-49}\right)^2\leq 312(x+y+z-78)$$

$$(x+y+z)^2\leq 312(x+y+z)-312\cdot 78$$

$$(x+y+z)^2\leq 312(x+y+z)-24336$$

$$(x+y+z)^2-312(x+y+z)+24336\leq 0$$

$$(x+y+z)^2-2\cdot 156\cdot(x+y+z)+156^2\le 0$$

$$(x+y+z-156)^2\le 0$$

$$\forall S\in\mathbb{R}_1(S-156)^2\ge 0\Rightarrow(x+y+z-156)^2=0\Rightarrow x+y+z=156$$

$$\text{Equality Condition}\Leftrightarrow\frac{\sqrt{x-4}}{2}=\frac{\sqrt{y-25}}{5}=\frac{\sqrt{z-49}}{7}=k$$

$$\Leftrightarrow\begin{cases}\sqrt{x-4}=2k\\ \sqrt{y-25}=5k\\ \sqrt{z-49}=7k\end{cases}\Leftrightarrow\begin{cases}x-4=4k^2\\ y-25=25k^2\\ z-49=49k^2\end{cases}\Leftrightarrow\begin{cases}x=4k^2+4\\ y=25k^2+25\\ z=49k^2+49\end{cases}$$

$$x+y+z=(4k^2+4)+(25k^2+25)+(49k^2+49)=78k^2+78$$

$$78k^2+78=156\Leftrightarrow 78k^2=78\Leftrightarrow k^2=1\Leftrightarrow k=1\ \ (k>0)$$

$$\Rightarrow\begin{cases}x=4(1)^2+4\\ y=25(1)^2+25\\ z=49(1)^2+49\end{cases}\Leftrightarrow\begin{cases}x=4+4\\ y=25+25\\ z=49+49\end{cases}\Leftrightarrow\begin{cases}x=8\\ y=50\\ z=98\end{cases}$$

$$(x,y,z)=(8,50,98)$$

JP.642 If $a,b,c>0,\sum a=3$ and $n\in\mathbb{N}$ then:

$$\sum\left(\frac{a^3}{a^2+ab+b^2}\right)^n\ge\frac{3}{3^n}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

For $n=0$ we obtain equality $3=3$.

For $n=1$ see Lemma.

For $n\ge 2$ we use Holder's inequality.

$$LHS=\sum\left(\frac{a^3}{a^2+ab+b^2}\right)^n\overset{Holder}{\ge}\frac{\left(\sum\frac{a^3}{a^2+ab+b^2}\right)^n}{3^{n-1}}\overset{Lemma}{\ge}\frac{1^n}{3^{n-1}}=\frac{3}{3^n}=RHS.$$

Lemma.

If $a,b,c>0,\sum a=3$ then:

$$\sum\frac{a^3}{a^2+ab+b^2}\ge 1.$$

Proof

If $a, b > 0$ then:

$$\frac{a^3}{a^2+ab+b^2} \geq \frac{2a-b}{3}$$

$$\frac{a^3}{a^2+ab+b^2} \geq \frac{2a-b}{3} \Leftrightarrow 3a^3 \geq (2a-b)(a^2+ab+b^2) \Leftrightarrow (a-b)^2(a+b) \geq 0$$

Let's get back to the main problem.

$$LHS = \sum \frac{a^3}{a^2+ab+b^2} \geq \sum \frac{2a-b}{3} = \frac{1}{3}\sum a = \frac{1}{3}\cdot 3 = 1 = RHS.$$

Equality holds if and only if $a = b = c = 1$.

Solution 2 by Jose Luis Diaz Barrero-Spain

For $a, b > 0$ we have

$$\frac{a^3}{a^2+ab+b^2} - \frac{2a-b}{3} = \frac{(a+b)(a-b)^2}{3(a^2+ab+b^2)} \geq 0.$$

Hence, adding cyclically and using $a+b+c=3$, we have

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3} = 1.$$

Write $x = \frac{a^3}{a^2+ab+b^2}, y = \frac{b^3}{b^2+bc+c^2}$ and $z = \frac{c^3}{c^2+ca+a^2}$. All three are positive. For $n \geq 1$ the function $t \to t^n$ is convex on $(0, \infty)$, so by Jensen's inequality, we have

$$x^n + y^n + z^n \geq 3\left(\frac{x+y+z}{3}\right)^n = \frac{S^n}{3^{n-1}} \geq \frac{1}{3^{n-1}} = \frac{3}{3^n},$$

where $S = x+y+z$. Equality holds if and only if $a = b = c = 1$.

Note. If $n = 0$ the inequality trivially holds, as can be immediately checked.

Solution 3 by Jenish Rijal-Bardiya-Nepal

For $n = 0$, the inequality becomes trivial. For $n = 1$, the inequality becomes:

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq 1$$

Here,

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{(a^2)^2}{a(a^2+ab+b^2)} \overset{\text{Bergström}}{\geq} \frac{(\sum a^2)^2}{\sum a(a^2+ab+b^2)} = \frac{(\sum a^2)^2}{(\sum a)(\sum a^2)} =$$

$$= \frac{\sum a^2}{\sum a} \overset{3\sum a^2 \geq (\sum a)^2}{\geq} \frac{\sum a}{3} = \frac{3}{3} = 1$$

For $n \geq 2$,

$$\sum_{cyc}\left(\frac{a^3}{a^2+ab+b^2}\right)^n \overset{\text{Power Mean}}{\geq} 3\cdot\left(\frac{1}{3}\cdot\sum_{cyc}\frac{a^3}{a^2+ab+b^2}\right)^n \geq 3\cdot\left(\frac{1}{3}\cdot 1\right)^n = \frac{3}{3^n}$$

$$\textit{Equality holds if and only if } \ a=b=c=1.$$

JP.643 Let $\lambda > 0$ fixed. If $x, y, z > 0, x^2+y^2+z^2 = \lambda$ then find the maximum value of

$$P = \frac{x}{\lambda - yz} + \frac{y}{\lambda - zx} + \frac{z}{\lambda - xy}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

With the substitution $\left(\frac{x}{\sqrt{\lambda}}, \frac{y}{\sqrt{\lambda}}, \frac{z}{\sqrt{\lambda}}\right) = (a, b, c)$ the problem can be reformulated.

If $a, b, c > 0, a^2+b^2+c^2 = 1$ then find the maximum value of

$$P = \frac{a}{1-bc} + \frac{b}{1-ca} + \frac{c}{1-ab}.$$

We prove: Lemma:

If $a, b, c > 0, a^2+b^2+c^2 = 1$ then:

$$\frac{a}{1-bc} \leq \frac{1}{\sqrt{3}}\left[1+\frac{1}{2}\left(\frac{a^2}{a^2+b^2}+\frac{a^2}{a^2+c^2}\right)\right]$$

Proof: We have:

$$\frac{1}{1-bc} \overset{AGM}{\leq} \frac{a}{1-\frac{b^2+c^2}{2}} = \frac{a}{1-\frac{1-a^2}{2}} = \frac{2a}{a^2+1} = \frac{1}{\sqrt{3}}\cdot\frac{2\sqrt{3}a}{a^2+1} \overset{AGM}{\leq} \frac{1}{\sqrt{3}}\cdot\frac{1+3a^2}{a^2+1} =$$

$$= \frac{1}{\sqrt{3}}\left(1+\frac{2a^2}{a^2+1}\right) = \frac{1}{\sqrt{3}}\left(1+\frac{2a^2}{a^2+(a^2+a^2+a^2)}\right) =$$

$$=\frac{1}{\sqrt{3}}\left(1+\frac{2a^2}{(a^2+b^2)+(a^2+c^2)}\right)=\frac{1}{\sqrt{3}}\left[1+\frac{1}{2}\left(\frac{a^2}{a^2+b^2}+\frac{a^2}{a^2+c^2}\right)\right].$$

Equality holds if and only if $a=b=c=\frac{1}{\sqrt{3}}$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$P=\sum\frac{a}{1-bc}\leq\sum\frac{1}{\sqrt{3}}\left[1+\frac{1}{2}\left(\frac{a^2}{a^2+b^2}+\frac{a^2}{a^2+c^2}\right)\right]=$$

$$=\frac{1}{\sqrt{3}}\left(3+\frac{1}{2}\sum\left(\frac{a^2}{a^2+b^2}+\frac{a^2}{a^2+c^2}\right)\right)=$$

$$=\frac{1}{\sqrt{3}}\left(3+\frac{1}{2}\sum\left(\frac{a^2}{a^2+b^2}+\frac{b^2}{b^2+a^2}\right)\right)=\frac{1}{\sqrt{3}}\left(3+\frac{1}{2}\sum 1\right)=\frac{1}{\sqrt{3}}\left(3+\frac{1}{2}\cdot 3\right)=\frac{3\sqrt{3}}{2}.$$

We deduce that $\max P=\frac{3\sqrt{3}}{2}$ and the maximum attained for $(a,b,c)=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Returning to the initial problem, we deduce that $\max P=\frac{3\sqrt{3}}{2}$ and the attained maximum for

$$(x,y,z)=\left(\sqrt{\frac{\lambda}{3}},\sqrt{\frac{\lambda}{3}},\sqrt{\frac{\lambda}{3}}\right).$$

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\text{Here, } P=\sum_{cyc}\frac{x}{\lambda-yz}\overset{\text{AM-GM}}{\leq}\sum_{cyc}\frac{x}{\lambda-\left(\frac{y^2+z^2}{2}\right)}=\sum_{cyc}\frac{2x}{2x^2+y^2+z^2}=\sum_{cyc}\frac{2x}{\lambda+x^2}$$

$$\text{Lemma: For any } t>0,\quad \frac{2t}{1+t^2}\leq\frac{3t+\sqrt{3}}{4}$$

$$\text{Proof: } 3t^3+\sqrt{3}t^2+\sqrt{3}=\frac{15t^3}{5}+\frac{5\sqrt{3}t^2}{5}+3\cdot\frac{1}{\sqrt{3}}\overset{\text{Weighted AM-GM}}{\geq}5\cdot\sqrt[5]{\frac{15t^3}{5}\cdot\frac{5\sqrt{3}t^2}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^3}=5t$$

$$\text{Thus, } \left(3t+\sqrt{3}\right)\left(1+t^2\right)=3t^3+\sqrt{3}t^2+\sqrt{3}+3t\geq 5t+3t=8t\ \therefore\frac{2t}{1+t^2}\leq\frac{3t+\sqrt{3}}{4}$$

$$\text{With } t=\frac{x}{\sqrt{\lambda}}\text{, we get: } \frac{2x}{\lambda+x^2}\leq\frac{3x+\sqrt{3\lambda}}{4\lambda}$$

Hence, $P \leq \sum_{cyc} \frac{2x}{\lambda + x^2} \leq \sum_{cyc} \frac{3x + \sqrt{3\lambda}}{4\lambda} = \frac{3(x+y+z) + 3\sqrt{3\lambda}}{4\lambda} \overset{(\sum x)^2 \leq 3(\sum x^2)}{\leq}$

$$\leq \frac{3\sqrt{3(x^2+y^2+z^2)} + 3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3(x^2+y^2+z^2)} + 3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3(\lambda)} + 3\sqrt{3\lambda}}{4\lambda} = \frac{3\sqrt{3}}{2\sqrt{\lambda}}$$

Equality holds if and only if $x = y = z = \sqrt{\frac{\lambda}{3}}$. **Remark.**

Alternatively, the inequality can be proved directly using *Jensen's* inequality:

$$P \leq \sum_{cyc} \frac{2x}{\lambda + x^2} \overset{\text{Jensen}}{\leq} 3f\left(\frac{x+y+z}{3}\right) \overset{(\sum x)^2 \leq 3(\sum x^2)}{\leq} 3f\left(\frac{\sqrt{3(x^2+y^2+z^2)}}{3}\right) = 3f\left(\sqrt{\frac{\lambda}{3}}\right) = \frac{3\sqrt{3}}{2\sqrt{\lambda}}$$

where $f(t) = \frac{2t}{\lambda + t^2}$, **which is increasing and concave on** $(0, \sqrt{\lambda}\,]$.

JP.644 **Solve for real numbers:**

$$\sqrt{x^2 + (4-x)^2} + \sqrt{y^2 + (8-y)^2} = 6\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\sqrt{\frac{x^2 + (4-x)^2}{2}} \overset{QM-AM}{\geq} \frac{x + (4-x)}{2} = 2$$

$$\sqrt{x^2 + (4-x)^2} \geq 2\sqrt{2} \qquad (1)$$

$$\sqrt{\frac{y^2 + (8-y)^2}{2}} \overset{QM-AM}{\geq} \frac{y + (8-y)}{2} = \frac{8}{2} = 4$$

$$\sqrt{y^2 + (8-y)^2} \geq 4\sqrt{2} \qquad (2)$$

By adding (1);(2): $LHS \geq 2\sqrt{2} + 4\sqrt{2} = 6\sqrt{2}$

Equality holds for: $x = 2; y = 4$.

Solution 2 by Marin Chirciu-Romania

$$LHS = \sqrt{x^2 + (4-x)^2}\sqrt{y^2 + (8-y^2)} \overset{Minkowski}{\geq} \sqrt{(x+y)^2 + (4-x+8-y)^2} =$$

$$= \sqrt{(x+y)^2 + (12-x-y)^2} \overset{SOS}{\geq} 6\sqrt{2} = RHS,$$

where $\sqrt{(x+y)^2 + (12-x-y)^2} \overset{SOS}{\geq} 6\sqrt{(x+y-6)^2} \geq 0$, with equal for

$\frac{x}{x-4} = \frac{y}{8-y}$ and $x + y = 6 \Leftrightarrow (x, y) = (2, 4)$.

We deduce that $(x, y) = (2, 4)$ is the unique solution of the equation.

Remark.

Let be $a, b > 0$. Solve for real numbers

$$\sqrt{x^2 + (2a-x)^2}\sqrt{y^2 + (2b-y)^2} = (a+b)\sqrt{2}$$

Marin Chirciu

Solution

$$LHS = \sqrt{x^2 + (2a-x)^2}\sqrt{y^2 + (2b-y)^2} \overset{Minkowski}{\geq} \sqrt{(x+y)^2 + (2a-x+2b-y)^2} =$$

$$= \sqrt{(x+y)^2 + (2a+2b-x-y)^2} \overset{SOS}{\geq} (a+b)\sqrt{2} = RHS,$$

where $\sqrt{(x+y)^2 + (2a+2b-x-y)^2} \overset{SOS}{\geq} (a+b)\sqrt{2} \Leftrightarrow$

$\Leftrightarrow (x+y-a-b)^2 \geq 0$, with equal for

$\frac{x}{2a-x} = \frac{y}{2b-y}$ and $x + y = a + b \Leftrightarrow (x, y) = (a, b)$.

We deduce that $(x, y) = (a, b)$ is the unique solution of the equation.

Note.

For $(a, b) = (2, 4)$ we obtain Problem JP.641 from RMM 43, Winter 2026.

Solution 3 by Jenish Rijal-Bardiya-Nepal

Here,

$$\sqrt{x^2 + (4-x)^2} + \sqrt{y^2 + (8-y)^2} = \sqrt{2x^2 - 8x + 16} + \sqrt{2y^2 - 16y + 64} =$$

$$= \sqrt{2(x-2)^2 + 8} + \sqrt{2(y-4)^2 + 32} \geq \sqrt{8} + \sqrt{32} = 6\sqrt{2}$$

Equality holds for $x = 2$ and $y = 4$.

Hence, $(x, y) = (2, 4)$ is the unique solution.

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JP.645 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=\sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)=$$

$$=\sqrt{3}\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)\overset{CBS}{\leq}\sqrt{3}\cdot\sqrt{(x+y+z)(xy+yz+zx)}=$$

$$=\sqrt{3}\cdot\sqrt{xy+yz+zx}\leq\sqrt{3}\cdot\sqrt{\frac{1}{3}(x+y+z)^2}=\sqrt{3}\cdot\frac{1}{\sqrt{3}}\cdot(x+y+z)=1$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}\leq 1$$

Equality holds for: $x=y=z=\frac{1}{3}$.

Solution 2 by Marin Chirciu-Romania

$$LHS=x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=\sum\sqrt{x}\sqrt{3xy}\overset{SOS}{\leq}\sqrt{\sum x\left(\sum x\right)^2}=$$

$$=\sqrt{1\cdot 1^2}=1=RHS,\ \text{with equal for}\ \frac{\sqrt{x}}{\sqrt{3xy}}=\frac{\sqrt{y}}{\sqrt{3yz}}=\frac{\sqrt{z}}{\sqrt{3zx}}\ \text{and}\ x+y+z=1\Leftrightarrow$$

$$\Leftrightarrow(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

We deduce that $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is the unique solution of the system.

Remark.

Solve for real numbers $\begin{cases} x,y,z,t>0 \\ x+y+z+t=1 \\ x\sqrt{\frac{8}{3}y}+y\sqrt{\frac{8}{3}z}+z\sqrt{\frac{8}{3}t}+t\sqrt{\frac{8}{3}x}=1 \end{cases}$

Marin Chirciu

Solution

$$LHS = x\sqrt{\frac{8}{3}y} + y\sqrt{\frac{8}{3}z} + z\sqrt{\frac{8}{3}t} + t\sqrt{\frac{8}{3}x} = \sum \sqrt{x}\sqrt{\frac{8}{3}xy} \overset{CBS}{\leq} \sqrt{\sum x \sum \frac{8}{3}xy} \overset{SOS}{\leq}$$

$$\leq \sqrt{\sum x (\sum x)^2} = \sqrt{1 \cdot 1^2} = 1 = RHS,$$ with equal for

$\frac{\sqrt{x}}{\sqrt{\frac{8}{3}xy}} = \frac{\sqrt{y}}{\sqrt{\frac{8}{3}yz}} = \frac{\sqrt{z}}{\sqrt{\frac{8}{3}zt}} = \frac{\sqrt{t}}{\sqrt{\frac{8}{3}tx}}$ and $x + y + z + t = 1 \Leftrightarrow (x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

$$\frac{8}{3}\sum xy \leq \left(\sum x\right)^2 \Leftrightarrow \sum (x-y)^2 \geq 0.$$

We denoted above $\sum xy = xy + xz + xt + yz + yt + zt$ and

$$\sum (x-y)^2 = (x-y)^2 + (x-z)^2 + (x-t)^2 + (y-z)^2 + (y-t)^2 + (z-t)^2$$

We deduce that $(x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the unique solution of the system.

Remark.

Solve for real numbers $\begin{cases} x, y, z, t > 0 \\ x^2 + y^2 + z^2 + t^2 = 1 \\ x\sqrt{\frac{3}{2}y} + y\sqrt{\frac{3}{2}z} + z\sqrt{\frac{3}{2}t} + t\sqrt{\frac{3}{2}x} = 1 \end{cases}$

Marin Chirciu

Solution

$$LHS = \sum \sqrt{x}\sqrt{\frac{3}{2}xy} \overset{CBS}{\leq} \sqrt{\sum x \sum \frac{3}{2}xy} \overset{SOS}{\leq} \sqrt{\sum x \left(\sum x^2\right)} = ???$$

$= \sqrt{1 \cdot 1^2} = 1 = RHS,$ with equal for $\frac{\sqrt{x}}{\sqrt{\frac{8}{3}xy}} = \frac{\sqrt{y}}{\sqrt{\frac{8}{3}yz}} = \frac{\sqrt{z}}{\sqrt{\frac{8}{3}zt}} = \frac{\sqrt{t}}{\sqrt{\frac{8}{3}tx}}$ and $x + y + z + t = 1 \Leftrightarrow$

$$\Leftrightarrow (x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

$$\frac{8}{3}\sum xy \leq \left(\sum x\right)^2 \Leftrightarrow \sum (x-y)^2 \geq 0.$$

We denoted above $\sum xy = xy + xz + xt + yz + yt + zt$ and

$$\sum (x-y)^2 = (x-y)^2 + (x-z)^2 + (x-t)^2 + (y-z)^2 + (y-t)^2 + (z-t)^2$$

We deduce that $(x, y, z, t) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is the unique solution of the system.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \Leftrightarrow \sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)=1$$

$$\Leftrightarrow \left[\sqrt{3}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)\right]^2=1^2 \Leftrightarrow 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=1$$

$$\left(\sqrt{x}\cdot\sqrt{xy}+\sqrt{y}\cdot\sqrt{yz}+\sqrt{z}\cdot\sqrt{zx}\right)^2 \leq$$

$$\leq \left(\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+\left(\sqrt{z}\right)^2\right)\left(\left(\sqrt{xy}\right)^2+\left(\sqrt{yz}\right)^2+\left(\sqrt{zx}\right)^2\right)$$

$$\Leftrightarrow \left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2 \leq (x+y+z)(xy+yz+zx)$$

$$\Rightarrow 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2 \leq 3(x+y+z)(xy+yz+zx)$$

$$x+y+z=1 \wedge 3\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=1 \Rightarrow 1 \leq 3(1)(xy+yz+zx) \Leftrightarrow$$

$$\Leftrightarrow 3(xy+yz+zx) \geq 1 \quad \textbf{(i)}$$

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$$

$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2+z^2)-2(xy+yz+zx) \geq 0$$

$$\Leftrightarrow x^2+y^2+z^2 \geq xy+yz+zx$$

$$\Leftrightarrow (x^2+y^2+z^2)+2(xy+yz+zx) \geq (xy+yz+zx)+2(xy+yz+zx) \Leftrightarrow$$

$$\Leftrightarrow (x+y+z)^2 \geq 3(xy+yz+zx)$$

$$x+y+z=1 \Rightarrow 1^2 \geq 3(xy+yz+zx) \Leftrightarrow 3(xy+yz+zx) \leq 1 \quad \textbf{(ii)}$$

$$\begin{cases} 3(xy+yz+zx) \geq 1 \\ 3(xy+yz+zx) \leq 1 \end{cases} \Rightarrow 3(xy+yz+zx)=1 \Rightarrow (x+y+z)^2=3(xy+yz+zx)$$

$$\Leftrightarrow 1=3(xy+yz+zx) \Leftrightarrow x^2+y^2+z^2+2xy+2yz+2zx=3xy+3yz+3zx$$

$$\Leftrightarrow x^2+y^2+z^2-xy-yz-zx=0 \Leftrightarrow \frac{1}{2}\left[(x-y)^2+(y-z)^2+(z-x)^2\right]=0$$

$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2=0 \Leftrightarrow x-y=0 \wedge y-z=0 \wedge z-x=0 \Leftrightarrow$$

$$\Leftrightarrow x=y=z$$

$$x+y+z=1 \wedge x=y=z \Rightarrow 3x=1 \Leftrightarrow x=\frac{1}{3} \Rightarrow x=y=z=\frac{1}{3}$$

$$\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}=\frac{1}{3}(1)+\frac{1}{3}(1)+\frac{1}{3}(1)=1$$

$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x,y,z>0 \\ x+y+z=1 \\ x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}=1 \end{cases}$$

$$x\sqrt{3y}=\sqrt{x}\cdot\sqrt{3xy}\le\frac{x+3xy}{2}$$

$$y\sqrt{3z}=\sqrt{y}\cdot\sqrt{3yz}\le\frac{y+3yz}{2}$$

$$z\sqrt{3x}=\sqrt{z}\cdot\sqrt{3zx}\le\frac{z+3zx}{2}$$

$$x\sqrt{3y}+y\sqrt{3z}+z\sqrt{3x}\le\frac{x+3xy}{2}+\frac{y+3yz}{2}+\frac{z+3zx}{2}$$

$$1\le\frac{(x+y+z)+3(xy+yz+zx)}{2}$$

$$x+y+z=1\Rightarrow 1\le\frac{1+3(xy+yz+zx)}{2}$$

$$\Leftrightarrow 2\le 1+3(xy+yz+zx)\Leftrightarrow 2-1\le 3(xy+yz+zx)\Leftrightarrow$$

$$\Leftrightarrow 3(xy+yz+zx)\ge 1 \quad \text{(i)}$$

$$(x-y)^2+(y-z)^2+(z-x)^2\ge 0$$

$$\Leftrightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\ge 0$$

$$\Leftrightarrow 2(x^2+y^2+z^2)-2(xy+yz+zx)\ge 0\Leftrightarrow x^2+y^2+z^2\ge xy+y+zx$$

$$\Leftrightarrow (x^2+y^2+z^2)+2(xy+yz+zx)\ge(xy+yz+zx)+2(xy+yz+zx)$$

$$\Leftrightarrow (x+y+z)^2\ge 3(xy+yz+zx)$$

$$x+y+z=1\Rightarrow 1^2\ge 3(xy+yz+zx)\Leftrightarrow 3(xy+yz+zx)\le 1 \quad \text{(ii)}$$

$$\begin{cases} 3(xy+yz+zx)\ge 1 \\ 3(xy+yz+zx)\le 1 \end{cases}\Rightarrow 3(xy+yz+zx)=1$$

$$\Rightarrow \begin{cases} \sqrt{x} = \sqrt{3xy} \\ \sqrt{y} = \sqrt{3yz} \\ \sqrt{z} = \sqrt{3zx} \end{cases} \Leftrightarrow \begin{cases} x = 3xy \\ y = 3yz \\ z = 3zx \end{cases} \Leftrightarrow \begin{cases} x(1-3y) = 0 \\ y(1-3z) = 0 \\ z(1-3x) = 0 \end{cases}$$

$$x, y, z > 0 \Rightarrow \begin{cases} 1-3y = 0 \\ 1-3z = 0 \\ 1-3x = 0 \end{cases} \Leftrightarrow \begin{cases} 3y = 1 \\ 3z = 1 \\ 3x = 1 \end{cases} \Leftrightarrow x = y = z = \frac{1}{3}$$

$$x + y + z = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\left(\frac{1}{3}\right)\sqrt{3 \cdot \frac{1}{3}} + \left(\frac{1}{3}\right)\sqrt{3 \cdot \frac{1}{3}} + \left(\frac{1}{3}\right)\sqrt{3 \cdot \frac{1}{3}} = \frac{1}{3}(1) + \frac{1}{3}(1) + \frac{1}{3}(1) = 1$$

$$(x, y, z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x, y, z > 0 \\ x + y + z = 1 \\ x\sqrt{3y} + y\sqrt{3z} + z\sqrt{3x} = 1 \end{cases}$$

$$x\sqrt{3y} + y\sqrt{3z} + z\sqrt{3x} = 1 \Leftrightarrow \sqrt{3}\left(x\sqrt{y} + y\sqrt{z} + z\sqrt{x}\right) = 1$$

$$\Leftrightarrow x\sqrt{y} + y\sqrt{z} + z\sqrt{x} = \frac{1}{\sqrt{3}}$$

$$\left(x\sqrt{y} + y\sqrt{z} + z\sqrt{x}\right)^2 = \left(\sqrt{x} \cdot \sqrt{xy} + \sqrt{y} \cdot \sqrt{yz} + \sqrt{z} \cdot \sqrt{zx}\right)^2$$

$$\leq \left[\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + \left(\sqrt{z}\right)^2\right]\left[\left(\sqrt{xy}\right)^2 + \left(\sqrt{yz}\right)^2 + \left(\sqrt{zx}\right)^2\right]$$

$$= (x + y + z)(xy + yz + zx) = 1 \cdot (xy + yz + zx) = xy + yz + zx$$

$$(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$$

$$\Leftrightarrow x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + z^2 - 2zx + x^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 2(x^2 + y^2 + z^2) - 2(xy + yz + zx) \geq 0$$

$$\Leftrightarrow x^2 + y^2 + z^2 \geq xy + yz + zx \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2 + z^2) + 2(xy + yz + zx) \geq 3(xy + yz + zx)$$

$$\Leftrightarrow (x + y + z)^2 \geq 3(xy + yz + zx) \Rightarrow 1^2 \geq 3(xy + yz + zx) \Leftrightarrow xy + yz + zx \leq \frac{1}{3}$$

$$\frac{1}{3} = \left(\frac{1}{\sqrt{3}}\right)^2 = \left(x\sqrt{y} + y\sqrt{z} + z\sqrt{x}\right)^2 \leq xy + yz + zx \leq \frac{1}{3}$$

$$\Rightarrow xy+yz+zx=\frac{1}{3}\wedge\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)^2=xy+yz+zx$$

$$\Rightarrow (x+y+z)^2=1=3\left(\frac{1}{3}\right)=3(xy+yz+zx)$$

$$\Leftrightarrow x^2+y^2+z^2+2xy+2yz+2zx=3xy+3yz+3zx$$

$$\Leftrightarrow x^2+y^2+z^2-xy-yz-zx=0\Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2+z^2-xy-yz-zx)=0$$

$$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2=0\Leftrightarrow$$

$$\Leftrightarrow x-y=0\wedge y-z=0\wedge z-x=0\Leftrightarrow x=y=z$$

$$x+y+z=1\wedge x=y=z\Rightarrow 3x=1\Leftrightarrow x=\frac{1}{3}\Rightarrow x=y=z=\frac{1}{3}$$

$$\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}+\left(\frac{1}{3}\right)\sqrt{3\cdot\frac{1}{3}}=\frac{1}{3}(1)+\frac{1}{3}(1)+\frac{1}{3}(1)=1$$

$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

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PROBLEMS FOR SENIORS

SP.631 Let a, b, c be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + ca = 3$. Prove that:

$$a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 3$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

Assume that $a \geq 1 \geq b \geq c$, and write the inequality as $F \geq 0$, where

$$F = a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - 3.$$

For fixed c, assume that a and F are functions of b. By differentiating the equality constraint, we get $(b + c)a' + a + c = 0$.

Therefore, we have

$$\frac{3F'(b)}{2} = b^{-\frac{1}{3}} + a^{-\frac{1}{3}}a' = b^{-\frac{1}{3}} - \frac{a^{-\frac{1}{3}}(a + c)}{b + c}.$$

We will show that $F'(b) \leq 0$, which is equivalent to

$$\frac{a + c}{b + c} \geq \left(\frac{a}{b}\right)^{\frac{1}{3}}.$$

This is true if

$$\left(\frac{a + c}{b + c}\right)^3 \geq \frac{a}{b}.$$

Since

$$\left(\frac{a + c}{b + c}\right)^3 \geq \left(\frac{a + c}{b + c}\right)^2 \geq \frac{2(a + c)}{b + c} - 1,$$

it suffices to show that

$$\frac{2(a + c)}{b + c} - 1 \geq \frac{a}{b},$$

which is equivalent to

$$\frac{2(a + c)}{b + c} - 2 \geq \frac{a}{b} - 1, \qquad \frac{2(a - b)}{b + c} \geq \frac{a - b}{b}, \qquad \frac{(a - b)(b - c)}{b(b + c)} \geq 0.$$

From $F'(b) \leq 0$, it follows that $F(b)$ is decreasing and has the minimum value when b is maximum, hence when $b = 1$. So, it suffices to consider the case $b = 1$, when we need to show that $a^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 2$ for $ac + a + c = 3$. Replacing a and c with a^3 and c^3, respectively, we need to prove that $a^2 + c^2 \geq 2$ for $a^3c^3 + a^3 + c^3 = 3$. By contradiction method, we will show that $a^2 + c^2 < 2$ involves $a^3c^3 + a^3 + c^3 < 3$. Since the expression $a^3c^3 + a^3 + c^3$ is strictly increasing in each of its variables a, b, c, it is enough to prove that $a^2 + c^2 = 2$ involves $a^3c^3 + a^3 + c^3 \leq 3$. Denoting $x = \frac{a+c}{2}$, we have $ac = 2x^2 - 1$. From

$2x^2-1=ac\geq 0$ and $2x^2-1=ac\leq x^2$, we get $\frac{1}{\sqrt{2}}\leq x\leq 1$, hence $\frac{2}{3}<x\leq 1$. Write the required inequality as follows:

$$a^3c^3+(a+c)^3-3ac(a+c)\leq 3,\ \ (2x^2-1)^3+8x^3-6x(2x^2-1)\leq 3,$$
$$4x^6-6x^4-2x^3+3x^2+3x-2\leq 0,\ \ \ (x-1)f(x)\leq 0,$$

where

$$f(x)=4x^5+4x^4-2x^3-4x^2-x+2.$$

We need to show that $f(x)\geq 0$. Indeed, we have

$$f(x)>3x^5+4x^4-2x^3-4x^2-x+2$$
$$=(3x-2)(x-1)^2(x^2+4x+1)+20x^3-22x^2+4$$
$$\geq 20x^3-22x^2+4=\frac{80x^3-88x^2+16}{4}=\frac{(4x-3)^2(5x+2)+3x-2}{4}>0.$$

The proof is completed. The equality occurs for $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

When one 1 variable equals to zero and WLOG we may assume $a=0$, then $bc=3\ (b,c>0)$ and b,c cannot be both ≤ 1 (since then : $3=bc\leq 1$)

$\therefore$ WLOG we may assume $b>1\geq c>0$; then : $b^{\frac{2}{3}}+c^{\frac{2}{3}}=c^{\frac{2}{3}}+\left(\frac{3}{c}\right)^{\frac{2}{3}}=f(c)$ (say)

and then : $f'(c)=\frac{2}{\alpha}\left(\frac{1}{3}-\frac{1}{\sqrt[3]{3}.\alpha^4}\right)\ (\alpha=\sqrt[3]{c}\leq 1)\overset{\alpha\leq 1}{\leq}\frac{2}{\alpha}\left(\frac{1}{3}-\frac{1}{\sqrt[3]{3}.}\right)<0\Rightarrow$

$f(c)$ is $\downarrow$ on $(0,1]\Rightarrow f(c)\geq f(1)>3$ and we now focus on the case when :

$a,b,c>0$ and if $a,b,c\leq 1$, then : $3=ab+bc+ca\leq 3\Rightarrow a=b=c=1$

$\Rightarrow a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}=3$ and now, since at most one of $a,b,c>1$, then :

WLOG we may assume $x^2>1\Rightarrow y^2,z^2\leq 1\ (\sqrt[3]{a}=x,\sqrt[3]{b}=y,\sqrt[3]{c}=z)$

$$\therefore\prod_{cyc}(x^2-1)\geq 0\Rightarrow\sum_{cyc}x^2\geq 1+\sum_{cyc}x^2y^2-x^2y^2z^2\overset{?}{\geq}3$$

$$\Leftrightarrow\sqrt[3]{\frac{\sum_{cyc}x^3y^3}{3}}.3\sum_{cyc}x^2y^2\overset{?}{\geq}2\sum_{cyc}x^3y^3+3x^2y^2z^2\left(\because\sum_{cyc}x^3y^3=3\right)$$

$$\Leftrightarrow 9\left(\sum_{cyc}u^3\right)\left(\sum_{cyc}u^2\right)^3\overset{?}{\geq}\left(2\sum_{cyc}u^3+3uvw\right)^3\ \ (u=xy,v=yz,w=zx)\ \ (*)$$

Assigning $v+w=a',w+x=b',u+v=c'\Rightarrow a'+b'-c'=2w>0,b'+c'-a'=2u>0$ and $c'+a'-b'=2v>0\Rightarrow a'+b'>c',b'+c'>a',c'+a'>b'$

$\Rightarrow a',b',c'$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $\sum_{cyc}u=s\Rightarrow uvw=r^2s,\sum_{cyc}u^2=s^2-8Rr-2r^2$,

$$\sum_{cyc} u^3 = s^3 - 12Rrs \text{ and then : } (*) \Leftrightarrow s^8 - (36Rr + 90r^2)s^6 +$$
$$r^2(864R^2 + 2376Rr + 54r^2)s^4 - r^3(11520R^3 + 19008R^2r + 1512Rr^2 + 99r^3)s^2$$
$$+Rr^4(55296R^3 + 41472R^2r + 10368Rr^2 + 864r^3) \overset{?}{\geq}_{(**)} 0$$

Case 1 $28R - 110r \geq 0$ and then : $P = (s^2 - 16Rr + 5r^2)^4 +$
$r(28R - 110r)(s^2 - 16Rr + 5r^2)^3 \overset{\text{via successive Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,
it suffices to prove : LHS of $(**) \overset{?}{\geq} P \Leftrightarrow (672R^2 - 2364Rr + 1554r^2)s^4 -$
$r(16640R^3 - 63552R^2r + 51612Rr^2 - 7651r^3)s^2 +$
$64r^2\left((1632R^4 - 6792R^3r + 6687R^2r^2 - 1979Rr^3 + 205r^4) + 30Rr^3 + 5r^4\right)$
$\overset{?}{\geq}_{(\bullet)} 0$ and $\because Q = \overbrace{(672R^2 - 2364Rr + 1554r^2)}^{>0\ \because 28R-110r\geq 0}(s^2 - 16Rr + 5r^2)^2 +$
$r\overbrace{(4864R^3 - 18816R^2r + 21756Rr^2 - 7889r^3)}^{>0\ \because 28R-110r\geq 0}(s^2 - 16Rr + 5r^2) \overset{\text{via successive Gerretsen}}{\geq} 0$
$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \overset{?}{\geq} Q$
$\Leftrightarrow (t-2)\left((t-2)(512t^2 - 320t + 536) + 729\right) \overset{?}{\geq} 0\ \left(t = \frac{R}{r}\right) \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2$
$\Rightarrow (\bullet) \Rightarrow (**)$ is true

Case 2 $28R - 110r < 0$ and then : $T = (s^2 - 16Rr + 5r^2)^4 +$
$r(28R - 110r)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) +$
$r\overbrace{(112R^3 - 552R^2r + 1080Rr^2 + 124r^3)}^{>0}(s^2 - 16Rr + 5r^2)^2$
$\overset{\text{via successive Gerretsen, Double-Rouche and Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,
it suffices to prove : LHS of $(**) \overset{?}{\geq} T \Leftrightarrow$
$\overbrace{(1792R^4 - 8224R^3r + 10656R^2r^2 - 2252Rr^3 - 1729r^4)}^{X}s^2 \overset{?}{\geq}_{(\bullet\bullet)}$
$r\left(\begin{matrix}64(448R^5 - 2328R^4r + 3815R^3r^2 - 1982R^2r^3 - 27Rr^4 + 58r^5) + \\ 48R^3r^2 + 24R^2r^3 + 24Rr^4 + 13r^5\end{matrix}\right)$

Case 2(a) $X \geq 0$ and then : LHS of $(\bullet\bullet) \overset{\text{Gerretsen}}{\geq} X(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(\bullet\bullet)$
$\Leftrightarrow (t-2)(704t^3 - 1308t^2 + 510t - 205) \overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2$

Case 2(b) $X < 0$; then : $LHS_{(\bullet\bullet)} \overset{\text{Gerretsen}}{\geq} X(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} RHS_{(\bullet\bullet)}$
$\Leftrightarrow (t-2)\left((t-2)\left((t-2)(1792t^3 - 2848t^2 + 2432t + 4288) + 9477\right) + 2916\right)$
$\overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \therefore (\bullet\bullet) \Rightarrow (**)$ is true $\therefore$ combining *all cases*,
$(**)$ is true $\forall XYZ_{s,R,r} \Rightarrow (*)$ is true & rounding up, $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} \geq 3\ \forall a, b, c \geq 0$

such that at most one of them is larger than 1 and $ab+bc+ca=3$,
$''=''$ iff $a=b=c=1$ (QED)

Solution 3 by Jenish Rijal-Bardiya-Nepal

Let $x=a^{\frac{2}{3}}$, $y=b^{\frac{2}{3}}$, and $z=c^{\frac{2}{3}}$. Then $(xy)^{\frac{3}{2}}+(yz)^{\frac{3}{2}}+(zx)^{\frac{3}{2}} \overset{(\blacksquare)}{=} 3$.

We need to prove $x+y+z\geq 3$. *Suppose, for a contradiction that* $x+y+z<3$.

Clearly if all $x,y,z\leq 1$, *then* $(\blacksquare)$ *forces* $x=y=z=1$.

Thus, due to symmetry, assume that:

$$x>1\geq y,z\geq 0.$$

Now, $2(xy)^{\frac{3}{2}} \overset{AM-GM}{\leq} xy(x+y)$ (*and analogs*)

Adding analogous inequalities, we get:

$$2(xy)^{\frac{3}{2}}+2(yz)^{\frac{3}{2}}+2(zx)^{\frac{3}{2}}\leq xy(x+y)+yz(y+z)+zx(z+x)$$

$$\Leftrightarrow 6\leq x^2(y+z)+x(y^2+z^2)+yz(y+z)$$

But, $6-x^2(y+z)-x(y^2+z^2)-yz(y+z)=$

$=(3-x-y-z)[(x+1)(y+z)+2]+(1-y)(1-z)(2x-y-z)$ *which is* >0.

which contradicts with $x^2(y+z)+x(y^2+z^2)+yz(y+z)\geq 6$.

So, our assumption that $x+y+z<3$ *is wrong*.

Therefore,

$$a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}\geq 3.$$

Equality holds if and only if $a=b=c=1$.

SP.632 Prove that 2 is the largest positive value of the constant k such that

$$a^2+b^2+c^2+d^2-4\geq k(ab+bc+cd+da-4)$$

for all positive real numbers a,b,c,d with at most one of them larger than 1 and $abcd=1$.

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by proposer

For $b = d = 1$ and $ac = 1$, the constraints are satisfied and the inequality becomes as follows:

$$a^2 + c^2 - 4 \geq 2k(a + c - 2),\ \ a^2 + \frac{1}{a^2} - 4 \geq 2k\left(a + \frac{1}{a} - 2\right),$$

$$\frac{(a^2-1)^2}{a^2} \geq \frac{2k(a-1)^2}{a},\ \ (a-1)^2 f(a) \geq 0,$$

where $f(a) = (a+1)^2 - 2ka$. The necessary condition $f(1) \geq 0$ implies $k \leq 2$. To show that 2 is the largest positive value of k, we need to prove that

$$a^2 + b^2 + c^2 + d^2 - 2(ab + bc + cd + da) + 4 \geq 0.$$

According to Lemma below, it suffices to show that $E \geq 0$ for $a \geq 1 \geq b \geq c \geq d$, where

$$E = a^2 + b^2 + c^2 + d^2 - 2(a + d)(b + c) + 4.$$

We have

$$E = (a + d - b - c)^2 - 2(ad + bc - 2).$$

Let $x = bc \leq 1$. From $(a-1)(d-1) \leq 0$, we get

$$a + d \geq ad + 1 = \frac{1}{x} + 1,$$

and from $(b-1)(c-1) \geq 0$, we get $b + c \leq bc + 1 = x + 1$. Since

$$a + d - b - c \geq \frac{1}{x} + 1 - (x + 1) = \frac{1 - x^2}{x} \geq 0,$$

we have

$$E \geq \frac{(1-x^2)^2}{x^2} - 2\left(\frac{1}{x} + x - 2\right) = \frac{(1-x^2)^2}{x^2} - \frac{2(1-x)^2}{x} = \frac{(1-x)^2(1+x^2)}{x^2} \geq 0.$$

The proof is completed. If $k = 2$, then the equality occurs for $a = b = c = d = 1$.

Lemma. Let x_1, x_2, x_3, x_4 be real numbers, and let (a, b, c, d) be a permutation of the sequence (x_1, x_2, x_3, x_4) such that $a \geq b \geq c \geq d$. Then,

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \leq (a + d)(b + c).$$

Proof. It suffices to show that

$$(a + d)(b + c) \geq (a + c)(b + d) \geq (a + b)(c + d).$$

The left inequality reduces to $(a - b)(c - d) \geq 0$,

and the right inequality to $(a-d)(b-c)\geq 0$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

Let $(a,b,c,d)=\left(t,1,\frac{1}{t},1\right);\quad t>1$

Notice: $\frac{a^2+b^2+c^2+d^2-4}{ab+bc+cd+da-4}=\frac{t^2+\frac{1}{t^2}-2}{2\left(t+\frac{1}{t}-2\right)}=2+\frac{(t-1)^2}{2t}\longrightarrow 2^+\quad(\text{as } t\longrightarrow 1^+)$

Hence no $k>2$ can work. It remains to prove that $k_{\max}=2$ works.

If all $a,b,c,d\leq 1$, then $a=b=c=d=1$ and the inequality becomes trivial.

Otherwise, due to symmetry, assume that: $a>1$ and $b,c,d\leq 1$.

Here, $ac=\frac{1}{bd}\geq 1\geq bd \Leftrightarrow ac\geq bd$. Now,

$(a-1)(1-c)\geq 0\Leftrightarrow a+c\geq ac+1$ and $(1-b)(1-d)\geq 0$

$\Leftrightarrow b+d\leq bd+1$ Thus, $a+c-b-d\geq ac-bd\geq 0$

Now,

$(a^2+b^2+c^2+d^2-4)-2(ab+bc+cd+da-4)=(a+c-b-d)^2-2(ac+bd-2)\geq$

$\overset{a+c-b-d\geq ac-bd}{\geqslant}(ac-bd)^2-2(ac+bd-2)\overset{abcd=1}{=}(ac+bd)(ac+bd-2)\overset{?}{\geqslant}0$

, which is true because $ac+bd\overset{\text{AM-GM}}{\geqslant}2\sqrt{abcd}=2$.

Therefore: $a^2+b^2+c^2+d^2-4\geq 2(ab+bc+cd+da-4)$

Hence: $k_{\max}=2$. *Equality holds for* $a=b=c=d=1$.

SP.633 Let $n\geq 3$ be an integer, and let $a_1,a_2,\ldots,a_n$ be positive real numbers such that

$$\sum_{r=1}^{n}a_r^2=n.$$

For every triple $1\leq i<j<k\leq n$, put

$$R_{ijk}=\frac{a_i}{3a_i+a_j}+\frac{a_j}{3a_j+a_k}+\frac{a_k}{3a_k+a_i}$$

and

$$D_{ijk}=\sqrt{\frac{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}{a_i+a_j+a_k}}.$$

Prove that

$$\frac{1}{\binom{n}{3}}\sum_{1\le i<j<k\le n}\left(\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\right)\ge 2.$$

Determine all cases of equality.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

First prove the homogeneous three-variable inequality on which the generalization is based.

Lemma. For all positive real numbers x,y,z, if

$$Q=x^2+y^2+z^2,\quad S=x+y+z,$$

then

$$\frac{2}{3}\left(\frac{x}{3x+y}+\frac{y}{3y+z}+\frac{z}{3z+x}\right)+\frac{3}{2}\sqrt{\frac{\sqrt{3Q}}{S}}\ge 2. \qquad (1)$$

Equality holds if and only if $x=y=z$.

Proof. Both terms on the left-hand side of (1) are unchanged when x,y,z are multiplied by the same positive number. We may therefore assume that

$$Q=x^2+y^2+z^2=3. \qquad (2)$$

By the Cauchy-Schwarz inequality in Engel form,

$$\frac{x}{3x+y}+\frac{y}{3y+z}+\frac{z}{3z+x}=\frac{x^2}{3x^2+xy}+\frac{y^2}{3y^2+yz}+\frac{z^2}{3z^2+zx}$$
$$\ge\frac{(x+y+z)^2}{3(x^2+y^2+z^2)+xy+yz+zx}.$$

Under (2),

$$xy+yz+zx=\frac{S^2-Q}{2}=\frac{S^2-3}{2}.$$

Consequently,

$$\frac{x}{3x+y}+\frac{y}{3y+z}+\frac{z}{3z+x}\geq\frac{2S^2}{S^2+15} \qquad (3)$$

Moreover, Cauchy – Schwarz gives $S\leq 3$. Set

$$t=\sqrt{\frac{3}{S}};$$

then $t\geq 1$ and $S=\frac{3}{t^2}$. From (3), the left-hand side of (1) is at least

$$\frac{4S^2}{3(S^2+15)}+\frac{3t}{2}=\frac{4}{3+5t^2}+\frac{3t}{2}.$$

It remains to verify that this last expression is at least 2. Indeed,

$$\frac{4}{3+5t^4}+\frac{3t}{2}-2=\frac{15t^5-20t^4+9t-4}{2(3+5t^4)}$$

$$=\frac{(t-1)(15t^4-5t^3-5t^2-5t+4)}{2(3+5t^4)}.$$

For $t>1$,

$$15t^4-5t^3-5t^2-5t+4>15t^4-15t^3+4$$

$$15t^3(t-1)+4>0.$$

Thus (1) follows,

Equality requires $t=1$, hence $S=3$. Together with $Q=3$, equality in Cauchy – Schwarz yields $x=y=z=1$ under normalization (2). Removing the normalization, equality holds exactly when $x=y=z$. We now apply the lemma to each triple (a_i,a_j,a_k) with $1\leq i<j<k\leq n$. For such a triple, the lemma states exactly that

$$\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq 2 \qquad (4)$$

There are $\binom{n}{3}$ triples. Summing (4) over all of them and dividing by $\binom{n}{3}$ proves the boxed inequality. Finally, equality in the average is possible only if equality holds for every triple. By the equality statement in the lemma, this means that

$a_i = a_j = a_k$ for every $1 \le i < j < k \le n$.

Hence $a_1 = a_2 = \cdots a_n$. The condition $\sum_{r=1}^{n} a_r^2 = n$ and positive then give

$$a_1 = a_2 = \cdots = a_n = 1.$$

Conversely, these values clearly give equality, so the equality case is completely determined.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\sum_{1\le i<j<k\le n} (a_i^2 + a_j^2 + a_k^2) = \sum_{1\le i<j<k\le n} a_i^2 + \sum_{1\le i<j<k\le n} a_j^2 + \sum_{1\le i<j<k\le n} a_k^2$$

$$= \binom{n-1}{2} a_1^2 + \binom{n-1}{2} a_2^2 + \cdots + \binom{n-1}{2} a_n^2 = \binom{n-1}{2} \sum_{r=1}^{n} a_r^2 = \binom{n-1}{2} n$$

$$\Rightarrow \frac{1}{\binom{n}{3}} \sum_{1\le i<j<k\le n} (a_i^2 + a_j^2 + a_k^2) = \frac{\binom{n-1}{2} n}{\frac{n(n-1)(n-2)}{6}} = \frac{\frac{(n-1)(n-2)}{2} \cdot n}{\frac{n(n-1)(n-2)}{6}} =$$

$$= \frac{(n-1)(n-2)n}{2} \cdot \frac{6}{n(n-1)(n-2)} = 3$$

$$R_{ijk} = \frac{a_i}{3a_i + a_j} + \frac{a_j}{3a_j + a_k} + \frac{a_k}{3a_k + a_i} = \frac{a_i^2}{3a_i^2 + a_i a_j} + \frac{a_j^2}{3a_j^2 + a_j a_k} + \frac{a_k^2}{3a_k^2 + a_k a_i}$$

$$\Rightarrow R_{ijk} \ge \frac{(a_i + a_j + a_k)^2}{(3a_i^2 + a_i a_j) + (3a_j^2 + aja_k) + (3a_k^2 + a_k a_i)} =$$

$$= \frac{(a_i + a_j + a_k)^2}{3(a_i^2 + a_j^2 + a_k^2) + (a_i a_j + a_j a_k + a_k a_i)}$$

$$(a_i - a_j)^2 + (a_j - a_k)^2 + (a_k - a_i)^2 \ge 0$$

$$\Rightarrow a_1^2 - 2a_i a_j + a_j^2 + a_j^2 - 2a_j a_k + a_k^2 + a_k^2 - 2a_k a_i + a_i^2 \ge 0$$

$$\Rightarrow 2(a_i^2 + a_j^2 + a_k^2) - 2(a_i a_j + a_j a_k + a_k a_i) \ge 0$$

$$\Rightarrow a_i a_j + a_j a_k + a_k a_i \le a_i^2 + a_j^2 + a_k^2 \Rightarrow 3(a_i^2 + a_j^2 + a_k^2) + (a_i a_j + a_j a_k + a_k a_i) \le$$

$$\le 3(a_i^2 + a_j^2 + a_k^2) + (a_i^2 + a_j^2 + a_k^2) = 4(a_i^2 + a_j^2 + a_k^2) \Rightarrow R_{ijk} \ge \frac{(a_i + a_j + a_k)^2}{4(a_i^2 + a_j^2 + a_k^2)}$$

$$D_{ijk}=\sqrt{\frac{3\left(a_i^2+a_j^2+a_k^2\right)}{a_i+a_j+a_k}}=\frac{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}{\sqrt{a_i+a_j+a_k}}$$

$$\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}}\geq\frac{a_i+a_j+a_k}{3}\Rightarrow a_i+a_j+a_k\leq 3\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}}=\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}$$

$$\Rightarrow D_{ijk}\geq\frac{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}{\sqrt{\sqrt{3\left(a_i^2+a_j^2+a_k^2\right)}}}=\sqrt[4]{3\left(a_i^2+a_j^2+a_k^2\right)}\geq 1$$

$$\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq 2\sqrt{\left(\frac{2}{3}R_{ijk}\right)\cdot\left(\frac{3}{2}D_{ijk}\right)}=2\sqrt{R_{ijk}\cdot D_{ijk}}$$

$$R_{ijk}\cdot D_{ijk}\geq\frac{\left(a_i+a_j+a_k\right)^2}{4\left(a_i^2+a_j^2+a_k^2\right)}\cdot\sqrt{\frac{3\left(a_i^2+a_j^2+a_k^2\right)}{a_i+a_j+a_k}}=\frac{\sqrt{3}}{4}\cdot\frac{\left(a_i+a_j+a_k\right)^{\frac{3}{2}}}{\left(a_i^2+a_j^2+a_k^2\right)^{\frac{1}{2}}}$$

$$a_i=a_j=a_k\Rightarrow R_{ijk}=\frac{a_i}{4a_i}+\frac{a_i}{4a_i}+\frac{a_i}{4a_i}=\frac{3}{4},\ D_{ijk}=\sqrt{\frac{3\left(3a_i^2\right)}{3a_i}}=\sqrt{3a_i}\overset{a_i=1}{=}\sqrt{3}$$

$$\Rightarrow\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\geq\frac{2}{3}\cdot\frac{3}{4}+\frac{3}{2}\cdot 1=\frac{1}{2}+\frac{3}{2}=2$$

$$\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}\left(\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk}\right)\geq\frac{1}{\binom{n}{3}}\sum_{1\leq i<j<k\leq n}2=\frac{2\cdot\binom{n}{3}}{\binom{n}{3}}=2$$

$$\Leftrightarrow a_1=a_2=\cdots=a_n\Rightarrow\sum_{r=1}^{n}a_r^2=n\cdot a_1^2=n\Rightarrow a_1^2=1\Rightarrow a_1=a_2=\cdots=a_n=1$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\sum_{1\leq i<j<k\leq n}\left(a_i^2+a_j^2+a_k^2\right)=\sum_{1\leq i<j<k\leq n}a_i^2+\sum_{1\leq i<j<k\leq n}a_j^2+\sum_{1\leq i<j<k\leq n}a_k^2$$

$$=\binom{n-1}{2}a_1^2+\binom{n-1}{2}a_2^2+\cdots+\binom{n-1}{2}a_n^2=\binom{n-1}{2}\sum_{r=1}^{n}a_r^2=\binom{n-1}{2}n$$

$$\Rightarrow \frac{1}{\binom{n}{3}} \sum_{1\le i<j<k\le n} (a_i^2+a_j^2+a_k^2) = \frac{\binom{n-1}{2} n}{\frac{n(n-1)(n-2)}{6}} = \frac{\frac{(n-1)(n-2)}{2}\cdot n}{\frac{n(n-1)(n-2)}{6}} =$$

$$= \frac{(n-1)(n-2)n}{2} \cdot \frac{6}{n(n-1)(n-2)} = 3$$

$$R_{ijk} = \frac{a_i}{3a_i+a_j} + \frac{a_j}{3a_j+a_k} + \frac{a_k}{3a_k+a_i} = \frac{a_i^2}{3a_i^2+a_ia_j} + \frac{a_j^2}{3a_j^2+a_jk} + \frac{a_k^2}{3a_k^2+a_ka_i}$$

$$\Rightarrow R_{ijk} \ge \frac{(a_i+a_j+a_k)^2}{(3a_i^2+a_ia_j)+(3a_j^2+a_ja_k)+(3a_k^2+a_ka_i)} =$$

$$= \frac{(a_i+a_j+a_k)^2}{3(a_i^2+a_j^2+a_k^2)+(a_ia_j+a_ja_k+a_ka_i)}$$

$$(a_i-a_j)^2+(a_j-a_k)^2+(a_k-a_i)^2 \ge 0 \Rightarrow$$

$$\Rightarrow a_i^2-2a_ia_j+a_j^2+a_j^2-2a_ja_k+a_k^2+a_k^2-2a_ka_i+a_i^2 \ge 0$$

$$\Rightarrow 2(a_i^2+a_j^2+a_k^2)-2(a_ia_j+a_ja_k+a_ka_i) \ge 0 \Rightarrow a_ia_j+a_ja_k+a_ka_i \le a_1^2+a_j^2+a_k^2$$

$$\Rightarrow 3(a_i^2+a_j^2+a_k^2)+(a_ia_j+a_ja_k+a_ka_i) \le$$

$$\le 3(a_i^2+a_j^2+a_k^2)+(a_i^2+a_j^2+a_k^2) = 4(a_i^2+a_j^2+a_k^2) \Rightarrow R_{ijk} \ge \frac{(a_i+a_j+a_k)^2}{4(a_i^2+a_j^2+a_k^2)}$$

$$\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}} \ge \frac{a_i+a_j+a_k}{3} \Rightarrow a_i+a_j+a_k \le 3\sqrt{\frac{a_i^2+a_j^2+a_k^2}{3}} = \sqrt{3(a_i^2+a_j^2+a_k^2)}$$

$$\Rightarrow D_{ijk} = \sqrt{\frac{3(a_i^2+a_j^2+a_k^2)}{a_i+a_j+a_k}} \ge \frac{\sqrt{3(a_i^2+a_j^2+a_k^2)}}{\sqrt{\sqrt{3(a_i^2+a_j^2+a_k^2)}}} = \sqrt[4]{3(a_i^2+a_j^2+a_k^2)} \ge 1$$

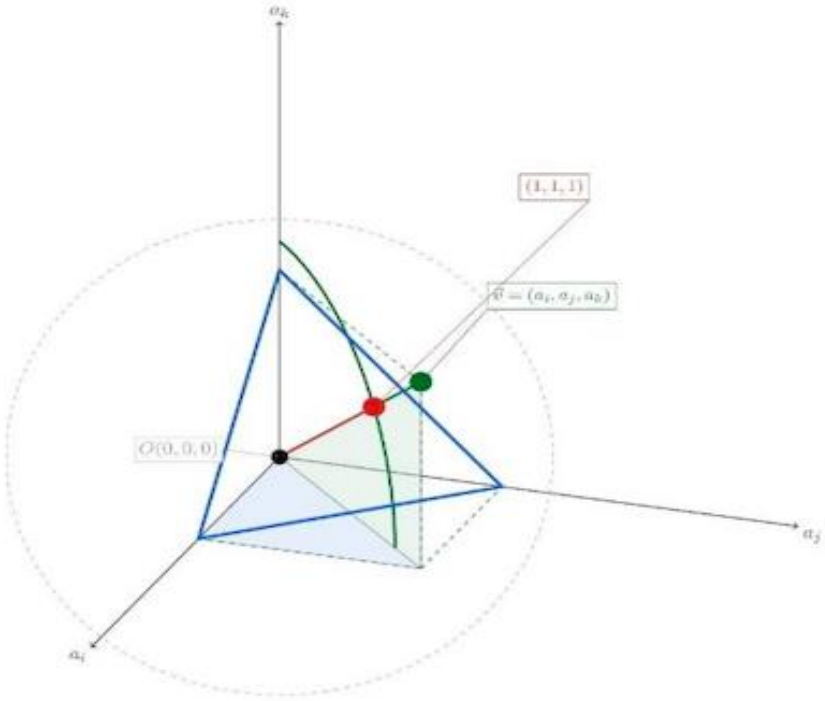
$$\frac{2}{3}R_{ijk}+\frac{3}{2}D_{ijk} \ge 2\sqrt{\left(\frac{2}{3}R_{ijk}\right)\cdot\left(\frac{3}{2}D_{ijk}\right)} = 2\sqrt{R_{ijk}\cdot D_{ijk}}$$

$$R_{ijk}\cdot D_{ijk} \ge \frac{(a_i+a_j+a_k)^2}{4(a_i^2+a_j^2+a_k^2)} \cdot \sqrt{\frac{3(a_i^2+a_j^2+a_k^2)}{a_i+a_j+a_k}} = \frac{\sqrt{3}}{5} \cdot \frac{(a_i+a_j+a_k)^{\frac{3}{2}}}{(a_i^2+a_j^2+a_k^2)^{\frac{1}{2}}}$$

$$a_i = a_j = a_k \Rightarrow R_{ijk} = \frac{3}{4}, \qquad D_{ijk} = \sqrt{3a_i} \overset{a_i=1}{=} \sqrt{3} \Rightarrow \frac{2}{3}\left(\frac{3}{4}\right) + \frac{3}{2}(1) = \frac{1}{2} + \frac{3}{2} = 2$$

$$\frac{1}{\binom{n}{3}} \sum_{1 \le i < j < k \le n} \left(\frac{2}{3} R_{ijk} + \frac{3}{2} D_{ijk}\right) \ge \frac{1}{\binom{n}{3}} \sum_{1 \le i < j < k \le n} 2 = \frac{2 \cdot \binom{n}{3}}{\binom{n}{3}} = 2$$

$$\Leftrightarrow a_1 = a_2 = \cdots = a_n \Rightarrow \sum_{r=1}^{n} a_r^2 = n \cdot a_1^2 = n \Rightarrow a_1^2 = 1 \Rightarrow a_1 = a_2 = \cdots = a_n = 1$$

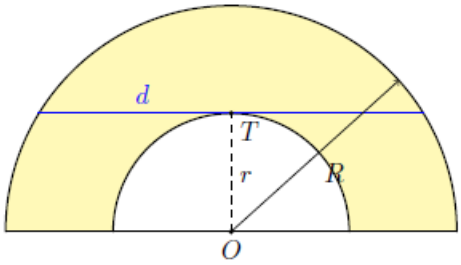


SP.634 Let two concentric spheres have radii R and r, where $R > r > 0$, and let their common centre be O. A plane parallel to their common equatorial plane is tangent to the smaller sphere in the upper half-space. Its intersection with the larger sphere is a circle of diameter d. The two upper solid hemispheres bound a hemispherical shell. Determine the volume V of this shell in terms of the general data r and d. Also determine the difference S of the curved surface areas of the two hemispheres.

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The figure is a meridian section of the three-dimensional configuration.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

Choose Cartesian coordinates with origin O and with the equatorial plane given by $z = 0$. The upper halves of the two spheres then satisfy

$$x^2 + y^2 + z^2 = R^2, \quad x^2 + y^2 + z^2 = r^2, \quad z \geq 0.$$

The tangent plane in the problem is $z = r$. Indeed, its distance from O is r, so it meets the smaller sphere only at the point $T = (0, 0, r)$. Substituting $z = r$ into the equation of the larger sphere gives

$$x^2 + y^2 = R^2 - r^2.$$

Thus the section of the larger sphere is a circle of radius $\frac{d}{2}$, and hence

$$\left(\frac{d}{2}\right)^2 = R^2 - r^2.$$

Consequently,

$$R = \sqrt{r^2 + \frac{d^2}{4}}. \qquad (1)$$

For completeness, we derive the hemisphere volume. At height $z \in [0, R]$, the horizontal section of the upper ball of radius R is a disk of area $\pi(R^2 - z^2)$. Therefore

$$Vol(H_r) = \int_0^R \pi(R^2 - z^2)\, dz = \pi\left[R^2 z - \frac{z^3}{3}\right]_0^R = \frac{2\pi R^3}{3}$$

The same calculation with radius r gives $Vol(H_r) = \frac{2\pi r^3}{3}$. The required shell is $\frac{H_R}{H_r}$, so it volume is

$$V = \frac{2\pi}{3}(R^3 - r^3).$$

Using (1), we obtain the answer entirely in terms of the given data:

$$V = \frac{2\pi}{3}\left[\left(r^2 + \frac{d^2}{4}\right)^{\frac{3}{2}} - r^3\right].$$

It remains to find the curved-area difference. The curved surface area of a hemisphere of radius a is $2\pi a^2$. Hence

$$S = 2\pi R^2 - 2\pi r^2 = 2\pi(R^2 - r^2) = 2\pi\left(\frac{d}{2}\right)^2,$$

and therefore

$$S = \frac{\pi d^2}{2}.$$

Notice that, unlike the planar area in the original configuration and the curved-area difference above, the volume is not determined by d alone. The formula contains r essentially: different inner radii with the same section diameter d produce shell volumes. This is why both r and d are included among the general data.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$R, r \in \mathbb{R}^+, R > r, d > 0$$

$$R^2 = r^2 + \left(\frac{d}{2}\right)^2 \Rightarrow R^2 - r^2 = \frac{d^2}{4} \Rightarrow R = \left(r^2 + \frac{d^2}{4}\right)^{\frac{1}{2}}$$

$$V = \int_0^r \pi[(R^2 - z^2) - (r^2 - z^2)]\,dz + \int_r^R \pi(R^2 - z^2)\,dz$$

$$\Rightarrow V = \pi\int_0^r (R^2 - z^2 - r^2 + z^2)\,dz + \pi\int_r^R (R^2 - z^2)\,dz$$

$$\Rightarrow V = \pi\int_0^r (R^2 - r^2)dz + \pi\int_r^R (R^2 - z^2)\,dz$$

$$\Rightarrow V = \pi(R^2 - r^2)[z]_0^r + \pi\left[R^2 z - \frac{z^3}{3}\right]_r^R$$

$$\Rightarrow V=\pi(R^2-r^2)(r-0)+\pi\left[\left(R^3-\frac{R^3}{3}\right)-\left(R^2r-\frac{r^3}{3}\right)\right]$$

$$\Rightarrow V=\pi(R^2-r^2)r+\pi\left[\frac{2R^3}{3}-R^2r+\frac{r^3}{3}\right]$$

$$\Rightarrow V=\pi R^2r-\pi r^3+\frac{2}{3}\pi R^3-\pi R^2r+\frac{1}{3}\pi r^3$$

$$\Rightarrow V=(\pi R^2r-\pi R^2r)+\left(\frac{1}{3}\pi r^3-\pi r^3\right)+\frac{2}{3}\pi R^3\Rightarrow V=0-\frac{2}{3}\pi r^3+\frac{2}{3}\pi R^3$$

$$\Rightarrow V=\frac{2}{3}\pi R^3-\frac{2}{3}\pi r^3\Rightarrow V=\frac{2}{3}\pi(R^3-r^3)\Rightarrow V=\frac{2}{3}\pi\left((R^2)^{\frac{3}{2}}-r^3\right)$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\left(r^2+\frac{d^2}{4}\right)^{\frac{3}{2}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\left(\frac{4r^2+d^2}{4}\right)^{\frac{3}{2}}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{4^{\frac{3}{2}}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{8}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}-8r^3}{8}\right]\Rightarrow V=\frac{2}{24}\pi\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]$$

$$\Rightarrow V=\frac{\pi}{12}\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]S=S_R-S_r\Rightarrow S=2\pi R^2-2\pi r^2$$

$$\Rightarrow S=2\pi(R^2-r^2)\Rightarrow S=2\pi\left(\frac{d^2}{4}\right)\Rightarrow S=\frac{2\pi d^2}{4}\Rightarrow S=\frac{\pi d^2}{2}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$R,r\in\mathbb{R}^+,R>r,d>0$$

$$R^2=r^2+\left(\frac{d}{2}\right)^2\Rightarrow R^2=r^2+\frac{d^2}{4}\Rightarrow R^2-r^2=\frac{d^2}{4}\Rightarrow R=\sqrt{r^2+\frac{d^2}{4}}\Rightarrow R=\left(r^2+\frac{d^2}{4}\right)^{\frac{1}{2}}$$

$$V=V_R-V_r\Rightarrow V=\frac{2}{3}\pi R^3-\frac{2}{3}\pi r^3\Rightarrow V=\frac{2}{3}\pi(R^3-r^3)\Rightarrow V=\frac{2}{3}\pi\left((R^2)^{\frac{3}{2}}-r^3\right)$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\left(r^2+\frac{d^2}{4}\right)^{\frac{3}{2}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\left(\frac{4r^2+d^2}{4}\right)^{\frac{3}{2}}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{4^{\frac{3}{2}}}-r^3\right]\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}}{8}-r^3\right]$$

$$\Rightarrow V=\frac{2}{3}\pi\left[\frac{(4r^2+d^2)^{\frac{3}{2}}-8r^3}{8}\right]\Rightarrow V=\frac{2}{24}\pi\left[(4r^2+d^3)^{\frac{3}{2}}-8r^3\right]$$

$$\Rightarrow V=\frac{\pi}{12}\left[(4r^2+d^2)^{\frac{3}{2}}-8r^3\right]$$

$$S=S_R-S_r\Rightarrow S=2\pi R^2-2\pi r^2\Rightarrow S=2\pi(R^2-r^2)\Rightarrow S=2\pi\left(\frac{d^2}{4}\right)$$

$$\Rightarrow S=\frac{2\pi d^2}{4}\Rightarrow S=\frac{\pi d^2}{2}$$

SP.635 Let $n\geq 2$, and let $a_1, a_2, \ldots, a_n$ be positive real numbers whose sum is $S=\sum_{i=1}^{n} a_i$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers satisfying $\sum_{i=1}^{n}\lambda_i=1$. Prove that:

$$\sum_{i=1}^{n}\lambda_i\frac{a_i}{S-a_i}\geq\frac{S}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}-1.$$

Determine when equality holds.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

Since all the a_i are positive, we have $S-a_i>0$ for every $i\in\{1,2,\ldots,n\}$. Moreover

$$\frac{a_i}{S-a_i}=\frac{S-(S-a_i)}{S-a_i}=\frac{S}{S-a_i}-1.$$

Multiplying this identity by λ_i, summing over all i, and using $\sum_{i=1}^{n}\lambda_i=1$, we obtain

$$\sum_{i=1}^{n}\lambda_i\frac{a_i}{S-a_i}=s\sum_{i=1}^{n}\frac{\lambda_i}{S-a_i}-\sum_{i=1}^{n}\lambda_i=S\sum_{i=1}^{n}\frac{\lambda_i}{S-a_i}-1.$$

By the weighed arithmetic-geometric mean inequality applied to the positive numbers $\frac{1}{S-a_i}$ with weights λ_i, we have

$$\sum_{i=1}^{n}\frac{\lambda_i}{S-a_i}\geq\prod_{i=1}^{n}\left(\frac{1}{S-a_i}\right)^{\lambda_i}=\frac{1}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}.$$

Therefore,

$$\sum_{i=1}^{n}\lambda_i\frac{a_i}{S-a_i}=S\sum_{i=1}^{n}\frac{\lambda_i}{S-a_i}-1\geq\frac{S}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}-1,$$

as required. Since every weight λ_i is positive, equality in the weighted arithmetic – geometric mean inequality holds if and only if

$$\frac{1}{S-a_i}=\frac{1}{S-a_2}=\cdots=\frac{1}{S-a_n}$$

This is equivalent to $a_1=a_2=\cdots=a_n$. Conversely, if $a_1=a_2=\cdots=a_n$, then all the numbers $\frac{1}{S-a_i}$ are equal, so equality indeed holds.

Thus equality occurs if and only if $a_1=a_2=\cdots a_n$.

Solution 2 by Jenish Rijal-Bardiya-Nepal

$$\sum_{i=1}^{n}\lambda_i\cdot\frac{a_i}{S-a_i}=\sum_{i=1}^{n}\lambda_i\cdot\left(\frac{S}{S-a_i}-1\right)=\sum_{i=1}^{n}\lambda_i\cdot\left(\frac{S}{S-a_i}\right)-\sum_{i=1}^{n}\lambda_i=S\sum_{i=1}^{n}\left(\frac{\lambda_i}{S-a_i}\right)-1$$

Now, $$\sum_{i=1}^{n}\left(\frac{\lambda_i}{S-a_i}\right)\overset{\text{Weighted AM-GM}}{\geq}\prod_{i=1}^{n}\left(\frac{1}{S-a_i}\right)^{\lambda_i}=\frac{1}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}$$

Therefore,

$$\sum_{i=1}^{n}\lambda_i\cdot\frac{a_i}{S-a_i}=S\sum_{i=1}^{n}\left(\frac{\lambda_i}{S-a_i}\right)-1\geq\frac{S}{\prod_{i=1}^{n}(S-a_i)^{\lambda_i}}-1$$

Equality holds if and only if $a_1=a_2=a_3=\cdots\cdots=a_n$

SP.636 Find all positive real numbers such that $ab+bc+ca=3abc$, and

$$\sqrt{\frac{a+b}{a^2+b^2}}+\sqrt{\frac{b+c}{b^2+c^2}}+\sqrt{\frac{c+a}{c^2+a^2}}=3$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

The condition $ab+bc+ca=3abc$ is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3.$$

Since $2(a^2+b^2)\geq(a+b)^2$ and, by AM-HM, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$, we have

$$\frac{a+b}{a^2+b^2}\leq\frac{2}{a+b}\leq\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right),$$

with equality in both if and only if $a=b$. Similarly, for the other two terms. By the Cauchy-Schwarz inequality,

$$\sum_{cyc}\sqrt{\frac{a+b}{a^2+b^2}}\leq\sqrt{3\sum_{cyc}\frac{a+b}{a^2+b^2}}\leq\sqrt{3\sum_{cyc}\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$=\sqrt{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}=\sqrt{9}=3.$$

Hence the given equation holds if and only if equality occurs throughout. This requires $a=b, b=c$ and $c=a$. Then $\frac{3}{a}=3$, that is, $a=b=c=1$.

Conversely, $a=b=c=1$ clearly satisfies both conditions. So, the only solution is $(a,b,c)=(1,1,1)$.

Solution 2 by Marin Chirciu-Romania

$$ab+bc+ca=3abc\Leftrightarrow\sum\frac{1}{a}=3.$$

$$LHS=\sum\sqrt{\frac{a+b}{a^2+b^2}}\overset{AM-GM}{\leq}\sum\sqrt{\frac{a+b}{2ab}}\overset{CBS}{\leq}\sqrt{3\sum\frac{a+b}{2ab}}=\sqrt{\frac{3}{2}\sum\left(\frac{1}{a}+\frac{1}{b}\right)}=$$

$$=\sqrt{\frac{3}{2}\cdot 2\sum\frac{1}{a}}=\sqrt{3\cdot 3}=3=RHS.$$

Equality holds if and only if $a=b=c=1$.

We deduce that $(a,b,c)=(1,1,1)$ is the unique solution of system of conditions.

Remark.

Let be $\lambda \geq 0$ fixed. Find $a, b, c > 0$ such that $ab + bc + ca = 3abc$ and

$$\sum \sqrt{\frac{a+\lambda b}{a^2+b^2}} = 3\sqrt{\frac{\lambda+1}{2}}$$

Marin Chirciu

Solution

$$ab + bc + ca = 3abc \Leftrightarrow \sum \frac{1}{a} = 3.$$

$$LHS = \sum \sqrt{\frac{a+\lambda b}{a^2+b^2}} \overset{AM-GM}{\leq} \sum \sqrt{\frac{a+\lambda b}{2ab}} \overset{CBS}{\leq} \sqrt{3\sum \frac{a+\lambda b}{2ab}} = \sqrt{\frac{3}{2}\sum\left(\lambda\frac{1}{a}+\frac{1}{b}\right)} =$$

$$= \sqrt{\frac{3}{2}\cdot(\lambda+1)\sum\frac{1}{a}} = \sqrt{\frac{3}{2}\cdot(\lambda+1)\cdot 3} = 3\sqrt{\frac{\lambda+1}{2}} = RHS.$$

Equality holds if and only if $a = b = c = 1$.

We deduce that $(a, b, c) = (1, 1, 1)$ is the unique solution of the system of conditions.

SP.637 If $a, b, c > 0, a + b + c = 3$ and $\lambda \geq 0$ then:

$$\frac{a+\lambda}{b^2+1} + \frac{b+\lambda}{c^2+1} + \frac{c+\lambda}{a^2+1} \geq \frac{3(\lambda+1)}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Lemma 1: If $a, b, c > 0, a + b + c = 3$ then:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{2}.$$

Proof: We have:

$$\sum \frac{a}{b^2+1} = \sum\left(1 - \frac{ab^2}{b^2+1}\right) \overset{AGM}{\geq} \sum\left(a - \frac{ab^2}{2b}\right) = \sum\left(a - \frac{ab}{2}\right) = \sum a - \frac{1}{2}\sum ab =$$

$$= 3 - \frac{1}{2}\cdot\frac{(\sum a)^2}{3} = 3 - \frac{1}{2}\cdot\frac{3^2}{3} = \frac{3}{2}.$$

Equality holds if and only if $a = b = c = 1$.

Lemma 2: If $a, b, c > 0, a + b + c = 3$ then:

$$\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{a^2+1}\geq\frac{3}{2}.$$

Proof: We have:

$$\sum\frac{1}{b^2+1}=\sum\left(1-\frac{b^2}{b^2+1}\right)\overset{AGM}{\geq}\sum\left(1-\frac{b^2}{2b}\right)=\sum\left(1-\frac{b}{2}\right)=3-\frac{1}{2}\sum a=3-\frac{1}{2}\cdot 3=\frac{3}{2}.$$

Equality holds if and only if $a=b=c=1$. Let's get back to the main problem.

Using the Lemmas, we obtain:

$$LHS=\sum\frac{a+\lambda}{b^2+1}=\sum\frac{a}{b^2+1}+\lambda\sum\frac{1}{b^2+1}\overset{Lemma\ 1+Lemma\ 2}{\geq}$$

$$\geq\frac{3}{2}+\lambda\frac{3}{2}=\frac{3(\lambda+1)}{2}=RHS$$

Equality holds if and only if $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a+\lambda}{b^2+1}=\sum_{cyc}\frac{(a+\lambda)(b^2+1-b^2)}{b^2+1}=\sum_{cyc}(a+\lambda)-\sum_{cyc}\frac{b^2(a+\lambda)}{b^2+1}$$

$$\overset{AM-GM}{\geq}\sum_{cyc}a+3\lambda-\sum_{cyc}\frac{b^2(a+\lambda)}{2b}=3(\lambda+1)-\frac{1}{2}\sum_{cyc}ab-\frac{\lambda}{2}\sum_{cyc}a\left(\because\sum_{cyc}a=3\right)$$

$$\geq 3(\lambda+1)-\frac{1}{6}\left(\sum_{cyc}a\right)^2-\frac{\lambda}{2}\sum_{cyc}a=3(\lambda+1)-\frac{9}{6}-\frac{3\lambda}{2}$$

$$\left(\because\sum_{cyc}a=3\right)=\frac{3(\lambda+1)}{2}\ \text{and so}$$

$$\frac{a+\lambda}{b^2+1}+\frac{b+\lambda}{c^2+1}+\frac{c+\lambda}{a^2+1}\geq\frac{3(\lambda+1)}{2}$$

$$\forall\, a,b,c>0\,|\,a+b+c=3\ and\ \lambda\geq 0,''='' \text{ iff } a=b=c=1\ (QED)$$

Solution 3 by Jenish Rijal-Bardiya-Nepal

$$\text{Lemma: } For\ any\ t>0,\qquad \frac{1}{t^2+1}\geq 1-\frac{t}{2}$$

$$\text{Proof: } \frac{1}{t^2+1}=1-\frac{t^2}{t^2+1}\overset{AM-GM}{\geq}1-\frac{t^2}{2t}=1-\frac{t}{2}\ \blacksquare$$

$$Thus, \sum_{cyc}\frac{a+\lambda}{b^2+1} \geq \sum_{cyc}(a+\lambda)\left(1-\frac{b}{2}\right) = \sum_{cyc}(a+\lambda) - \frac{1}{2}\sum_{cyc}(ab+b\lambda)$$

$$Now, \sum_{cyc}(ab+b\lambda) \overset{\sum a=3}{=} \sum_{cyc} ab + 3\lambda \overset{3\sum ab\leq(\sum a)^2}{\leq} \frac{(\sum a)^2}{3} + 3\lambda = 3+3\lambda$$

$$Hence,$$

$$\sum_{cyc}\frac{a+\lambda}{b^2+1} \geq \sum_{cyc}(a+\lambda) - \frac{1}{2}(3+3\lambda) = 3+3\lambda-\frac{1}{2}(3+3\lambda) = \frac{3(\lambda+1)}{2}$$

$$Equality\ holds\ if\ and\ only\ if\ \ a=b=c=1.$$

SP.638 If $x, y, z \geq 1$ and $\lambda \geq 0$ then:

$$\frac{x^3+x+\lambda}{y+z+\lambda} + \frac{y^3+y+\lambda}{z+x+\lambda} + \frac{z^3+z+\lambda}{x+y+\lambda} \geq 3$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We have:

$$LHS = \sum \frac{x^3+x+\lambda}{y+z+\lambda} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod\frac{x^3+x+\lambda}{y+z+\lambda}} \overset{(1)}{\geq} 3 = RHS,$$

where (1) $\Leftrightarrow \sqrt[3]{\prod\frac{x^3+x+\lambda}{y+z+\lambda}} \geq 1 \Leftrightarrow \prod\frac{x^3+x+\lambda}{y+z+\lambda} \geq 1$, **which follows from:**

$$\frac{x^3+x+\lambda}{x+y+\lambda}\cdot\frac{y^3+y+\lambda}{x+y+\lambda} \geq 1 \Leftrightarrow (x^3+x+\lambda)(y^3+y+\lambda) \geq (x+y+\lambda)^2 \Leftrightarrow$$

$\Leftrightarrow \lambda(x^3+y^3)+xy(x^2+y^2)+x^3y^3 \geq \lambda(x+y)+(x^2+y^2)+xy$, **which follows from:**

$\lambda(x^3+y^3) \geq \lambda(x+y), xy(x^2+y^2) \geq (x^2+y^2)$ **and** $x^3y^3 \geq xy$, **true for** $x, y \geq 1$ **and** $\lambda \geq 0$.

From $\frac{x^3+x+\lambda}{x+y+\lambda}\cdot\frac{y^3+y+\lambda}{x+y+\lambda} \geq 1$ **and the analogs, by multiplying member by member, we obtain:**

$\prod\frac{x^3+x+\lambda}{y+z+\lambda} \geq 1$, **with equality for** $x=y=z=1$.

Equality holds if and only if $x=y=z=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x^3+x+\lambda}{y+z+\lambda}\overset{AM-GM}{\geq}2.\sum_{cyc}\frac{x^2}{y+z+\lambda}+\lambda.\sum_{cyc}\frac{1}{y+z+\lambda}\overset{Bergstrom}{\geq}$$

$$\frac{2\left(\sum_{cyc}x\right)^2}{2\sum_{cyc}x+3\lambda}+\frac{9\lambda}{2\sum_{cyc}x+3\lambda}\ (\because\lambda\geq 0)\overset{?}{\geq}3\Leftrightarrow 2\left(\sum_{cyc}x\right)^2+9\lambda\overset{?}{\geq}6\sum_{cyc}x$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}x-3\right)\overset{?}{\geq}0\rightarrow\text{true}\because x,y,z\geq 1\Rightarrow\sum_{cyc}x\geq 3\ \textit{and so,}$$

$$\frac{x^3+x+\lambda}{y+z+\lambda}+\frac{y^3+y+\lambda}{z+x+\lambda}+\frac{z^3+z+\lambda}{x+y+\lambda}\geq 3\ \forall\ x,y,z\geq 1\ \textit{and}\ \lambda\geq 0,$$

$$''=''\ \text{iff}\ x=y=z=1\ (\text{QED})$$

Solution 3 by Jenish Rijal-Bardiya-Nepal

$$\text{Let } S:=x+y+z\geq 3\ \ (\text{as } x,y,z\geq 1).$$

$$\sum_{cyc}\frac{x^3+x+\lambda}{y+z+\lambda}\overset{\text{Bergström}}{\geq}\frac{\left(\sum\sqrt{x^3+x+\lambda}\right)^2}{2(x+y+z)+3\lambda}\overset{AM-GM}{\geq}\frac{\left(\sum\sqrt{2x^2+\lambda}\right)^2}{2S+3\lambda}$$

$$=\frac{\left(\sum\sqrt{\left(\sqrt{2}x\right)^2+\sqrt{\lambda}^2}\right)^2}{2S+3\lambda}\geq\overset{\text{Minkowski}}{\geq}\frac{\left(\sqrt{\left[\sqrt{2}(x+y+z)\right]^2+\left(3\sqrt{\lambda}\right)^2}\right)^2}{2S+3\lambda}=$$

$$=\frac{2S^2+9\lambda}{2S+3\lambda}=\frac{2S\cdot S+9\lambda}{2S+3\lambda}\overset{S\geq 3}{\geq}\frac{2S\cdot 3+9\lambda}{2S+3\lambda}=3$$

$$\textit{Equality holds if and only if}\ \ x=y=z=1.$$

SP.639 If $a,b,c>0$ and $1\leq\lambda\leq 2$ then:

$$\sum\frac{xy}{x^2+\lambda y^2}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

$$LHS=\sum\frac{xy}{x^2+\lambda y^2}=\sum\frac{xy}{x^2+y^2+(\lambda-1)y^2}\leq\sum\frac{xy}{2xy+(\lambda-1)y^2}=$$

$$=\sum\frac{x}{2x+(\lambda-1)y}=$$

$$=\sum\frac{1}{2}\left(1-\frac{(\lambda-1)y}{2x+(\lambda-1)y}\right)=\frac{1}{2}\left(3-\sum\frac{(\lambda-1)y}{2x+(\lambda-1)y}\right)=$$

$$=\frac{3}{2}-\frac{1}{2}\sum\frac{(\lambda-1)y}{2x+(\lambda-1)y}\leq\frac{3}{\lambda+1}=RHS,$$

where $\frac{3}{2}-\frac{1}{2}\sum\frac{(\lambda-1)y}{2x+(\lambda-1)y}\leq\frac{3}{\lambda+1}\Leftrightarrow\frac{1}{2}\sum\frac{(\lambda-1)y}{2x+(\lambda-1)y}\geq\frac{3(\lambda-1)}{2(\lambda+1)}\Leftrightarrow$

$$\Leftrightarrow\sum\frac{y}{2x+(\lambda-1)y}\geq\frac{3}{\lambda+1},$$

see $\sum\frac{y}{2x+(\lambda-1)y}=\sum\frac{y^2}{2xy+(\lambda-1)y^2}\overset{CS}{\geq}\frac{(\sum y)^2}{\sum(2xy+(\lambda-1)y^2)}=\frac{\sum x^2+2\sum xy}{2\sum xy+(\lambda-1)\sum x^2}\geq\frac{3}{\lambda+1}$ where

$$\frac{\sum x^2+2\sum xy}{2\sum xy+(\lambda-1)\sum x^2}\geq\frac{3}{\lambda+1}\Leftrightarrow$$

$$\Leftrightarrow(\lambda+1)\sum x^2+2(\lambda+1)\sum xy\geq 6\sum xy+3(\lambda-1)\sum x^2\Leftrightarrow$$

$$\Leftrightarrow(4-2\lambda)\sum x^2\geq(4-2\lambda)\sum xy,\text{ see }\lambda\leq 2\text{ and }\sum x^2\geq\sum xy.$$

Equality holds if and only if $x=y=z$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{xy}{x^2+\lambda y^2}=\sum_{cyc}\frac{xy}{x^2+(\lambda-1)y^2+y^2}\overset{AM-GM}{\leq}$$

$$\frac{1}{2}\cdot\sum_{cyc}\frac{(2xy+(\lambda-1)y^2)-(\lambda-1)y^2}{2xy+(\lambda-1)y^2}=\frac{3}{2}-\frac{\lambda-1}{2}\cdot\sum_{cyc}\frac{y^2}{2xy+(\lambda-1)y^2}$$

$$\overset{Bergstrom}{\leq}\frac{3}{2}-\frac{\lambda-1}{2}\cdot\frac{\left(\sum_{cyc}x\right)^2}{2\sum_{cyc}xy+(\lambda-1)\sum_{cyc}x^2}\ (\because\lambda\geq 1)\overset{?}{\leq}\frac{3}{\lambda+1}$$

$$\Leftrightarrow\frac{3(\lambda-1)}{2(\lambda+1)}\overset{?}{\leq}\frac{\lambda-1}{2}\cdot\frac{\left(\sum_{cyc}x\right)^2}{2\sum_{cyc}xy+(\lambda-1)\sum_{cyc}x^2}$$

$$\Leftrightarrow(\lambda+1)\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)\overset{?}{\geq}6\sum_{cyc}xy+(3\lambda-3)\sum_{cyc}x^2\ (\because\lambda-1\geq 0)$$

$$\Leftrightarrow 2(2-\lambda)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\overset{?}{\geq}0\rightarrow\text{true}\because 2\geq\lambda\ and\ \sum_{cyc}x^2\geq\sum_{cyc}xy$$

$$\therefore \sum_{cyc} \frac{xy}{x^2+\lambda y^2} \leq \frac{3}{\lambda+1} \ \forall\, x,y,z>0 \ and \ 1 \leq \lambda \leq 2, '' = '' \ iff \ x=y=z \ (QED)$$

SP.640 Find all prime numbers p, q, r with $q < 32$ such that

$$p^q + q^p = r(4p+1).$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

We show that the solutions are $(p,q,r) = (2,11,241)$ and $(p,q,r) = (2,13,929)$. Note that $4p+1$ is always odd. We distinguish three cases.

Case 1: p and q are both odd. Then $p^q + q^p$ is even while $4p+1$ is odd, so $r = 2$ and

$$p^q + q^p = 8p + 2$$

But $p \geq 3$ and $q \geq 3$ give $q^p \geq 3^p$, and $3^p > 8p+2$ for every $p \geq 3$ (it holds for $p = 3$, since $27 > 26$, and the left side more than triples while the right side increases by 8 when p increases by 1). Hence $p^q + q^p > 8p + 2$, a contradiction.

Case 2: $q = 2$ and p is odd. Let $m = 4p+1$. Then $4p \equiv -1 (mod\ m)$, so $16p^2 \equiv 1 (mod\ m)$. Multiplying the congruence $p^2 + 2^p \equiv 0 (mod\ m)$ by 16 we get

$$2^{p+4} \equiv -1\ (mod\ m).$$

Since $p+4$ is odd, we can write $2^{p+4} = 2y^2$ with $y = 2^{\frac{p+3}{2}}$. Then $2y^2 \equiv -1\ (mod\ m)$, and multiplying by 2 gives

$$(2y)^2 \equiv -2\ (mod\ m).$$

Hence -2 is q quadratic residue modulo every prime divisor of m (note that m is odd and coprime to $2y$), and therefore the Jacobi symbol satisfies $\left(\frac{-2}{m}\right) = 1$. On the other hand, p is odd, so $p \equiv 1$ or $3\ (mod\ 4)$ and $m = 4p+1 \equiv 5$ or $13\ \ (\text{mod } 16)$; in both cases $m \equiv 5\ (\text{mod } 8)$. Therefore

$$\left(\frac{-1}{m}\right) = 1, \quad \left(\frac{2}{m}\right) = -1, \quad \left(\frac{-2}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{2}{m}\right) = -1,$$

a contradiction. So, there are no solutions in this case.

Case 3: $p = 2$. Now $4p+1 = 9$ and the equation becomes

$$2^q + q^2 = 9r.$$

For $q = 2$ and $q = 3$ the left side equals 8 and 17, which are not divisible by 9. Let $q \geq 5$. Since $2^6 \equiv 1 \pmod 9$ and $q \equiv 1$ or $5 \pmod 6$, we have

$$2^q \equiv 2 \text{ if } q \equiv 1 \pmod 6, \quad 2^q \equiv 5 \text{ if } q \equiv 5 \pmod 6.$$

Also $q^2 \equiv 1, 4, 7 \pmod 9$ according as $q \equiv \pm 1, \pm 2, \pm 4 \pmod 9$. Hence $9 | 2^q + q^2$ requires either $q \equiv 1 \pmod 6$ and $q^2 \equiv 7$, that is $q \equiv \pm 4 \pmod 9$, or $q \equiv 5 \pmod 6$ and $q^2 \equiv 4$, that is, $q \equiv \pm 2 \pmod 9$. Combining these conditions gives

$$9 | 2^q + q^2 \Leftrightarrow q \equiv 11 \text{ or } 13 \pmod{18}.$$

The primes $q < 32$ with this property are $11, 13, 29$ and 31.

- $q = 11$: $2^{11} + 11^2 = 2048 + 121 = 2169 = 9 \cdot 241$, and 241 is prime (it is not divisible by $2, 3, 5, 7, 11, 13$ and $17^2 > 241$).
- $q = 13$: $2^{13} + 13^2 = 8192 + 169 = 8361 = 9 \cdot 929$, and 929 is prime (it is not divisible by $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$, and $31^2 > 929$).
- $q = 29$ and $q = 31$: we show that $27 | 2^q + q^2$, so that r is divisible by 3; since $r > 3$, it is not prime. As $2^{18} \equiv 1 \pmod{27}$,

$$2^{29} + 29^2 \equiv 2^{11} + 2^2 = 2048 + 4 = 2052 = 27 \cdot 76 \equiv 0 \pmod{27},$$

$$2^{31} + 31^2 \equiv 2^{13} + 4^2 = 8192 + 16 = 8208 = 27 \cdot 304 \equiv 0 \pmod{27}.$$

Therefore, the solutions are

$$(p, q, r) = (2, 11, 241) \text{ and } (p, q, r) = (2, 13, 929).$$

Solution 2 by Jenish Rijal-Bardiya-Nepal

Case I: q is odd.

If p is odd, then $p^q + q^p$ is even, forcing $r = 2$.

So, $$p^q + q^p = 8p + 2$$

But $$p^q + q^p \geq p^3 > 8p + 2 \quad \text{for } p \geq 3.$$

Thus, $p = 2$ and the equation now becomes:

$$q^2 + 2^q = 9r$$

As $q < 32$ is a prime, mod 9 gives,

q	3	5	7	11	13	17	19	23	29	31
$q^2+2^q \pmod 9$	8	3	6	0	0	6	3	3	0	0

Hence, $$q \in \{11, 13, 29, 31\}$$

Since $2^{18} \equiv 1 \pmod{27}$,

$$29^2 + 2^{29} \equiv 2^2 + 2^{11} \equiv 4 + 23 \equiv 0 \pmod{27}$$

and $$31^2 + 2^{31} \equiv 4^2 + 2^{13} \equiv 16 + 11 \equiv 0 \pmod{27}$$

Thus, for $q = 29$ or 31, we have $3 \mid r$, forcing $r = 3$ which is impossible since $q^2 + 2^q > 27$.

Now, for $q \in \{11, 13\}$, we get,

$$(p, q, r) = (2, 11, 241) \text{ or } (2, 13, 929).$$

as the valid solutions of Case I.

Solution 1 for *Case II*:

Case II: $q = 2$.

The equation now becomes: $p^2 + 2^p = r(4p + 1)$ If $p = 2$, we get $8 = 9r$ (impossible)

Hence p is odd. Let $m = 4p + 1$. We get, $m \mid p^2 + 2^p$

Since $4p \equiv -1 \pmod{m}$,

$$16p^2 \equiv 1 \pmod{m}$$

$$16 \cdot 2^p \equiv -1 \pmod{m}$$

Now $m \equiv 5 \pmod 8$, so

$$\left(\frac{2}{m}\right) = -1 \text{ and } \left(\frac{-1}{m}\right) = 1$$

Taking Jacobi symbols mod m, we have,

$$1 = \left(\frac{-1}{m}\right) = \left(\frac{16 \cdot 2^p}{m}\right) = \left(\frac{16}{m}\right)\left(\frac{2}{m}\right)^p = (-1)^p = -1, \text{ a contradiction.}$$

Therefore $q \neq 2$, which completes the proof!

Solution 2 for *Case II*:

Case II: $q = 2$.

If $p = 2$, we get $8 = 9r$ (impossible) The equation now becomes: $p^2 + 2^p = r(4p + 1)$

Hence p is odd. For any $p \geq 3$ we have $p^2 \equiv 1 \pmod 8$ and $4p + 1 \equiv 5 \pmod 8$.

So, $p^2 + 2^p \equiv 1 + 0 \equiv 1 \pmod 8 \implies 1 \equiv r \cdot 5 \pmod 8 \quad \therefore r \equiv 5 \pmod 8$

Let $r = 8k + 5$ for $k \geq 0$. Let us consider the following even numbers:

$$2, 4, 6, 8, \ldots\ldots, 2(4k + 2)$$

Taking mod r, $2 \equiv 2 \pmod r, 4 \equiv 4 \pmod r, 6 \equiv 6 \pmod r \ldots\ldots, 4k + 2 \equiv 4k + 2 \pmod r$ and $4k + 4 \equiv -(4k + 1) \pmod r, 4k + 6 \equiv -(4k - 1) \pmod r, \ldots\ldots, 2(4k + 2) \equiv -1 \pmod r$

Multiplying all the congruences, we get,

$$2^{4k+2} \cdot (4k + 2)! \equiv -(4k + 2)! \pmod r$$

Since r is a prime number $> 4k + 2$, we can cancel $(4k + 2)!$ and get,

$$2^{4k+2} \equiv -1 \pmod r \implies 2^{\frac{r-1}{2}} \equiv -1 \pmod r$$

We have, $p^2 \equiv -2^p \pmod r \implies (p^2)^{\frac{r-1}{2}} \equiv (-2^p)^{\frac{r-1}{2}} \pmod r \implies p^{r-1} \equiv (-1)^{4k+2}\left(2^{4k+2}\right)^p \pmod r$

$$\implies p^{r-1} \equiv 1(-1)^p \equiv -1 \pmod r$$

, which is a contradiction because by Fermat's little theorem, $p^{r-1} \equiv 1 \pmod r$

$[r \nmid p$ since $p^2 \equiv -2^p \not\equiv 0 \pmod r]$

Hence, the required triples are $(p, q, r) = (2, 11, 241)$ or $(2, 13, 929)$.

SP.641 Let $A(z)$ be a polynomial with complex coefficients, and let $a, b, c \in \mathbb{C}$. Suppose that $A(z)$ leaves the same remainder upon division by $(z - a)(z - b)$ and upon division by $(z - a)(z - c)$. Determine the value of

$$(a - b)A(c) + (b - c)A(a) + (c - a)A(b).$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Since both divisors have degree 2, the common remainder has degree at most 1. Write is as $R(z) = pz + q$ with $p, q \in \mathbb{C}$. Then there exist polynomials $Q_1(z)$ and $Q_2(z)$ such that

$A(z) = (z - a)(z - b)Q_1(z) + pz + q$ and $A(z) = (z - a)(z - c)Q_2(z) + pz + q$.

Substituting $z = a$ and $z = b$ in the first identity, and $z = c$ in the second, we obtain

$$A(a) = pa + q, \quad A(b) = pb + q, \quad A(c) = pc + q.$$

Hence: $S=(a-b)(pc+q)+(b-c)(pa+q)+(c-a)(pb+q)$

$$=p[(a-b)c+(b-c)a+(c-a)b]+q[(a-b)+(b-c)+(c-a)]$$

$$=p(ac-bc+ab-ac+bc-ab)+q\cdot 0=0.$$

Therefore: $(a-b)A(c)+(b-c)A(a)+(c-a)A(b)=0$.

Remarks.

1. The expression equals, up to sign, the determinant

$$\begin{vmatrix} a & A(a) & 1 \\ b & A(b) & 1 \\ c & A(c) & 1 \end{vmatrix},$$

which vanishes because the points $(a,A(a)),(b,A(b)),(c,A(c))$ all satisfy the linear relation $w=pz+q$. The hypothesis on the reminder is precisely what forces this "collinearity".

2. The result remains valid when a,b,c are not distinct. For instance, if $b=a$, the first divisor is $(z-a)^2$ and we still have $A(a)=A(b)=R(a)$. If $b=c$, the expression vanishes identically.

SP.642 It is possible to write each positive integer as a sum of distinct Lucas numbers such that no two summands are neighbors in the Lucas sequence? The same question for Fibonacci numbers.

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

(a) Lucas numbers. Recall that $L_0=2, L_1=1$ and $L_{k+1}=L_k+L_{k-1}$ for $k\geq 1$, so the sequence is $2,1,3,4,7,11,18,29,\ldots$ Note that $L_1<L_2<L_3<\cdots$, and that

$$L_{k+1}-L_k=L_{k-1}\quad (k\geq 1).$$

We prove by strong induction on n that every positive integer n is a sum of distinct Lucas numbers, no two of them with consecutive indices.

For $n\leq 3$ the statement is immediate: $1=L_1, 2=L_0$ and $3=L_2$ are themselves Lucas numbers, so each of them is a sum with a single summand, and the condition that no two summands are neighbors in the sequence is satisfied trivially.

Let $n \geq 4$ and assume the statement holds for all positive integers less than n. Since $L_3 = 4 \leq n$ and $L_k \to \infty$, there is a unique $k \geq 3$ with

$$L_k \leq n < L_{k+1}$$

Put $m = n - L_k$. Then $0 \leq m < L_{k+1} - L_k = L_{k-1}$. If $m = 0$, then $n = L_k$ and we are done. Otherwise $1 \leq m < n$, and by the induction hypothesis m is a sum of distinct Lucas numbers with no two consecutive indices. Every monotonicity of $L_1, L_2, \ldots$ gives $j \leq k - 2$; if $j = 0$, then j is not adjacent to k because $k \geq 3$. Hence adding the summand L_k to the representation of neighbors in the sequence. This completes the induction.

For instance, $5 = L_2 + L_0 = 3 + 2, 16 = L_5 + L_3 + L_1 = 11 + 4 + 1$ and

$$17 = L_5 + L_3 + L_0 = 11 + 4 + 2.$$

(b) Fibonacci numbers. Let $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$. Since $F_1 = F_2$, we use only $F_2, F_3, F_4, \ldots = 1, 2, 3, 5, 8, \ldots$, which is strictly increasing, and $F_{k+1} - F_k = F_{k-1}$. The argument is the same. We have $1 = F_2$. For $n \geq 2$, let $k \geq 3$ be such that $F_k \leq n < F_{k+1}$, and put

$m = n - F_k$, so that $0 \leq m < F_{k-1}$. If $m = 0$ we are done; otherwise, by the induction hypothesis m is a sum of distinct, pairwise non-consecutive Fibonacci numbers, all of them at most $m < F_{k-1}$, hence with indices at most $k - 2$. Adding F_k gives the required representation of n.

Thus, every positive integer is a sum of distinct Fibonacci numbers, no two of them consecutive. This is Zeckendorf's theorem.

Remark. The Fibonacci representation is unique. The Lucas representation is not: for example,

$5 = L_2 + L_0 = L_3 + L_1$. Uniqueness is restored if we also forbid using L_0 and L_2 together.

SP.643 **Find the equation of the common perpendicular line and the minimum distance between the lines r and s, given by**

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$r:\begin{cases}2x-y-2=0\\y-2z+6=0\end{cases}$ $s:\begin{cases}x-y-2=0\\2y-z+1=0\end{cases}$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

We have

1. Line r: direction vector via $\overrightarrow{n_1}\times\overrightarrow{n_2}$, where $\overrightarrow{n_1}=(2,-1,0),\overrightarrow{n_2}=(0,1,-2)$:

$$\vec{u}=\overrightarrow{n_1}\times\overrightarrow{n_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-1&0\\0&1&-2\end{vmatrix}=(2,4,2)\parallel(1,2,1)$$

Point on r: $A(1,0,3)$. So r: point $A(1,0,3)$, direction $\vec{u}(1,2,1)$.

2. Line s: direction vector via $\overrightarrow{n_1}\times\overrightarrow{n_2}$, where $\overrightarrow{n_1}=(1,-1,0),\overrightarrow{n_2}=(0,2,-1)$:

$$\vec{v}=\overrightarrow{n_1}\times\overrightarrow{n_2}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&-1&0\\0&2&-1\end{vmatrix}=(1,1,2)$$

Point on s: $B(2,0,1)$. So s: point $B(2,0,1)$, direction $\vec{v}(1,1,2)$.

3. Vector between point:

$$\overrightarrow{AB}=B-A=(1,0,-2)$$

4. Cross product $\vec{u}\times\vec{v}$:

$$\vec{u}\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&2&1\\1&1&2\end{vmatrix}=(3,-1,-1)$$

5. Scalar triple product (check skew lines):

$$\overrightarrow{AB}\cdot(\vec{u}\times\vec{v})=(1)(3)+(0)(-1)+(-2)(-1)=5\neq0$$

Since this is nonzero, r and s are skew lines. Then, we have

- Line r: point $A(1,0,3)$, direction $\vec{u}(1,2,1)$
- Line s: point $B(2,0,1)$, direction $\vec{v}(1,1,2)$
- Common perpendicular direction: $\vec{w}=\vec{u}\times\vec{v}=(3,-1,-1)$

Next, we find the feet of the common perpendicular. Let $P=A+t\vec{u}=(1+t,2t,3+t)$ on r, and $Q=B+s\vec{v}=(2+s,s,1+2s)$ on s. We need $\overrightarrow{PQ}\perp\vec{u}$ and $\overrightarrow{PQ}\perp\vec{v}$, where

$\overrightarrow{PQ} = (1+s-t, s-2t, -2+2s-t)$.

$$\overrightarrow{PQ} \cdot \vec{u} = 0 \Rightarrow 5s - 6t = 1$$

$$\overrightarrow{PQ} \cdot \vec{v} = 0 \Rightarrow 6s - 5t = 3$$

Solving: $s = \frac{13}{11}, t = \frac{9}{11}$.

Now, we compute the feet of the perpendicular:

$$P = \left(\frac{20}{11}, \frac{18}{11}, \frac{42}{11}\right), \quad Q = \left(\frac{35}{11}, \frac{13}{11}, \frac{37}{11}\right)$$

Check: $\overrightarrow{PQ} = \left(\frac{15}{11}, -\frac{5}{11}, -\frac{5}{11}\right) = \frac{5}{11}(3,-1,-1)$ parallel to $\vec{w}$.

Equation of the common perpendicular line. Passes through $P\left(\frac{20}{11}, \frac{18}{11}, \frac{42}{11}\right)$ with direction $\vec{w}(3,-1,-1)$:

$$\frac{x - \frac{20}{11}}{3} = \frac{y - \frac{18}{11}}{-1} = \frac{z - \frac{42}{11}}{-1}$$

- Plane containing r and $\vec{w}$: normal $= \vec{u} \times \vec{w} = (-1, 4, -7)$, through $A(1,0,3)$ is $x - 4y + 7z - 22 = 0$.
- Plane containing s and $\vec{w}$: normal $= \vec{v} \times \vec{w} = (1, 7, -4)$, through $B(2,0,1)$ is $x + 7y - 4z + 2 = 0$. Thus, the equation of common perpendicular is:

$$\begin{cases} x - 4y + 7z - 22 = 0 \\ x + 7y - 4z + 2 = 0 \end{cases}$$

Finally, the minimum distance is

$$d(r,s) = \left|\overrightarrow{PQ}\right| = \frac{5}{11}\sqrt{3^2 + (-1)^2 + (-1)^2} = \frac{5}{11}\sqrt{11} = \frac{5\sqrt{11}}{11}.$$

SP.644 Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\frac{1}{2a+1} + \frac{1}{2b+1} + \frac{1}{2c+1} + \frac{1}{2d+1} \geq \frac{a+b+c+d}{3}$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality as $E \geq 0$, where

$$E=\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}+\frac{1}{2d+1}-\frac{a+b+c+d}{3}.$$

For fixed a and b, assume that c and E are functions of d. We have $c'=\frac{-(a+b+c)}{a+b+d}$ and

$$3E'(d)=-\left[\frac{6}{(2c+1)^2}+1\right]c'-\frac{6}{(2d+1)^2}-1=$$

$$=\frac{(2c+1)^2+6}{(2c+1)^2}\cdot\frac{a+b+c}{a+b+d}-\frac{(2d+1)^2+6}{(2d+1)^2}$$

We claim that $E'(d)\leq 0$. Write this inequality as follows:

$$\frac{a+b+c}{a+b+d}\leq\frac{[(2d+1)^2+6](2c+1)^2}{[(2c+1)^2+6](2d+1)^2},\ \frac{a+b+c}{a+b+d}-1$$

$$\leq\frac{[(2d+1)^2+6](2c+1)^2}{[(2c+1)^2+6](2d+1)^2}-1,$$

$$\frac{c-d}{a+b+d}\leq\frac{24(c-d)(c+d+1)}{[(2c+1)^2+6](2d+1)^2}.$$

It is true if

$$\frac{1}{a+b+d}\leq\frac{24(c+d+1)}{[(2c+1)^2+6](2d+1)^2}.$$

Since $(2c+1)^2+6\leq 15$ and $a+b+d\geq 2+d\geq 2d+1$, it suffices to show that

$$1\leq\frac{8(c+d+1)}{5(2d+1)},$$

which is equivalent to the obvious inequality $8c-2d+3\geq 0$. From $E'(d)\leq 0$, it follows that $E(d)$ is decreasing and has the minimum value when d is maximum, hence when $d=c$. So, it suffices to consider $d=c$, when we need to show that $F\geq 0$ for

$$ab+2(a+b)c+c^2=6,\quad a\geq b\geq 1\geq c,$$

where

$$F=\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{2}{2c+1}-\frac{a+b+2c}{3}.$$

From $6\geq ab\geq b^2$, we get $b\leq\sqrt{6}$.

For fixed a, assume that c and F are functions of b. We have $2(a+b+c)c'+a+2c=0$ and

$$3F'(b) = -\left[\frac{12}{(2c+1)^2}+2\right]c' - \frac{6}{(2b+1)^2} - 1 =$$

$$= \frac{(2c+1)^2+6}{(2c+1)^2}\cdot\frac{a+2c}{a+b+c} - \frac{(2b+1)^2+6}{(2b+1)^2}.$$

We will show that $F'(b) \geq 0$. Write this inequality as follows:

$$\frac{[(2c+1)^2+6](2b+1)^2}{[(2b+1)^2+6](2c+1)^2} \geq \frac{a+b+c}{a+2c},$$

$$\frac{[(2c+1)^2+6](2b+1)^2}{[(2b+1)^2+6](2c+1)^2} - 1 \geq \frac{a+b+c}{a+2c} - 1,$$

$$\frac{24(b-c)(b+c+1)}{(4b^2+4b+7)(2c+1)^2} \geq \frac{b-c}{a+2c}.$$

It is true if

$$\frac{24(b+c+1)}{(4b^2+4b+7)(2c+1)^2} \geq \frac{1}{a+2c}.$$

Since

$$(b+c+1)(a+2c) \geq (b+2c)(a+2c) \geq (b+2c)^2,$$

it suffices to show that

$$\frac{24(b+2c)^2}{(2c+1)^2} \geq 4b^2+4b+7.$$

Since

$$\frac{b+2c}{2c+1} = 1 + \frac{b-1}{2c+1} \geq 1 + \frac{b-1}{3} = \frac{b+2}{3},$$

we have

$$\frac{24(b+2c)^2}{(2c+1)^2} - (4b^2+4b+7) \geq \frac{8(b+2)^2}{3} - (4b^2+4b+7) =$$

$$= \frac{4(6-b^2)+20b-13}{3} > 0$$

From $F'(b) \geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. So, it suffices to consider this case, when we need to show that

$$\frac{1}{2a+1} + \frac{1}{3} + \frac{2}{2c+1} \geq \frac{a+1+2c}{3}$$

for $a=\frac{6-2c-c^2}{2c+1}$. Since

$$2a+1=\frac{13-2c-2c^2}{2c+1},\qquad \frac{1}{3}+\frac{2}{2c+1}=\frac{7+2c}{3(2c+1)},\quad a+1+2c=\frac{7+2c+3c^2}{2c+1},$$

the required inequality becomes as follows:

$$\frac{2c+1}{13-2c-2c^2}+\frac{7+2c}{3(2c+1)}\geq\frac{7+2c+3c^2}{3(2c+1)},\qquad \frac{2c+1}{13-2c-2c^2}\geq\frac{c^2}{2c+1},$$

$$2c^4+2c^3-9c^2+4c+1\geq 0,\ \ (c-1)^2(2c^2+6c+1)\geq 0.$$

The proof is finished. The equality occurs for $(a,b,c,d)=(1,1,1,1)$.

Remark. Actually, the following generalization is true:

- If $k\geq k_0=\frac{7+\sqrt{65}}{8}\approx 1.8828$, then

$$\frac{1}{ka+1}+\frac{1}{kb+1}+\frac{1}{kc+1}+\frac{1}{kd+1}\geq\frac{a+b+c+d}{k+1}$$

for all nonnegative real numbers a,b,c,d such that $a\geq b\geq 1\geq c\geq d$ and

$$ab+ac+ad+bc+bd+cd=6.$$

If $k=k_0$, then the equality occurs for $(a,b,c,d)=(1,1,1,1)$ and also for $(a,b,c,d)=(6,1,0,0)$.

SP.645 **Under stereographic projection, what condition must two points z_1,z_2 in the plane satisfy for them to correspond to a pair of antipodal (diametrically opposite) points on the sphere?**

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Let the sphere be the unit sphere

$$x^2+y^2+z^2=1,$$

and let the stereographic projection be taken from the north pole $N=(0,0,1)$ onto the equatorial plane $z=0$, which we identify with the complex plane $\mathbb{C}$. For a point $P=(x,y,z)$ on the sphere (different from N), the projection is the point where the line NP meets the plane $z=0$. That is,

$$\zeta = \frac{x+iy}{1-z} \quad (1)$$

If $P = (x, y, z)$ has stereographic image $z_1 = \frac{x+iy}{1-z}$, its antipodal point is $P' = (-x, -y, -z)$, whose image is

$$z_2 = \frac{-x - iy}{1 + z}.$$

Now, we find a relation between z_1 and z_2. Computing $z_1\overline{z_2}$, we have

$$\overline{z_2} = \frac{-x + iy}{1 + z},$$

so

$$z_1\overline{z_2} = \frac{(x + iy)(-x + iy)}{(1 - z)(1 + z)} = \frac{-(x^2 + y^2)}{1 - z^2}.$$

Since $x^2 + y^2 = 1 - z^2$ on the sphere, this simplifies to

$$z_1\overline{z_2} = \frac{-(1 - z^2)}{1 - z^2} = -1.$$

Thus, the condition is $z_1\overline{z_2} = -1$ or equivalently $z_2 = -\frac{1}{z_1}$.

Geometrically this says:

- $|z_1||z_2| = 1$ – the two points are related by inversion in the unit circle.
- $\arg z_2 = \pi + \arg z_1 \ (mod\ 2\pi)$ – they lie in opposite directions from the origin

So z_1 and z_2 are stereographic projections of antipodal points on the sphere iff $z_1\overline{z_2} = -1$.

Special cases are:

- The projection of the north pole itself is ∞, and its antipode, the south pole, projects to 0.
- If $z_1 = 0$ then $z_2 = \infty$; if z_1 lies on the unit circle $(|z_1| = 1)$, then $z_2 = \frac{1}{\overline{z_1}} = -z_1$, its antipode on the same circle.

Finally, we derive the well – known expression (1). Indeed, let $N = (0, 0, 1)$ be the north pole of the unit sphere $x^2 + y^2 + z^2 = 1$, and let $P = (x, y, z)$ be any point on the sphere

other than N. We want the point where the line through N and P meets the equatorial plane

$z = 0$. A point on the line through N and P can be written as

$$Q(t) = N + t(P - N) = (0,0,1) + t\big((x,y,z) - (0,0,1)\big), \quad t \in \mathbb{R}.$$

Writing this out in coordinates:

$$Q(t) = \big(tx, ty, 1 + t(z-1)\big).$$

At $t = 0$ we get N, and at $t = 1$ we get P.

Now, we impose the condition $z = 0$. We seek t such that the third coordinate of $Q(t)$ vanishes:

$$1 + t(z-1) = 0.$$

Since $P \neq N$, we have $z \neq 1$, so

$$t = \frac{1}{1-z}.$$

Substituting into the first two coordinates yields

$$Q\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Identifying $(u, v, 0)$ with $u + iv$, the intersection point corresponds to

$$\zeta = \frac{x}{1-z} + i\frac{y}{1-z} = \frac{x+iy}{1-z}.$$

UNDERGRADUATE PROBLEMS

UP.631 Prove that:

$$\int_0^1 \frac{(\log x)^2}{x^2 - x - 1}\,dx = \frac{2}{2\varphi - 1}\left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right)\right)$$

$$where\ \varphi = \frac{1+\sqrt{5}}{2}\ is\ the\ Golden\ Ratio.$$

Proposed by Omkumar Rao-India

Solution 1 by proposer

$$x^2 - x - 1 \Rightarrow x_1 = \varphi\ , x_2 = -\frac{1}{\varphi}$$

$$Let\ \Psi = \int_0^1 \frac{(\log x)^2}{x^2 - x - 1} dx$$

$$\int_0^1 \frac{(\log x)^2}{x^2 - x - 1} dx = \int_0^1 (\log x)^2 \left(\frac{1}{\varphi + \frac{1}{\varphi}} \left(\frac{1}{x - \varphi} - \frac{1}{x + \frac{1}{\varphi}} \right) \right) dx$$

$$= \int_0^1 (\log x)^2 \left(\frac{1}{2\varphi - 1} \left(\frac{1}{x - \varphi} - \frac{1}{x + \frac{1}{\varphi}} \right) \right) dx$$

$$\Psi = \frac{1}{2\varphi - 1} \left[\underbrace{\int_0^1 \frac{(\log x)^2}{x - \varphi} dx}_{-2Li_3\left(\frac{1}{\varphi}\right)} - \underbrace{\int_0^1 \frac{(\log x)^2}{x + \frac{1}{\varphi}} dx}_{-2Li_3(-\varphi)} \right] = \frac{2}{2\varphi - 1} \left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right) \right)$$

$$\boxed{\int_0^1 \frac{(\log x)^2}{x^2 - x - 1} dx = \frac{2}{2\varphi - 1} \left(Li_3(-\varphi) - Li_3\left(\frac{1}{\varphi}\right) \right)}$$

Solution 2 by Said Attaoui-Algeria

We have $\varphi^2 = \varphi + 1, \varphi - 1 = \varphi^{-1}$ and $\varphi + \varphi^{-1} = 2\varphi - 1 = \sqrt{5}$. Hence

$x^2 - x - 1 = (x - \varphi)(x + \varphi^{-1})$ and

$$\frac{1}{x^2 - x - 1} = \frac{1}{\sqrt{5}} \left(\frac{1}{x - \varphi} - \frac{1}{x + \varphi^{-1}} \right).$$

For $z \in \mathbb{C} \setminus [1, \infty)$ define

$$Li_3(z)=\frac{z}{2}\int_0^1\frac{\ln^2 t}{1-zt}dt.$$

For $|z|<1$, expanding $\frac{1}{1-zt}$ as a geometric series and using $\int_0^1 t^{n-1}\ln^2 t\,dt=\frac{2}{n^3}$ gives $\sum_{n\geq 1}\frac{z^n}{n^3}$. The right-hand side is analytic on $\mathbb{C}\setminus[1,\infty)$, so by the identity theorem it is the principal-branch analytic continuation of the trilogarithm. In particular it is valid at $z=\varphi^{-1}$ (inside the unit disc) and at $z=-\varphi$ (on the negative real axis).

Since

$$\frac{1}{x-\varphi}=-\frac{1}{\varphi}\cdot\frac{1}{1-\frac{x}{\varphi}},\qquad \frac{1}{x+\varphi^{-1}}=\frac{\varphi}{1+\varphi x},$$

the definition with $z=\varphi^{-1}$ and $z=-\varphi$ yields

$$\int_0^1\frac{\ln^2 x}{x-\varphi}dx=-\frac{1}{\varphi}\cdot 2\varphi Li_3(\varphi^{-1})=-2Li_3(\varphi^{-1}),$$

$$\int_0^1\frac{\ln^2 x}{x+\varphi^{-1}}dx=\varphi\cdot\frac{2}{-\varphi}Li_3(-\varphi)=-2Li_3(-\varphi)$$

Subtracting and dividing by $\sqrt{5}=2\varphi-1$,

$$\int_0^1\frac{\ln^2 x}{x^2-x-1}dx=\frac{2}{2\varphi-1}\left(Li_3(-\varphi)-Li_3\left(\frac{1}{\varphi}\right)\right).$$

Remark. The relations $\varphi^2=\varphi+1$ and $\varphi^{-1}=\varphi-1$ make the poles φ and $-\varphi^{-1}$ conjugates in $\mathbb{Q}(\sqrt{5})$, with difference $\sqrt{5}=2\varphi-1$. This is the source of the prefactor $\frac{2}{2\varphi-1}$.

Solution 3 by Lara Gangler-Hungary

I. Obtaining two simpler integrals J and K

$$I=\int_0^1\frac{\ln^2(x)}{x^2-x-1}dx=\int_0^1\frac{\ln^2(x)}{(x-\varphi)\left(x+\frac{1}{\varphi}\right)}dx=\frac{1}{\varphi+\frac{1}{\varphi}}\int_0^1\left(\frac{\ln^2(x)}{x-\varphi}-\frac{\ln^2(x)}{x+\frac{1}{\varphi}}\right)dx=$$

$$=\frac{\varphi}{\varphi^2+1}(K-J)$$

II. Obtaining two integrals J_1 and J_2 from J by splitting the range of integration

$$J=\int_0^1 \frac{\ln^2(x)}{x+\frac{1}{\varphi}}dx=\underbrace{\int_0^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw}_{w=-\ln(x)\to dx=-e^w dw}$$

Problem

The denominator has to be transformed into the form $\frac{1}{1+a}$ with $|a|\le 1$ so that it can be expressed as an infinite sum but it can't be done so uniformly over the whole interval

→ the interval is split in two and the denominator slightly changed and is transformed into the form $\frac{1}{1+a}$

$$J=\int_0^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_0^{\ln(\varphi)} \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw+\int_{\ln(\varphi)}^\infty \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=J_1+J_2$$

$$0\le w\le \ln(\varphi)\to 1\le e^w\le \varphi\to \frac{1}{\varphi}\le \frac{1}{\varphi}e^w\le 1$$

→ the denominator can be expressed as an infinite sum in form $1+\frac{1}{\varphi}e^w$

$$\ln(\varphi)\le w\le \infty\to 0\le e^{-w}\le \frac{1}{\varphi}\to 0\le \varphi e^{-w}\le 1$$

→ the denominator can be expressed as an infinite sum in form $1+\varphi e^{-w}$

III. Calculating J_1

$$J_1=\int_0^{\ln(\varphi)} \frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_0^{\ln(\varphi)} \frac{w^2}{1+\frac{1}{\varphi}e^w}dw=\int_0^{\ln(\varphi)} \frac{w^2}{1+\frac{1}{\varphi}e^w}dw$$

$$=\int_0^{\ln(\varphi)} w^2\sum_{k=0}^\infty e^{wk}\left(-\frac{1}{\varphi}\right)^k dw=\int_0^{\ln(\varphi)} w^2dw+\int_0^{\ln(\varphi)} w^2\sum_{k=1}^\infty e^{wk}\left(-\frac{1}{\varphi}\right)^k dw=$$

$$=\left[\frac{w^3}{3}\right]_0^{\ln(\varphi)}+\sum_{k=1}^\infty \left(-\frac{1}{\varphi}\right)^k\int_0^{\ln(\varphi)} w^2e^{wk}\,dw=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k=1}^\infty \left(-\frac{1}{\varphi}\right)^k\left[\left(\frac{w^2}{k}-\frac{2w}{k^2}+\frac{2}{k^3}\right)\right]_0^{\ln(\varphi)}=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k}^{\infty}\left(-\frac{1}{\varphi}\right)^k\left[\left(\frac{\ln^2(\varphi)}{k}-\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)\varphi^k-\frac{2}{k^3}\right]=$$

$$=\frac{\ln^3(\varphi)}{3}+\sum_{k=1}^{\infty}\left(\ln^2(\varphi)\frac{(-1)^k}{k}-2\ln(\varphi)\frac{(-1)^k}{k^2}+2\frac{(-1)^k}{k^3}-2\frac{\left(-\frac{1}{\varphi}\right)^k}{k^3}\right)=$$

$$=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+2\ln(\varphi)\times\frac{\pi^2}{12}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)=$$

$$=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)$$

IV. Calculating J_2

$$J_2=\int_{\ln(\varphi)}^{\infty}\frac{w^2e^{-w}}{e^{-w}+\frac{1}{\varphi}}dw=\int_{\ln(\varphi)}^{\infty}\frac{\varphi w^2e^{-w}}{1+\varphi e^{-w}}dw=\int_{\ln(\varphi)}^{\infty}\varphi w^2\,e^{-w}\sum_{k=0}^{\infty}(-\varphi)^k\,e^{-wk}dw$$

$$=\int_{\ln(\varphi)}^{\infty}-w^2\sum_{k=1}^{\infty}(-\varphi)^k\,e^{-wk}dw=-\sum_{k=1}^{\infty}(-\varphi)^k\int_{\ln(\varphi)}^{\infty}w^2e^{-wk}\,dw$$

$$=\sum_{k=1}^{\infty}(-\varphi)^k\left[\left(\frac{w^2}{k}+\frac{2w}{k^2}+\frac{2}{k^3}\right)e^{-wk}\right]_{\ln(\varphi)}^{\infty}$$

$$=\sum_{k=1}^{\infty}(-\varphi)^k\left[0-\left(\frac{\ln^2(\varphi)}{k}+\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)\left(\frac{1}{\varphi}\right)^k\right]$$

$$=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\ln^2(\varphi)}{k}+\frac{2\ln(\varphi)}{k^2}+\frac{2}{k^3}\right)$$

$$=\sum_{k}^{\infty}\left(\ln^2(\varphi)\frac{(-1)^{k+1}}{k}+2\ln(\varphi)\frac{(-1)^{k+1}}{k^2}+2\frac{(-1)^{k+1}}{k^3}\right)$$

$$=\ln^2(\varphi)\ln(2)+2\ln(\varphi)\times\frac{\pi^2}{12}+2\times\frac{3}{4}\zeta(3)=\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}+\frac{3}{2}\zeta(3)$$

V. Calculating J

$$J=J_1+J_2=\frac{\ln^3(\varphi)}{3}-\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}-\frac{3}{2}\zeta(3)-2Li_3\left(-\frac{1}{\varphi}\right)+$$

$$+\ln^2(\varphi)\ln(2)+\frac{\pi^2\ln(\varphi)}{6}+\frac{3}{2}\zeta(3)=$$

$$=\frac{\ln^3(\varphi)}{3}+\frac{\pi^2\ln(\varphi)}{3}-2Li_3\left(-\frac{1}{\varphi}\right)=$$

$$=\frac{\ln^3(\varphi)}{3}+\frac{\pi^2\ln(\varphi)}{3}-2\underbrace{\left(Li_3(-\varphi)+\frac{\ln^3(\varphi)}{6}+\frac{\pi^2\ln(\varphi)}{6}\right)}_{\substack{\textit{Trilogarithm identity,}\\ \textit{see on the Trilogarithm page}\\ \textit{of Wolfram Alpha}}}=-2Li_3(-\varphi)$$

VI. Calculating K

$$K=\int_0^1\frac{\ln^2(x)}{x-\varphi}dx=\int_0^1\frac{\ln^2(x)}{x-\varphi}dx=\int_0^1\frac{\ln^2(x)}{x-\varphi}dx=-\frac{1}{\varphi}\int_0^1\frac{\ln^2(x)}{1-\frac{x}{\varphi}}dx=$$

$$=-\frac{1}{\varphi}\int_0^1\ln^2(x)\underbrace{\sum_{k=0}^{\infty}\left(\frac{x}{\varphi}\right)^k}_{0\le\frac{x}{\varphi}\le1}dx$$

$$=-\frac{1}{\varphi}\sum_{k=0}^{\infty}\left(\frac{1}{\varphi}\right)^k\int_0^1x^k\ln^2(x)dx=-\frac{1}{\varphi}\sum_{k=0}^{\infty}\left(\frac{1}{\varphi}\right)^k\frac{\Gamma(3)}{(k+1)^3}=-\frac{2\left(\frac{1}{\varphi}\right)^{k+1}}{(k+1)^3}=-2Li_3\left(\frac{1}{\varphi}\right)$$

VII. End Game

$$I=\frac{\varphi}{\varphi^2+1}(K-J)=\frac{\varphi}{\varphi+2}\left(-2LI_3\left(\frac{1}{\varphi}\right)+2Li_3(-\varphi)\right)=$$

$$=\frac{2\varphi}{\varphi+2}\left(Li_3(-\varphi)-Li_3\left(\frac{1}{\varphi}\right)\right)$$

$$=\frac{2}{1+\frac{2}{\varphi}}\left(Li_3(-\varphi)-Li_3\left(\frac{1}{\varphi}\right)\right)=\frac{2}{2\varphi-1}\left(Li_3(-\varphi)-Li_3\left(\frac{1}{\varphi}\right)\right)$$

UP.632 Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x+2y+3z}{e^{x+y+z}}dxdydz$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\Omega=\int_0^1\int_0^1\int_0^1(x+2y+3z)\,e^{-x-y-z}\,dxdydz=$$

$$=\int_0^1\int_0^1\int_0^1 xe^{-x-y-z}\,dxdydz+2\int_0^1\int_0^1\int_0^1 ye^{-x-y-z}\,dxdydz+$$

$$+3\int_0^1\int_0^1\int_0^1 ze^{-x-y-z}\,dxdydz=6\int_0^1\int_0^1\int_0^1 xe^{-x}e^{-y}e^{-z}\,dxdydz$$

$$=6\left(\int_0^1 xe^{-x}\,dx\right)\left(\int_0^1 e^{-y}dy\right)\left(\int_0^1 e^{-z}dz\right)=$$

$$=6\left(-xe^{-x}|_0^1+\int_0^1 e^{-x}dx\right)\left(-e^{-y}|_0^1\right)\left(-e^{-z}|_0^1\right)=$$

$$=6\left(-\frac{1}{e}-\frac{1}{e}+e^0\right)(-e^{-1}+e^0)^2=6\left(1-\frac{2}{e}\right)\left(1-\frac{1}{e}\right)^2$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Since $e^{-(x+y+z)}=e^{-x}e^{-y}e^{-z}$, the integrand separates. We need the two one-variable integrals

$$S=\int_0^1 e^{-t}\,dt=1-\frac{1}{e},\qquad B=\int_0^1 te^{-t}=[-(t+1)e^{-t}]_0^1=1-\frac{2}{e}.$$

Each term of the numerator gives a product of one-variable integrals. For instance,

$$\int_0^1\int_0^1\int_0^1 xe^{-x}e^{-y}e^{-z}\,dx\,dy\,dz=BA^2,$$

and the terms in y and z give the same value by symmetry. Therefore

$$\Omega=(1+2+3)BA^2=6\left(1-\frac{2}{e}\right)\left(1-\frac{1}{e}\right)^2=\frac{6(e-2)(e-1)^2}{e^3}.$$

UP.633 Prove the equality:

$$\int_0^\infty\frac{\arctan x\cdot|\ln x|}{x^2-x+1}\,dx=\frac{\pi}{30}\left[\psi_1\left(\frac{1}{6}\right)-2\pi^2\right]$$

where $\psi_1(x)$ is trigamma function.

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_0^{\infty}\frac{\arctan x\cdot|\ln x|}{x^2-x+1}dx$$

We also consider the integral:

$$J=\int_0^{\infty}\frac{\arctan x\cdot|\ln x|}{x^2-x+1}dx$$

We have

$$I+J=\frac{\pi}{2}\int_0^{\infty}\frac{|\ln x|}{x^2-x+1}dx$$

In the integral J we make the change of variable $x=\frac{1}{t}$ and immediately obtain that $J=I$.

Therefore:

$$I=\frac{\pi}{4}\int_0^{\infty}\frac{|\ln x|}{x^2-x+1}dx$$

We will calculate:

$$K=\int_0^{\infty}\frac{|\ln x|}{x^2-x+1}dx=\int_0^{1}\frac{|\ln x|}{x^2-x+1}dx+\int_1^{\infty}\frac{|\ln x|}{x^2-x+1}dx$$

Let us denote:

$$A=\int_0^{1}\frac{|\ln x|}{x^2-x+1}dx;\quad B=\int_1^{\infty}\frac{|\ln x|}{x^2-x+1}dx$$

We can write:

$$A=-\int_0^{1}\frac{(1+x)\ln x}{1+x^3}dx$$

We note:

$$C=\int_0^{1}\frac{(1+x)\ln x}{1+x^3}dx=\int_0^{1}\frac{\ln x}{1+x^3}dx+\int_0^{1}\frac{x\ln x}{1+x^3}dx$$

We will calculate:

$$D=\int_0^{1}\frac{\ln x}{1+x^3}dx$$

We make the change of variable:

$$y=x^3;x=y^{\frac{1}{3}}$$

We obtain:

$$D=\frac{1}{9}\int_0^1 \frac{y^{-\frac{2}{3}}\ln y}{1+y}dy$$

We will use the following known relationship:

$$\int_0^1 \frac{z^a \ln z}{1+z}dz=\frac{1}{4}\left[\psi_1\left(\frac{a+2}{2}\right)-\psi_1\left(\frac{a+1}{2}\right)\right], a\in\mathbb{R}\setminus\{-1,-2,\dots\}$$

where $\psi_1(x)$ is the trigamma function.

We can therefore write:

$$D=\frac{1}{36}\left[\psi_1\left(\frac{4}{6}\right)-\psi_1\left(\frac{1}{6}\right)\right]$$

By proceeding similarly to the integral D, we obtain:

$$E=\int_0^1 \frac{x\ln x}{1+x^3}dx\frac{1}{36}\left[\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{2}{6}\right)\right]$$

We obtained the value of the integral C:

$$C=D+E=\frac{1}{36}\left[\psi_1\left(\frac{4}{6}\right)-\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{2}{6}\right)\right]$$

So:

$$A=\frac{1}{36}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We consider the integral B. We make the variable change: $x=\frac{1}{t}$. We obtain: $B=A$

Result:

$$K=A+B=2A$$

$$K=\frac{1}{18}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We have:

$$I=\frac{\pi}{4}K,\qquad I=\frac{\pi}{72}\left[-\psi_1\left(\frac{4}{6}\right)+\psi_1\left(\frac{1}{6}\right)-\psi_1\left(\frac{5}{6}\right)+\psi_1\left(\frac{2}{6}\right)\right]$$

We use the reflection formula:

$$\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{5}{6}\right)=\frac{\pi^2}{\sin^2\frac{\pi}{6}}=4\pi^2,\ \ \psi_1\left(\frac{2}{6}\right)+\psi_1\left(\frac{4}{6}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{6}}=\frac{4\pi^2}{3}$$

We can write:

$$I=\frac{\pi}{36}\left[\psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{2}{6}\right)-\frac{8\pi^2}{3}\right]$$

We will now use the duplication formula:

$$4\psi_1(2x)=\psi_1(x)+\psi_1\left(x+\frac{1}{2}\right)$$

We obtain:

$$\psi_1\left(\frac{2}{6}\right)=\frac{1}{5}\psi_1\left(\frac{1}{6}\right)+\frac{4\pi^2}{15}$$

Result:

$$I=\frac{\pi}{30}\left[\psi_1\left(\frac{1}{6}\right)-2\pi^2\right]$$

UP.634 Calculate the integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\operatorname{arccot}x\,(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}\,dx$$

In this problem we will consider that definition of the function $\operatorname{arccot}(x)$ which has the image the interval $(0,\pi)$.

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

We use the following definition of Euler's Beta function

$$B(p,q)=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2p-1}\,(\cos\theta)^{2q-1}d\theta \qquad (1)$$

where $p>0, q>0$. Or, equivalently:

$$B(p,q)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(|\sin\theta|)^{2p-1}\,(\cos\theta)^{2q-1}d\theta \qquad (2)$$

We will demonstrate that relations (1) and (2) are equivalent.

We have:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(|\sin\theta|)^{2p-1}(\cos\theta)^{2q-1}d\theta=\int_{-\frac{\pi}{2}}^{0}(|\sin\theta|)^{2p-1}(\cos\theta)^{2q-1}d\theta+$$

$$+\int_{0}^{\frac{\pi}{2}}(|\sin\theta|)^{2p-1}(\cos\theta)^{2q-1}d\theta=B_1+B_2$$

$$B_1=\int_{-\frac{\pi}{2}}^{0}(|\sin\theta|)^{2p-1}(\cos\theta)^{2q-1}d\theta$$

Let us denote: $\theta=-t, t>0$

$$B_1=\int_{\frac{\pi}{2}}^{0}(\sin t)^{2p-1}(\cos t)^{2q-1}(-dt)=\int_{0}^{\frac{\pi}{2}}(\sin t)^{2p-1}(\cos t)^{2q-1}dt$$

$$B_2=\int_{0}^{\frac{\pi}{2}}(|\sin\theta|)^{2p-1}(\cos\theta)^{2q-1}d\theta=\int_{0}^{\frac{\pi}{2}}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta$$

I have demonstrated the statement above.

We consider: $2p-1=\frac{1}{3}; 2q-1=\frac{5}{3}$, so: $p=\frac{2}{3}, q=\frac{4}{3}$

Let us denote:

$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\operatorname{arccot}x\,(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx,\qquad B=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\arctan x\,(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx$$

We have:

$$A+B=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(|\sin x|)^{\frac{1}{3}}(\cos x)^{\frac{5}{3}}dx$$

But $B=0$ because the function to be integrated is odd.

So:

$$A=\frac{\pi}{2}B\left(\frac{2}{3},\frac{4}{3}\right)$$

We use the known relationship:

$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where $\Gamma(a)$ is the Euler's Gamma function.

$$B\left(\frac{2}{3},\frac{4}{3}\right)=\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$$

We will use the known relations:

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha);\ \ \Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha};\Gamma(2)=1$$

We obtain:

$$B\left(\frac{2}{3},\frac{4}{3}\right)=\frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1}=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi\sqrt{3}}{9}$$

But:

$$A=\frac{\pi}{2}B\left(\frac{2}{3},\frac{4}{3}\right),\qquad A=\frac{\pi^2\sqrt{3}}{9}$$

We obtained the value of the integral A: Thus, the problem is solved.

UP.635 Prove that:

$$\int_0^{\frac{\pi}{2}}(\cos x)^2e^{\cos(\tan(x))}\cos(\sin(\tan(x)))\,dx=\frac{\pi(e+1)}{4e}\sqrt{e}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^{\frac{\pi}{2}}(\cos x)^2e^{\cos(\tan(x))}\cos(\sin(\tan(x)))\,dx\overset{\tan(x)\to x}{=}\int_0^{\infty}\frac{1}{1+x^2}.e^{\cos(x)}.\cos(\sin(x)).\frac{dx}{1+x^2}$$

$$\Rightarrow\int_0^{\infty}\frac{e^{\cos(x)}.\cos(\sin(x))}{(1+x^2)^2}dx=\Re\left(\int_0^{\infty}\frac{e^{\cos(x)}e^{i\sin(x)}}{(1+x^2)^2}dx\right)=\Re\left(\int_0^{\infty}\frac{e^{e^{ix}}}{(1+x^2)^2}dx\right)$$

$$=\Re\left(\int_0^{\infty}\frac{1}{(1+x^2)^2}\left(\sum_{n=0}^{\infty}\frac{(e^{ix})^n}{n!}\right)dx\right)$$

$$\Rightarrow\int_0^{\infty}\frac{1}{(1+x^2)^2}\left(\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}\right)dx=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx$$

$$f(k)=\int_0^{\infty}\frac{\cos(nx)}{k^2+x^2}dx=\frac{\pi}{2k}e^{-kx}$$

$$f'(k)=\int_0^{\infty}\frac{-2k\cos(nx)}{(k^2+x^2)^2}dx=\frac{\pi}{2}\left(-\frac{1}{k^2e^{kn}}-\frac{n}{ke^{kn}}\right)$$

$$\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx=\frac{\pi}{4}(1+n)e^{-n}$$

$$\Rightarrow\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos(nx)}{(1+x^2)^2}dx=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\pi}{4}(1+n)e^{-n}\right]=\frac{\pi}{4}\sum_{n=0}^{\infty}\left[\frac{n+1}{n!}(e^{-1})^n\right]$$

$$\Rightarrow\frac{\pi}{4}\sum_{n=0}^{\infty}\left[\frac{n+1}{n!}(e^{-1})^n\right]=\frac{\pi}{4}\left[\sum_{n=0}^{\infty}\frac{(e^{-1})^n}{n!}+e^{-1}\sum_{n=1}^{\infty}\frac{(e^{-1})^{n-1}}{(n-1)!}\right]=$$

$$=\frac{\pi}{4}\left[e^{e^{-1}}+e^{-1}e^{e^{-1}}\right]=\frac{\pi(e+1)}{4e}\sqrt{e}$$

UP.636 Solve the linear differential equation

$$y^{IV}-5y'''+10y''-10y'+4y=x^4-5x^3+10x^2-10x+4.$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

We first solve the associated homogeneous equation

$$y^{IV}-5y'''+10y''-10y'+4y=0.$$

Its characteristic polynomial is $r^4-5r^3+10r^2-10r+4=0$.

This polynomial equation factors as $(r-1)(r-2)(r-1-i)(r-1+i)=0$,

and has the roots $r_1=1, r_2=2, r_{3,4}=1\pm i$. We have two distinct real roots and one pair of complex conjugate roots (with real part $\alpha=1$ and imaginary part $\beta=1$). Then the homogeneous solution $y_h(x)$ is given by

$$y_h(x)=C_1e^x+C_2e^{2x}+e^x(C_3\cos x+C_4\sin x),$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Now, we find a particular solution. The right-hand side is a polynomial of degree 4:

$$Q(x)=x^4-5x^3+10x^2-10x+4.$$

Since $r=0$ is not a root of the characteristic equation (the constant term is $4\neq 0$), we look for a particular solution of the same polynomial form

$$y_p(x)=Ax^4+Bx^3+Cx^2+Dx+E.$$

Computing its derivatives, we obtain

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D, y_p'' = 12Ax^2 + 6Bx + 2C,$$

$$y_p''' = 24Ax + 6B, y_p^{IV} = 24A.$$

Substituting into $y^{IV} - 5y''' + 10y'' - 10y' + 4y$, and collecting terms by powers of x, we obtain

$$4Ax^4 + (-40A + 4B)x^3 + (120A - 30B + 4C)x^2$$
$$+(-120A + 60B - 20C + 4D)x + (24A - 30B + 20C - 10D + 4E).$$

Equating with $x^4 - 5x^3 + 10x^2 - 10x + 4$, we have

$$\begin{aligned} x^4: 4A &= 1 \\ x^3: -40A + 4B &= -5 \\ x^2: 120A - 30B + 4C &= 10 \\ x^1: -120A + 60B - 20C + 4D &= -10 \\ x^0: 24A - 30B + 20C - 10D + 4E &= 4 \end{aligned}$$

From the first equation, we get $A = \frac{1}{4}$. Substituting into the second we have

$-10 + 4B = -5 \Rightarrow B = \frac{5}{4}$.

Substituting A, B into the third yields $30 - \frac{75}{2} + 4C = 10 \Rightarrow C = \frac{35}{8}$.

Substituting A, B, C into the fourth:

$$-30 + 75 - \frac{175}{2} + 4D = -10 \Rightarrow D = \frac{65}{8}$$

Substituting A, B, C, D into the fifth:

$$6 - \frac{75}{2} + \frac{175}{2} - \frac{325}{4} + 4E = 4 \Rightarrow E = \frac{117}{16}.$$

Thus, a particular solution is

$$y_p(x) = \frac{1}{4}x^4 + \frac{5}{4}x^3 + \frac{35}{8}x^2 + \frac{65}{8}x + \frac{117}{16}.$$

Combining the homogeneous and particular solutions, $y = y_h + y_p$, we obtain the general solution

$$y(x) = C_1e^x + C_2e^{2x} + e^x(C_3\cos x + C_4\sin x)$$

$$+\frac{1}{4}x^4+\frac{5}{4}x^3+\frac{35}{8}x^2+\frac{65}{8}x+\frac{117}{16}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants, determined by initial or boundary conditions if specified.

UP.637 Find the general solution of the system

$$x' = x + 2y + 3z,$$

$$y' = 3x + y + 2z,$$

$$z' = 2x + 3y + z.$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Writing $X = (x, y, z)^T$, the system is $X' = AX$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Notice that A is a circulant matrix, each row is a cyclic shift of the previous one. This structure will simplify finding eigenvalues and eigenvectors. Its characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0.$$

Expanding along the first row, we have

$$\det(A - \lambda I) = (1-\lambda)[(1-\lambda)^2 - 6] - 2[3(1-\lambda) - 4] + 3[9 - 2(1-\lambda)]$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 5) - 2(-3\lambda - 1) + 3(2\lambda + 7).$$

Expanding $(1-\lambda)(\lambda^2 - 2\lambda - 5) = -\lambda^3 + 3\lambda^2 + 3\lambda - 5$, we get

$$\det(A - \lambda I) = -\lambda^3 + 3\lambda^2 + 3\lambda - 5 + 6\lambda + 2 + 6\lambda + 21 = -\lambda^3 + 3\lambda^2 + 15\lambda + 18.$$

Setting this to zero and multiplying by -1, we obtain

$$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0.$$

Factoring, yields

$$\lambda^3 - 3\lambda^2 - 15\lambda - 18 = (\lambda - 6)(\lambda^2 + 3\lambda + 3),$$

which roots are

$$\lambda_1 = 6,\ \ \lambda_{2,3} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

Now, we compute the corresponding eigenvectors:

1. Eigenvector for $\lambda_1 = 6$. Since every row of A sums to 6, the vector $(1,1,1)^T$ satisfies $A(1,1,1)^T = 6(1,1,1)^T$, and $v_1 = (1,1,1)^T$ is the eigenvector for $\lambda_1 = 6$.

2. Eigenvector for $\lambda = -\frac{3}{2} + i\frac{\sqrt{3}}{2}$. Using the circulant structure, with $\omega = e^{\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the eigenvector associated with this eigenvalue is

$$v = (1, \omega^2, \omega) = \left(1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = p + i,$$

where

$$p = \left(1, -\frac{1}{2}, -\frac{1}{2}\right),\ \ q = \left(0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right),$$

It can be easily checked that $Av = \lambda v$.

Now, we write the real solutions from the complex eigen pair.

For a complex eigenvalue $\lambda = \alpha + i\beta$ with eigenvector $v = p + iq$, two independent real solutions are given by

$$u_1(t) = e^{\alpha t}[p\cos\beta t - q\sin\beta t], \qquad u_2(t) = e^{\alpha t}[p\sin\beta t + q\cos\beta t].$$

Here $\alpha = -\frac{3}{2}, \beta = \frac{\sqrt{3}}{2}$. Define

$$P(t) = e^{-\frac{3t}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}t + C_3\sin\frac{\sqrt{3}}{2}t\right),$$

$$Q(t) = e^{-\frac{3t}{2}}\left(C_2\sin\frac{\sqrt{3}}{2}t - C_3\cos\frac{\sqrt{3}}{2}t\right),$$

so that the contribution $C_2u_1 + C_2u_2$ has

- First component $(p_x = 1, q_x = 0)$: equal to $P(t)$.
- Second component $\left(p_y = -\frac{1}{2}, q_y = -\frac{\sqrt{3}}{2}\right)$:

$$-\frac{1}{2}(C_2\cos\beta t + C_3\sin\beta t) + \frac{\sqrt{3}}{2}(C_2\sin\beta T - C_3\cos\beta t)\cdot e^{-\frac{3t}{2}} = -\frac{1}{2}P(t) + \frac{\sqrt{3}}{2}Q(t).$$

- Third component $\left(p_z = -\frac{1}{2}, q_z = \frac{\sqrt{3}}{2}\right)$: $-\frac{1}{2}P(t) - \frac{\sqrt{3}}{2}Q(t)$.

Combining the solution from $\lambda_1 = 6$ (along $v_1 = (1,1,1)^T$) with the two real solutions above, we obtain the general solution:

$$x(t) = C_1 e^{6t} + P(t),$$

$$y(t) = C_1 e^{6t} - \frac{1}{2}P(t) + \frac{\sqrt{3}}{2}Q(t),$$

$$z(t) = C_1 e^{6t} - \frac{1}{2}P(t) - \frac{\sqrt{3}}{2}Q(t),$$

where

$$P(t) = e^{-\frac{3t}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}t + C_3 \sin\frac{\sqrt{3}}{2}t\right),$$

$$Q(t) = e^{-\frac{3t}{2}}\left(C_2 \sin\frac{\sqrt{3}}{2}t - C_3 \cos\frac{\sqrt{3}}{2}t\right),$$

and $C_1, C_2, C_3 \in \mathbb{R}$ are arbitrary constants.

Remark. Adding the three equations gives $x + y + z = 3C_1 e^{6t}$.

The sum grows (or decays) purely along the eigen direction $(1,1,1)^T$, reflecting the symmetry of the circulant matrix A. The components $y(t)$ and $z(t)$ are mirror images about the shared term $-\frac{1}{2}P(t)$, a direct consequence of the same cyclic symmetry.

UP.638 Compute the square root of the following matrix

$$X = \frac{1}{14}\begin{pmatrix} 41 & 23 & -25 & 1 \\ 19 & 46 & -29 & 2 \\ 21 & -7 & 7 & 7 \\ 17 & -6 & -23 & 46 \end{pmatrix}$$

Proposed by José Luis Díaz – Barrero – Spain

Solution 1 by proposer

We are given

$$X=\frac{1}{14}\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix}=\begin{pmatrix}\frac{41}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{23}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{23}{7}\end{pmatrix}$$

and we want a matrix Y such that $Y^2=X$. We proceed using the standard diagonalization approach. We carry out the following steps:

1. Computing the characteristic polynomial and eigenvalues:

Computing $\det(X-\lambda I)$ (via cofactor expansion or row reduction of the symbolic matrix $X-\lambda I$) yields factored characteristic polynomial

$$\det(X-\lambda I)=(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

Hence the eigenvalues of X are

$$\lambda_1=1,\lambda_2=2,\lambda_3=3,\lambda_4=4.$$

All four eigenvalues are real, distinct, and strictly positive. Since X is a 4×4 matrix with 4 distinct eigenvalues, it is automatically diagonalizable. Because the spectrum is positive, a real square root of X exists.

2. Computing the eigenvectors: For each eigenvalue λ, we solve the homogeneous system $(x-\lambda I)v=0$, i.e. find $\ker(X-\lambda I)$. In every case the coefficient matrix has rank 3, confirming a one-dimensional eigenspace, consistent with the eigenvalues being simple.

$$\lambda=1:$$

$$X-I=\begin{pmatrix}\frac{27}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{16}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{16}{7}\end{pmatrix},\ker(X-I)=span\{(1,2,3,2)^T\}.$$

$$\lambda=2:$$

$$X-2I=\begin{pmatrix}\frac{13}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{9}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{9}{7}\end{pmatrix},\ker(X-2I)=span\{(2,1,2,1)^T\}.$$

$\lambda=3$:

$$X-3I=\begin{pmatrix}-\frac{1}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & \frac{2}{7} & \frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{5}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & \frac{2}{7}\end{pmatrix},\ker(X-3I)=span\{(1,1,1,3)^T\}.$$

$\lambda=4$:

$$X-4I=\begin{pmatrix}-\frac{15}{14} & \frac{23}{14} & -\frac{25}{14} & \frac{1}{14}\\ \frac{19}{14} & -\frac{5}{7} & -\frac{29}{14} & \frac{1}{7}\\ \frac{3}{2} & -\frac{1}{2} & -\frac{7}{2} & \frac{1}{2}\\ \frac{17}{14} & -\frac{3}{7} & -\frac{23}{14} & -\frac{5}{7}\end{pmatrix},\ker(X-4I)=span\{(3,3,1,1)^T\}.$$

3. Assemble the eigenvector matrix P and invert it: Placing the eigenvectors as columns, in the order $\lambda=1,2,3,4$, we have

$$P=\begin{pmatrix}1 & 2 & 1 & 3\\ 2 & 1 & 1 & 3\\ 3 & 2 & 1 & 1\\ 2 & 1 & 3 & 1\end{pmatrix}.$$

One computes $\det(P)=-28\neq 0$, confirming P is invertible (as it must be, since the eigenvectors of a matrix with distinct eigenvalues are automatically linearly independent). Its inverse is

$$P^{-1}=\frac{1}{14}\begin{pmatrix}-6 & 5 & 4 & -1\\ 8 & -9 & 4 & -1\\ 1 & -2 & -3 & 6\\ 1 & 5 & -3 & -1\end{pmatrix}=\begin{pmatrix}-\frac{3}{7} & \frac{5}{14} & \frac{2}{7} & -\frac{1}{14}\\ \frac{4}{7} & -\frac{9}{14} & \frac{2}{7} & -\frac{1}{14}\\ \frac{1}{14} & -\frac{1}{7} & -\frac{3}{14} & \frac{3}{7}\\ \frac{1}{14} & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{14}\end{pmatrix}.$$

By construction,

$$X=P\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix}P^{-1}.$$

(This can be verified directly by multiplying out the right-hand side and checking it matches the given X.)

4. Take the square root of the diagonal part: Since every eigenvalue is positive, we may take positive square roots:

$$D^{\frac{1}{2}}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & 2\end{pmatrix}.$$

5. Assemble the square root of X: Define

$$Y=P\,D^{\frac{1}{2}}P^{-1}.$$

Then

$$Y^2=PD^{\frac{1}{2}}\underbrace{P^{-1}P}_{=I}D^{\frac{1}{2}}P^{-1}=P\,D^{\frac{1}{2}}D^{\frac{1}{2}}P^{-1}=PDP^{-1}=X,$$

so $Y=X^{\frac{1}{2}}$. Carrying out the multiplication $PD^{\frac{1}{2}}P^{-1}$ symbolically gives the exact result

$$X^{\frac{1}{2}}=\begin{bmatrix} \frac{\sqrt{3}}{14}+\frac{8\sqrt{2}}{7} & -\frac{9\sqrt{2}}{7}-\frac{\sqrt{3}}{7}+\frac{5}{2} & -1-\frac{3\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{1}{2}-\frac{\sqrt{2}}{7}+\frac{3\sqrt{3}}{7} \\ -\frac{3}{7}+\frac{\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{9\sqrt{2}}{14}-\frac{\sqrt{3}}{7}+\frac{20}{7} & -\frac{5}{7}-\frac{3\sqrt{3}}{14}+\frac{2\sqrt{2}}{7} & -\frac{4}{7}-\frac{\sqrt{2}}{14}+\frac{3\sqrt{3}}{7} \\ -\frac{8}{7}+\frac{\sqrt{3}}{14}+\frac{8\sqrt{2}}{7} & -\frac{9\sqrt{2}}{7}-\frac{\sqrt{3}}{7}+\frac{24}{14} & -\frac{3\sqrt{3}}{14}+\frac{3}{7}+\frac{4\sqrt{2}}{7} & -\frac{5}{14}-\frac{\sqrt{2}}{7}+\frac{3\sqrt{3}}{7} \\ -\frac{5}{7}+\frac{3\sqrt{3}}{14}+\frac{4\sqrt{2}}{7} & -\frac{9\sqrt{2}}{14}-\frac{3\sqrt{3}}{7}+\frac{10}{7} & -\frac{9\sqrt{3}}{14}+\frac{1}{7}+\frac{2\sqrt{2}}{7} & -\frac{2}{7}-\frac{\sqrt{2}}{14}+\frac{9\sqrt{3}}{7} \end{bmatrix}.$$

It can be approximated numerically, to obtain

$$X^{\frac{1}{2}}\approx\begin{pmatrix} 1.73996 & 0.43429 & -0.56303 & 0.04028 \\ 0.50327 & 1.70057 & -0.68138 & 0.06986 \\ 0.59711 & -0.28000 & 0.86554 & 0.18313 \\ 0.46499 & -0.22287 & -0.56654 & 1.84019 \end{pmatrix}.$$

1. Summary

Given only the numerical matrix X, the procedure is:

1. **Compute the characteristic polynomial and its roots (the eigenvalues).**
2. **If all eigenvalues are real and positive (and X is diagonalizable), a real square root exist.**
3. **Find an eigenvector for each eigenvalue and assemble them as columns of P.**
4. **Form $D^{\frac{1}{2}}$ by taking positive square roots of the eigenvalues.**
5. **The square root is $X^{\frac{1}{2}}=PD^{\frac{1}{2}}P^{-1}$.**

This is precisely the general diagonalization method for matrix square roots, applicable whenever the matrix is diagonalizable with a non-negative spectrum.

Solution 2 by Hikmat Mammadov-Azerbaijan

$$X=\begin{pmatrix} 41 & 23 & -25 & 1 \\ 19 & 46 & -29 & 2 \\ 21 & -7 & 7 & 7 \\ 17 & -6 & -23 & 46 \end{pmatrix}$$

$$A-14X=\begin{pmatrix} 41 & 23 & -25 & 1 \\ 19 & 46 & 29 & 2 \\ 21 & -7 & 7 & 7 \\ 17 & -6 & -23 & 46 \end{pmatrix}$$

$$\det(A-\lambda I)=\begin{vmatrix}41-\lambda & 23 & -25 & 1\\ 19 & 46-\lambda & -29 & 2\\ 21 & 7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow R_1\to R_1-R_4,\qquad R_2\to R_2-R_3$$

$$\det(A-\lambda I)=\begin{vmatrix}24-\lambda & 29 & -2 & -(45-\lambda)\\ -2 & 53-\lambda & -(36-\lambda) & -5\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow tr(A)=41+46+7+46=140$$

$$\Rightarrow tr(A^2)=1610+2744-434+1960=5880$$

$$\Rightarrow P_A(\lambda)=(\lambda-14)(\lambda-28)(\lambda-42)(\lambda-56)=0$$

$$\Rightarrow \sigma(A)=\{14,28,42,56\}\Rightarrow \sigma(X)=\left\{\frac{14}{14},\frac{28}{14},\frac{42}{14},\frac{56}{14}\right\}=\{1,2,3,4\}$$

$$Y^2=X\Rightarrow \sigma(Y)=\{\sqrt{1},\sqrt{2},\sqrt{3},\sqrt{4}\}=\{1,\sqrt{2},\sqrt{3},2\}$$

$$p(\lambda)=a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0$$

$$\begin{cases}p(1)=a_3(1)^3+a_2(1)^2+a_1(1)+a_0=1\\ p(2)=a_3(2)^3+a_2(2)^2+a_1(2)+a_0=\sqrt{2}\\ p(3)=a_3(3)^3+a_2(3)^2+a_1(3)+a_0=\sqrt{3}\\ p(4)=a_3(4)^3+a_2(4)^2+a_1(4)+a_0=2\end{cases}\Rightarrow$$

$$\Rightarrow\begin{pmatrix}1 & 1 & 1 & 1\\ 8 & 4 & 2 & 1\\ 27 & 9 & 3 & 1\\ 64 & 16 & 4 & 1\end{pmatrix}\begin{pmatrix}a_3\\ a_2\\ a_1\\ a_0\end{pmatrix}-\begin{pmatrix}1\\ \sqrt{2}\\ \sqrt{3}\\ 2\end{pmatrix}$$

$$L_1(\lambda)=\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{(1-2)(1-3)(1-4)}=-\frac{1}{6}(\lambda^3-9\lambda^2+26\lambda-24)$$

$$L_2(\lambda)=\frac{(\lambda-1)(\lambda-3)(\lambda-4)}{(2-1)(2-3)(2-4)}=\frac{1}{2}(\lambda^3-8\lambda^2+19\lambda-12)$$

$$L_3(\lambda)=\frac{(\lambda-1)(\lambda-2)(\lambda-4)}{(3-1)(3-2)(3-4)}=-\frac{1}{2}(\lambda^3-7\lambda^2+14\lambda-8)$$

$$L_4(\lambda)=\frac{(\lambda-1)(\lambda-2)(\lambda-3)}{(4-1)(4-2)(4-3)}=\frac{1}{6}(\lambda^3-6\lambda^2+11-6)$$

$$a_3=\frac{1+3\sqrt{2}-3\sqrt{3}}{6},\quad a_2=\frac{-1-8\sqrt{2}+7\sqrt{3}}{2}$$

$$a_1=\frac{-2+57\sqrt{2}-42\sqrt{3}}{6},\ \ a_0=2-6\sqrt{2}+4\sqrt{3}$$

$$\Rightarrow \sqrt{X}=a_3X^3+a_2X^2+a_1X+a_0I$$

Solution 3 by Hikmat Mammadov-Azerbaijana

$$X=\frac{1}{14}\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix},\ \ A=14X=\begin{pmatrix}41 & 23 & -25 & 1\\ 19 & 46 & -29 & 2\\ 21 & -7 & 7 & 7\\ 17 & -6 & -23 & 46\end{pmatrix}$$

$$\det(A-\lambda I)=\begin{vmatrix}41-\lambda & 23 & -25 & 1\\ 19 & 46-\lambda & -29 & 2\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow R_1\to R_1-R_4,\ \ R_2\to R_2-R_3$$

$$\det(A-\lambda I)=\begin{vmatrix}24-\lambda & 29 & -2 & -(45-\lambda)\\ -2 & 53-\lambda & -(36-\lambda) & -5\\ 21 & -7 & 7-\lambda & 7\\ 17 & -6 & -23 & 46-\lambda\end{vmatrix}=0$$

$$\Rightarrow tr(A)=41+46+7+46=140$$

$$\Rightarrow tr\,(A^2)=1610+2744-434+1960=5880$$

$$\Rightarrow P_A(\lambda)=(\lambda-14)(\lambda-28)(\lambda-42)(\lambda-56)=0$$

$$\Rightarrow \sigma(A)=\{14,28,42,56\}\Rightarrow \sigma(X)=\{\mu_1,\mu_2,\mu_3,\mu_4\}=\{1,2,3,4\}$$

$$Y^2=X\Rightarrow \sigma(Y)=\{\sqrt{1},\sqrt{2},\sqrt{3},\sqrt{4}\}=\{1,\sqrt{2},\sqrt{3},2\}$$

$$\Rightarrow Y=\sqrt{x}=\sum_{k=1}^{4}\sqrt{\mu_k}\,P_k=1\cdot P_1+\sqrt{2}\cdot P_2+\sqrt{3}\cdot P_3+2\cdot P_4$$

$$P_k=\prod_{j\neq k}\frac{X-\mu_j I}{\mu_k-\mu_j}\ \ (k=1,2,3,4)$$

$$P_1=\frac{(X-2I)(X-3I)(X-4I)}{(1-2)(1-3)(1-4)}=-\frac{1}{6}(X^3-9X^2+26X-24I)$$

$$P_2=\frac{(X-I)(X-3I)(X-4I)}{(2-1)(2-3)(2-4)}=\frac{1}{2}(X^3-8X^2+19X-12I)$$

$$P_3=\frac{(X-I)(X-2I)(X-4I)}{(3-1)(3-2)(3-4)}=-\frac{1}{2}(X^3-7X^2+14X-8I)$$

$$P_4=\frac{(X-I)(X-2I)(X-3I)}{(4-1)(4-2)(4-3)}=\frac{1}{6}(X^3-6X^2+11X-6I)$$

$$\Rightarrow\sqrt{X}=1\cdot P_1+\sqrt{2}\cdot P_2+\sqrt{3}\cdot P_3+2\cdot P_4$$

$$a_3=-\frac{1}{6}(1)+\frac{1}{2}(\sqrt{2})-\frac{1}{2}(\sqrt{3})+\frac{1}{6}(2)=\frac{1+3\sqrt{2}-3\sqrt{3}}{6}$$

$$a_2=-\frac{1}{6}(-9)+\frac{1}{2}(-8\sqrt{2})-\frac{1}{2}(-7\sqrt{3})+\frac{1}{6}(-12)=\frac{-1-8\sqrt{2}+7\sqrt{3}}{2}$$

$$a_1=-\frac{1}{6}(26)+\frac{1}{2}(19\sqrt{2})-\frac{1}{2}(14\sqrt{3})+\frac{1}{6}(22)=\frac{-2+57\sqrt{2}-42\sqrt{3}}{6}$$

$$a_0=-\frac{1}{6}(-24)+\frac{1}{2}(-12\sqrt{2})-\frac{1}{2}(-8\sqrt{3})+\frac{1}{6}(-12)=2-6\sqrt{2}+4\sqrt{3}$$

$$\Rightarrow\sqrt{X}=a_3X^3+a_2X^2+a_1X$$

UP.639 **Find:**

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)\right]^{-1}$$

Proposed by Omkumar Rao-India

Solution 1 by proposer

$$\sum_{i=1}^{k}\left(i^2+\frac{i}{3}-\frac{1}{12}\right)=\frac{k(k+1)(2k+1)}{6}+\frac{k(k+1)}{6}-\frac{k}{12}=\frac{k(2k+1)(2k+3)}{12}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{12}{k(2k+1)(2k+3)}=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4}{k}-\frac{12}{2k+1}+\frac{4}{2k+3}\right)=4\lim_{n\to\infty}H_n-12\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+1}+4\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+3}}_{\sum_{k=2}^{n+1}\frac{1}{2k+1}}$$

$$\Omega=4\lim_{n\to\infty}H_n-12\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k+1}+4\lim_{n\to\infty}\sum_{k=2}^{n+1}\frac{1}{2k+1}$$

$$=4\lim_{n\to\infty}H_n-\frac{12}{3}+\frac{4}{2n+3}-8\lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{2k+1}$$

$$\Omega = 4\lim_{n\to\infty} H_n - \frac{12}{3} + \lim_{n\to\infty}\frac{4}{2n+3} - 8\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{2k+1} - \frac{1}{3}\right)$$

$$= 4\lim_{n\to\infty} H_n - \frac{4}{3} + \underbrace{\lim_{n\to\infty}\frac{4}{2n+3}}_{0} - 8\lim_{n\to\infty}\underbrace{\sum_{k=1}^{n}\frac{1}{2k+1}}_{H_{2n+1}-1-\frac{1}{2}H_n}$$

$$\Omega = 4\lim_{n\to\infty} H_n - \frac{4}{3} - 8\lim_{n\to\infty}\left(H_{2n+1} - 1 - \frac{1}{2}H_n\right) = 8\lim_{n\to\infty} H_n - 8\lim_{n\to\infty} H_{2n+1} + \frac{20}{3}$$

$$\Omega \overset{H_n=\ln(n)+\gamma+O\left(\frac{1}{n}\right)}{=} 8\lim_{n\to\infty}\left(\left(\ln(n)+\gamma+O\left(\frac{1}{n}\right)\right) - \left(\ln(2n+1)+\gamma+O\left(\frac{1}{2n+1}\right)\right)\right) + \frac{20}{3}$$

$$\Omega = 8\lim_{n\to\infty}\ln\left(\frac{n}{2n+1}\right) + \frac{20}{3} = \frac{20}{3} - 8\ln 2$$

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i^2 + \frac{i}{3} - \frac{1}{12}\right)\right]^{-1} = \frac{20}{3} - 8\ln 2$$

Solution 2 by Jose Luis Diaz Barrero-Spain

First we compute the inner sum:

$$\sum_{i=1}^{k}\left(i^2 + \frac{i}{3} - \frac{1}{12}\right) = \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{6} - \frac{k}{12}$$

$$= \frac{k(4k^2+8k+3)}{12} = \frac{k(2k+1)(2k+3)}{12}.$$

Decomposing its inverse into partial fractions, we get

$$\frac{12}{k(2k+1)(2k+3)} = \frac{4}{k} - \frac{12}{2k+1} + \frac{4}{2k+3} = 8\left(\frac{1}{2k} - \frac{1}{2k+1}\right) - 4\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$$

Since

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{2k+1}\right) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = 1 - \ln 2$$

and

$$\sum_{k=1}^{\infty}\left(\frac{1}{2k+1}-\frac{1}{2k+3}\right)=\frac{1}{3}$$

(telescoping), it follows that

$$\Omega = 8(1-\ln 2)-\frac{4}{3}=\frac{20}{3}-8\ln 2.$$

UP.640 Compute the following sum

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n-x^{n+1}+x^{2n+1}}$$

Proposed by José Luis Díaz – Barrero – Spain

Solution by proposer

Notice that

$$1-x^n-x^{n+1}+x^{2n+1}=(1-x^n)-x^{n+1}(1-x^n)=(1-x^n)(1-x^{n+1}).$$

So each term of the series becomes

$$\frac{x^n}{(1-x^n)(1-x^{n+1})}.$$

Consider the difference

$$\frac{1}{1-x^n}-\frac{1}{1-x^{n+1}}=\frac{(1-x^{n+1})-(1-x^n)}{(1-x^n)(1-x^{n+1})}=\frac{x^n-x^{n+1}}{(1-x^n)(1-x^{n+1})}=\frac{x^n(1-x)}{(1-x^n)(1-x^{n+1})}.$$

Dividing both sides by $(1-x)$ (assuming $x\neq 1$), we have

$$\frac{x^n}{(1-x^n)(1-x^{n+1})}=\frac{1}{1-x}\left(\frac{1}{1-x^n}-\frac{1}{1-x^{n+1}}\right).$$

So the general term is exactly a telescoping difference.

Next, we sum the telescoping series:

$$S_N=\sum_{n=1}^{N}\frac{x^n}{(1-x^n)(1-x^{n+1})}=\frac{1}{1-x}\sum_{n=1}^{N}\left(\frac{1}{1-x^n}-\frac{1}{1-x^{n+1}}\right)$$

$$=\frac{1}{1-x}\left(\frac{1}{1-x}-\frac{1}{1-x^{N+1}}\right).$$

For $|x|<1$, we have $x^{N+1}\to 0$, so $\frac{1}{1-x^{N+1}}\to 1$. Therefore

$$S_\infty = \frac{1}{1-x}\left(\frac{1}{1-x}-1\right) = \frac{1}{1-x}\cdot\frac{x}{1-x} = \frac{x}{(1-x)^2}.$$

$$\sum_{n=1}^{\infty}\frac{x^n}{1-x^n-x^{n+1}+x^{2n+1}} = \frac{x}{(1-x)^2},\quad |x|<1.$$

UP.641 **Find the line integral of the total differentials:**

$$\Omega = \int_{(3,4,5)}^{(6,8,10)} \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz = dU$$

$$\left.\begin{aligned}\frac{\partial U}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\\ \frac{\partial U}{\partial y} &= \frac{y}{\sqrt{x^2+y^2+z^2}}\\ \frac{\partial U}{\partial z} &= \frac{z}{\sqrt{x^2+y^2+z^2}}\end{aligned}\right\} \Rightarrow U(x,y,z) = \sqrt{x^2+y^2+z^2}$$

$$\Omega = U(6,8,10) - U(3,4,5) =$$

$$= \sqrt{6^2+8^2+10^2} - \sqrt{3^2+4^2+5^2} = \sqrt{200} - \sqrt{50} = 10\sqrt{2} - 5\sqrt{2} = 5\sqrt{2}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Find the integral of the total differential

$$\Omega = \int_{(3,4,5)}^{(6,8,10)} \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}.$$

Let $r = \sqrt{x^2+y^2+z^2}$**. For** $(x,y,z) \neq (0,0,0)$ **we have**

$$\frac{\partial r}{\partial x} = \frac{x}{r},\qquad \frac{\partial r}{\partial y} = \frac{y}{r},\qquad \frac{\partial r}{\partial z} = \frac{z}{r},$$

hence

$$\frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}} = dr,$$

that is, the integrand is the total differential of the potential $F = r$. Since r is of class C^1 on $D = \mathbb{R}^3 \setminus \{(0,0,0)\}$ (it is not differentiable at the origin), the form is exact on D. Consequently, if $\gamma(t), a \le t \le b$ is any piecewise C^1 curve in D joining A to B, the chain rule gives

$$\int_\gamma dr = \int_a^b \nabla r\big(\gamma(t)\big) \cdot \gamma'(t) dt = \int_a^b \frac{d}{dt} r\big(\gamma(t)\big)\, dt = r(B) - r(A),$$

so the integral depends only on the endpoints, provided the path avoids the origin. For instance, the segment $\gamma(t) = t(3,4,5), 1 \le t \le 2$, lies in D, and along it $r = 5\sqrt{2}t$, which gives

$$\Omega = \int_1^2 5\sqrt{2}\, dt = r(2) - r(1) = 5\sqrt{2}.$$

UP.642 Find the line integral of the total differentials:

$$\Omega = \int_{(0,0,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)} 2\cos x\, dx + 3\cos y\ dy + 5\cos z\ dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$2\cos x\ dx + 3\cos y\ dy + 5\cos z\ dz = dU$$

$$\frac{\partial U}{\partial x} = 2\cos x; \frac{\partial U}{\partial y} = 3\cos y; \frac{\partial U}{\partial z} = 5\cos z$$

$$U(x,y,z) = 2\sin x + 3\sin y + 5\sin z$$

$$U\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) = 2\cdot 1 + 3\cdot 1 + 5\cdot 1 = 10$$

$$U(0,0,0) = 0, \qquad \Omega = U\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) - U(0,0,0) = 10 - 0 = 10$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_{(0,0,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)} 2\cos x\, dx + 3\cos y\, dy + 5\cos z\, dz, \qquad C = C_1 \cup C_2 \cup C_3$$

$$P_0(0,0,0) \to P_1\left(\frac{\pi}{2},0,0\right) \to P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right) \to P_3\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$C_1: P_0(0,0,0) \to P_1\left(\frac{\pi}{2},0,0\right), \quad \begin{cases} x \in \left[0,\frac{\pi}{2}\right] \\ y=0 \Rightarrow dy=0 \\ z=0 \Rightarrow dz=0 \end{cases}$$

$$\Omega_1 = \int_0^{\frac{\pi}{2}} 2\cos x\,dx + 3\cos(0) + 5\cos(0)\cdot 0 = \int_0^{\frac{\pi}{2}} 2\cos x\,dx$$

$$= [2\sin x]_0^{\frac{\pi}{2}} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) = 2(1) - 2(0) = 2$$

$$C_2: P_1\left(\frac{\pi}{2},0,0\right) \to P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right), \quad \begin{cases} x=\frac{\pi}{2} \Rightarrow dx=0 \\ y \in \left[0,\frac{\pi}{2}\right] \\ z=0 \Rightarrow dz=0 \end{cases}$$

$$\Omega_2 = \int_0^{\frac{\pi}{2}} 2\cos\left(\frac{\pi}{2}\right)\cdot 0 + 3\cos y\,dy + 5\cos(0)\cdot 0 = \int_0^{\frac{\pi}{2}} 3\cos y\,dy$$

$$= [3\sin y]_0^{\frac{\pi}{2}} = 3\sin\left(\frac{\pi}{2}\right) - 3\sin(0) = 3(1) - 3(0) = 3$$

$$C_3: P_2\left(\frac{\pi}{2},\frac{\pi}{2},0\right) \to P_3\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right), \quad \begin{cases} x=\frac{\pi}{2} \Rightarrow dx=0 \\ y=\frac{\pi}{2} \Rightarrow dy=0 \\ z \in \left[0,\frac{\pi}{2}\right] \end{cases}$$

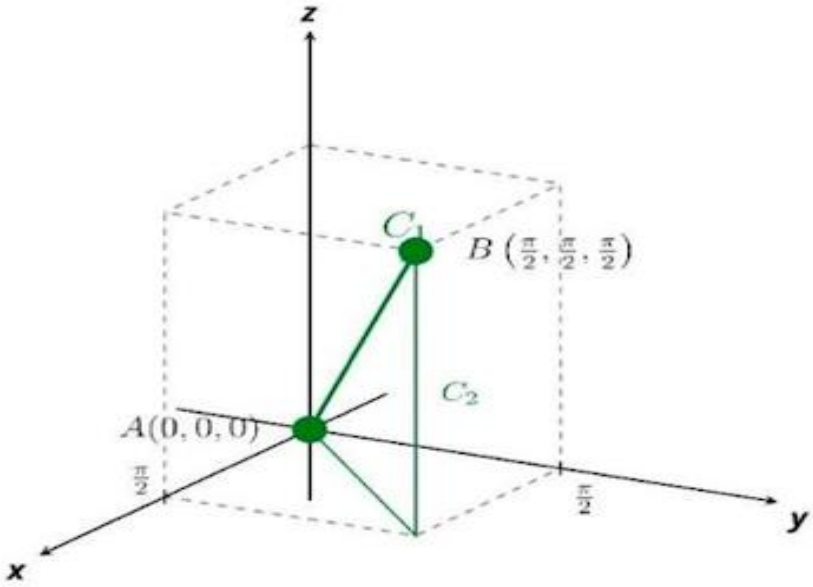
$$\Omega_3 = \int_0^{\frac{\pi}{2}} 2\cos\left(\frac{\pi}{2}\right)\cdot 0 + 3\cos\left(\frac{\pi}{2}\right)\cdot 0 + 5\cos z\,dz = \int_0^{\frac{\pi}{2}} 5\cos z\,dz$$

$$= [5\sin z]_0^{\frac{\pi}{2}} = 5\sin\left(\frac{\pi}{2}\right) - 5\sin(0) = 5(1) - 5(0) = 5$$

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 = 2 + 3 = 5, \qquad \Omega = 10$$

Observation:

$$\nabla \times \vec{F} = \vec{0} \Rightarrow \int_{C_1} \omega = \int_{C_2} \omega = f(B) - f(A)$$



$$\Omega = \int_C df = f(B) - f(A) = 10$$

Solution 3 by Jose Luis Diaz Barrero-Spain

Write the integrand as $P\,dx + Q\,dy + R\,dz$, where

$$P = 2\cos x, \quad Q = 3\cos y, \quad R = 5\cos z.$$

Each coefficient depends only on its own variable, so

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 0.$$

Thus curl $(P, Q, R) = 0$ and the form is closed. Since $\mathbb{R}^3$ is star-shaped, Poincare's lemma ensures that the form is exact, that is $P\,dx + Q\,dy + R\,dz = dF$ for some F of class C^1 on $\mathbb{R}^3$.

Now, we look for F with $F_x = P, F_y = Q, F_z = R$. Integrating $F_x = 2\cos x$ with respect to x,

$$F(x, y, z) = 2\sin x + g(y, z).$$

From $F_y = g_y = 3\cos y$ we get $g(y, z) = 3\sin y + h(z)$, and from $F_z = h'(z) = 5\cos z$ we get $h(z) = 5\sin z + C$. Hence

$$F(x, y, z) = 2\sin x + 3\sin y + 5\sin z + C$$

Let $\gamma(t), a \le t \le b$, be any piecewise C^1 curve joining $A=(0,0,0)$ to $B=\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)$. By the chain rule,

$$\int_\gamma dF = \int_a^b \nabla F(\gamma(t))\cdot\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t))dt = F(B)-F(A),$$

so the integral depends only on the endpoints.

Finally, we have

$$\Omega = F\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right) - F(0,0,0) = (2+3+5)-0 = 10.$$

Along the segment $\gamma(t)=(t,t,t), 0\le t\le\frac{\pi}{2}$, we have $dx=dy=dz=dt$, and

$$\Omega = \int_0^{\frac{\pi}{2}} (2+3+5)\cos t\, dt = 10[\sin t]_0^{\frac{\pi}{2}} = 10.$$

UP.643 Find:

$$\Omega = \int_0^\infty\int_0^\infty\int_0^\infty (2x+3y+5z)e^{-x^2-y^2-z^2}\,dx\,dy\,dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$\Omega = 2\int_0^\infty\int_0^\infty\int_0^\infty xe^{-x^2-y^2-z^2}dx\,dy\,dz + 3\int_0^\infty\int_0^\infty\int_0^\infty ye^{-x^2-y^2-z^2}dx\,dy\,dz +$$

$$+5\int_0^\infty\int_0^\infty\int_0^\infty ze^{-x^2-y^2-z^2}dx\,dy\,dz = 10\int_0^\infty\int_0^\infty\int_0^\infty xe^{-x^2-y^2-z^2}dx\,dy\,dz =$$

$$= 10\left(\int_0^\infty xe^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)\left(\int_0^\infty e^{-z^2}dz\right) = \frac{10}{-2}\int_0^\infty (e^{-x^2})'dx\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2} =$$

$$= \frac{-5\pi}{4}e^{-x^2}\Big|_0^\infty = -\frac{5\pi}{4}(e^{-\infty}-e^0) = -\frac{5\pi}{4}(0-1) = \frac{5\pi}{4}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

Let

$$G=\int_0^\infty e^{-t^2}dt, M=\int_0^\infty te^{-t^2}dt.$$

Using polar coordinates in the first quadrant,

$$G^2=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,dr\,d\theta=\frac{\pi}{2}\cdot\frac{1}{2}=\frac{\pi}{4},$$

so $G=\frac{\sqrt{\pi}}{2}$, as it is well-known. Moreover: $M=\left[-\frac{1}{2}e^{-t^2}\right]_0^\infty=\frac{1}{2}$.

Since $e^{-x^2-y^2-z^2}=e^{-x^2}e^{-y^2}e^{-z^2}$ and the inegrand is nonnegative, by Tonelli's theorem we may integrate term by term and factor each term. For instance,

$$\int_0^\infty\int_0^\infty\int_0^\infty xe^{-x^2}e^{-y^2}e^{-z^2}\,dx\,dy\,dz=MG^2=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8},$$

and, by symmetry, the terms in y and z give the same value. Finally, we have

$$\Omega=(2+3+5)MG^2=10\cdot\frac{\pi}{8}=\frac{5\pi}{4}.$$

UP.644 Let be:

$$D=\{(x,y,z)|x,y,z\in\mathbb{R};0\le x\le 1;0\le y\le x;0\le z\le xy\}$$

Find:

$$\Omega=\iiint_D xyz\,dx\,dy\,dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

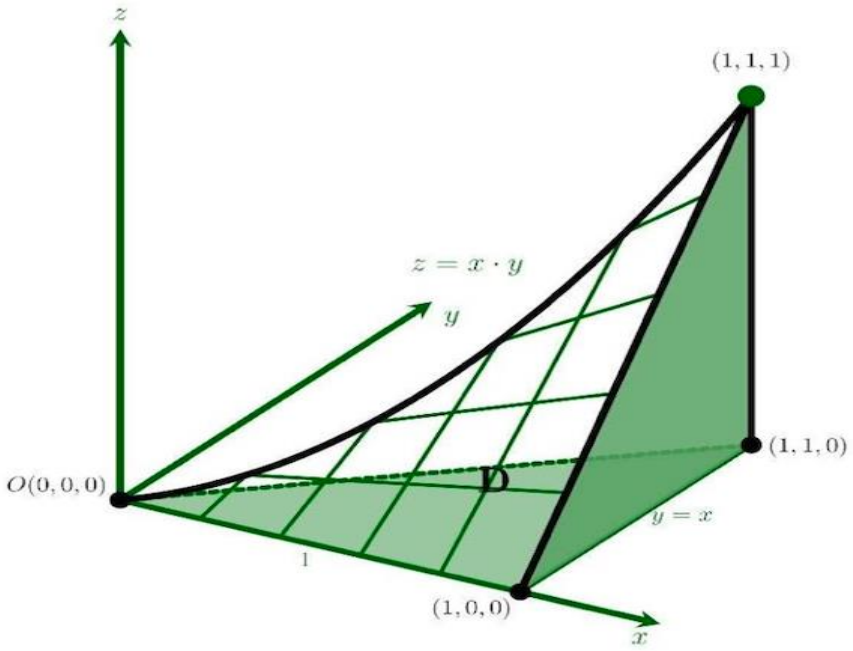
$$\Omega=\iiint_D xyz\,dx\,dy\,dz=\int_0^1\left(\int_0^x\left(\int_0^{xy}xyz\,dz\right)dy\right)dx=$$

$$=\int_0^1\left(\int_0^x\left(xy\cdot\frac{z^2}{2}\Big|_0^{xy}\right)dy\right)dz=\int_0^1\left(\int_0^x\frac{(xy)^3}{x}dy\right)dx=\int_0^1\left(\frac{x^3}{2}\int_0^x y^3\,dy\right)dx=$$

$$=\int_0^1\left(\frac{x^3}{2}\cdot\frac{y^4}{4}\Big|_0^x\right)dx=\int_0^1\left(\frac{x^3}{2}\cdot\frac{x^4}{4}\right)dx=\frac{1}{8}\int_0^1 x^7\,dx=\frac{1}{8}\cdot\frac{x^8}{8}\Big|_0^1=\frac{1}{64}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$D=\{(x,y,z)\in\mathbb{R}^3:0\le x\le 1,0\le y\le x,0\le z\le x\cdot y\}$$



$$\Omega = \int\int\int_D x \cdot y \cdot z \, dx\, dy\, dz, \qquad \Omega = \int_0^1 \left(\int_0^x \left(\int_0^{x \cdot y} x \cdot y \cdot z \, dz \right) dy \right) dx$$

$$\Omega = \int_0^1 x \cdot \left(\int_0^x y \cdot \left(\int_0^{x \cdot y} z \, dz \right) dy \right) dx$$

$$\int_0^{x \cdot y} z \, dz = \left[\frac{z^{1+1}}{1+1} \right]_0^{x \cdot y} = \left[\frac{z^2}{2} \right]_0^{x \cdot y} = \frac{(x \cdot y)^2}{2} - \frac{0^2}{2} = \frac{x^2 \cdot y^2}{2} - 0 = \frac{x^2 \cdot y^2}{2}$$

$$\Omega = \int_0^1 x \cdot \left(\int_0^x y \cdot \left(\frac{x^2 \cdot y^2}{2} \right) dy \right) dx$$

$$\Omega = \int_0^1 x \cdot \left(\int_0^x \frac{x^2 \cdot y^{1+2}}{2} dy \right) dx = \int_0^1 x \cdot \left(\int_0^x \frac{x^2 \cdot y^3}{2} dy \right) dx$$

$$\Omega = \int_0^1 x \cdot \left(\frac{x^2}{2} \cdot \int_0^x y^3 \, dy \right) dx$$

$$\int_0^x y^3 \, dy = \left[\frac{y^{3+1}}{3+1} \right]_0^x = \left[\frac{y^4}{4} \right]_0^x = \frac{x^4}{4} - \frac{0^4}{4} = \frac{x^4}{4} - 0 = \frac{x^4}{4}, \qquad \Omega = \int_0^1 x \cdot \left(\frac{x^2}{2} \cdot \frac{x^4}{4} \right) dx$$

$$\Omega = \int_0^1 x \cdot \left(\frac{x^2 \cdot x^4}{2 \cdot 4} \right) dx = \int_0^1 x \cdot \left(\frac{x^{2+4}}{8} \right) dx = \int_0^1 x \cdot \frac{x^6}{8} dx$$

$$\Omega = \int_0^1 \frac{x^{1+6}}{8} dx = \int_0^1 \frac{x^7}{8} dx = \frac{1}{8} \int_0^1 x^7 \, dx$$

$$\int_0^1 x^7\,dx=\left[\frac{x^{7+1}}{7+1}\right]_0^1=\left[\frac{x^8}{8}\right]_0^1=\frac{1^8}{8}-\frac{0^8}{8}=\frac{1}{8}-0=\frac{1}{8}$$

$$\Omega=\frac{1}{8}\cdot\frac{1}{8}=\frac{1\cdot 1}{8\cdot 8}=\frac{1}{64}\to\Omega=\frac{1}{64}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\begin{cases} x=x \\ y=u\cdot x \\ z=v\cdot x\cdot y=v\cdot u\cdot x^2 \end{cases}$$

$$\begin{cases} 0\le x\le 1 \\ 0\le y\le x\Rightarrow 0\le u\cdot x\le x\Rightarrow 0\le u\le 1 \\ 0\le z\le x\cdot y\Rightarrow 0\le v\cdot u\cdot x^2\le u\cdot x^2\Rightarrow 0\le v\le 1 \end{cases}$$

$$J=\frac{\partial(x,y,z)}{\partial(x,u,v)}=\begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\begin{cases} \frac{\partial x}{\partial x}=1, & \frac{\partial x}{\partial u}=0, & \frac{\partial x}{\partial v}=0 \\ \frac{\partial y}{\partial x}=u, & \frac{\partial y}{\partial u}=x, & \frac{\partial y}{\partial v}=0 \\ \frac{\partial z}{\partial x}=2\cdot u\cdot v\cdot x, & \frac{\partial z}{\partial u}=v\cdot x^2, & \frac{\partial z}{\partial v}=u\cdot x^2 \end{cases}$$

$$J=\begin{vmatrix} 1 & 0 & 0 \\ u & x & 0 \\ 2uvx & vx^2 & ux^2 \end{vmatrix}$$

$$J=1\cdot\begin{vmatrix} x & 0 \\ vx^2 & ux^2 \end{vmatrix}-0\cdot\begin{vmatrix} u & 0 \\ 2uvx & ux^2 \end{vmatrix}+0\cdot\begin{vmatrix} u & x \\ 2uvx & vx^2 \end{vmatrix}$$

$$J=1\cdot(x\cdot ux^2-0\cdot vx^2)-0+0=x\cdot ux^2=u\cdot x^3$$

$$x\cdot y\cdot z=x\cdot(u\cdot x)\cdot(v\cdot u\cdot x^2)=u^2\cdot v\cdot x^4$$

$$\Omega=\int_0^1\int_0^1\int_0^1(u^2\cdot v\cdot x^4)\cdot|J|dx\,du\,dv$$

$$\Omega=\int_0^1\int_0^1\int_0^1(u^2\cdot v\cdot x^4)\cdot(u\cdot x^3)dx\,du\,dv,\qquad \Omega=\int_0^1\int_0^1\int_0^1 x^7\cdot u^3\cdot v\,dx\,du\,dv$$

$$\Omega = \left(\int_0^1 x^7 dx\right) \cdot \left(\int_0^1 u^3\, du\right) \cdot \left(\int_0^1 v\, dv\right)$$

$$\int_0^1 x^7\, dx = \left[\frac{x^{7+1}}{7+1}\right]_0^1 = \left[\frac{x^8}{8}\right]_0^1 = \frac{1^8}{8} - \frac{0^8}{8} = \frac{1}{8} - 0 = \frac{1}{8}$$

$$\int_0^1 u^3\, du = \left[\frac{u^{3+1}}{3+1}\right]_0^1 = \left[\frac{u^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\int_0^1 v\, dv = \left[\frac{v^{1+1}}{1+1}\right]_0^1 = \left[\frac{v^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\Omega = \frac{1}{8} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8 \cdot 4 \cdot 2} \to \Omega = \frac{1}{64}$$

Solution 4 by Jose Luis Diaz Barrero-Spain

The region D is described by

$$0 \le x \le 1,\ \ 0 \le y \le x,\ \ 0 \le z \le xy,$$

So, by Fubini's theorem, the triple integral can be written as the iterated integral

$$\Omega = \int_0^1 \left(\int_0^x \left(\int_0^{xy} xyz\, dz\right) dy\right) dx.$$

Since x and y are constant with respect to z, then

$$\int_0^{xy} xyz\, dz = xy\left[\frac{z^2}{2}\right]_{z=0}^{z=xy} = xy \cdot \frac{(xy)^2}{2} = \frac{x^2y^3}{2}.$$

Now x is constant with respect to y, hence

$$\int_0^x \frac{x^3y^3}{2} dy = \frac{x^3}{2}\left[\frac{y^4}{4}\right]_{y=0}^{y=x} = \frac{x^3}{2} \cdot \frac{x^4}{4} = \frac{x^7}{8}.$$

Finally,

$$\Omega = \int_0^1 \frac{x^7}{8} dx = \frac{1}{8}\left[\frac{x^8}{8}\right]_{x=0}^{x=1} = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}.$$

Therefore,

$$\Omega = \int\int\int_D xyz\, dx\, dy\, dz = \frac{1}{64}.$$

UP.645 Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}dx\,dy\,dz$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}dx\,dy\,dz=$$

$$=\int_0^1\int_0^1\int_0^1 e^{2x-x-y-z}\,dx\,dy\,dz+\int_0^1\int_0^1\int_0^1 e^{-x+3y-y-z}\,dx\,dy\,dz+$$

$$+\int_0^1\int_0^1\int_0^1 e^{-x-y+3z}\,dx\,dy\,dz=$$

$$=\left(\int_0^1 e^x dx\right)\left(\int_0^1 e^{-y}dy\right)\left(\int_0^1 e^{-z}dz\right)+\left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{2y}dy\right)\left(\int_0^1 e^{-z}dz\right)+$$

$$+\left(\int_0^1 e^{-x}dx\right)\left(\int_0^1 e^{-y}\,dy\right)\left(\int_0^1 e^{3z}\,dz\right)=$$

$$=(e-1)(e^{-1}-e^0)^2+\left(\frac{1}{2}e^2-\frac{1}{2}\right)(e^{-1}-e^0)^2+\left(\frac{1}{3}e^3-\frac{1}{3}\right)(e^{-1}-e^0)^2=$$

$$=\left(\frac{1}{e}-1\right)^2\left(e-1+\frac{e^2-1}{2}+\frac{e^3-1}{3}\right)=$$

$$=\frac{(1-e)^2}{e^2}\cdot\frac{6e-6+3e^2-3+2e^3-2}{6}=\frac{(1-e)^2(2e^3+3e^2+6e-11)}{6e^2}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

First we split the integrand into three terms

$$\frac{e^{2x}+e^{3y}+e^{4z}}{e^{x+y+z}}=e^xe^{-y}e^{-z}+e^{-x}e^{2y}e^{-z}+e^{-x}e^{-y}e^{3z}.$$

Each term is a product of functions of a single variable and the domain of integration is the cube $[0,1]^3$. Therefore, by Fubini's theorem, the integral of each term is the product of three one-variable integrals. We need the following integrals:

$$\int_0^1 e^{-t}\,dt=\frac{e-1}{e},\quad\int_0^1 e^t\,dt=e-1,$$

$$\int_0^1 e^{2t}\,dt = \frac{e^2-1}{2},\ \int_0^1 e^{3t}\,dt = \frac{e^3-1}{3}.$$

Now, we evaluate the first term:

$$\int_0^1\int_0^1\int_0^1 e^x e^{-y} e^{-z}\,dx\,dy\,dz = (e-1)\left(\frac{e-1}{2}\right)^2 = \frac{(e-1)^3}{e^2}.$$

The second term:

$$\int_0^1\int_0^1\int_0^1 e^{-x}\,e^{2y}\,e^{-z}\,dx\,dy\,dz = \frac{e^2-1}{2}\left(\frac{e-1}{e}\right)^2 = \frac{(e^2-1)(e-1)^2}{2e^2}.$$

The third term:

$$\int_0^1\int_0^1\int_0^1 e^{-x}e^{-y}e^{3z}\,dx\,dy\,dz = \frac{e^3-1}{3}\left(\frac{e-1}{e}\right)^2 = \frac{(e^3-1)(e-1)^2}{3e^2}.$$

Adding up, yields

$$\Omega = \frac{(e-1)^2}{e^2}\left[(e-1)+\frac{e^2-1}{2}+\frac{e^3-1}{3}\right] =$$

$$= \frac{(e-1)^2}{6e^2}[6(e-1)+3(e^2-1)+2(e^3-1)].$$

Hence,

$$\Omega = \frac{(e-1)^2(2e^3+3e^2+6e-11)}{6e^2}$$