ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\pi} \frac{\sin^2 x}{.(5-3cosx)^3}dx$$

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$$Using\ Feynman's\ Technique:$$

$$Let\ I(a)=\int_0^{\pi}\frac{dx}{a+bcosx}, a>|b|$$

$$Let\ t=\tan\frac{x}{2}\ then\cos x=\frac{1-t^2}{1+t^2}\ \&\ dx=\frac{2dt}{1+t^2}$$

$$I(a)=\int_0^{\infty}\frac{2dt}{a(1+t^2)+b(1-t^2)}=2\int_0^{\infty}\frac{dt}{(a+b)+(a-b)t^2}=$$

$$=\frac{2}{a-b}\int_0^{\infty}\frac{dt}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2+t^2}=$$

$$=\frac{2}{\sqrt{a^2-b^2}}\left(\arctan\left(t\sqrt{\frac{a-b}{a+b}}\right)\right)_0^{\infty}=\frac{\pi}{\sqrt{a^2-b^2}}\ (A)$$

$$now\ I'(a)=\frac{d}{da}\int_0^{\pi}\frac{dx}{a+bcosx}=\frac{d}{da}\left(\frac{\pi}{\sqrt{a^2-b^2}}\right)$$

$$-\int_0^{\pi}\frac{dx}{(a+bcosx)^2}=-\frac{\pi}{2}.\frac{2a}{(a^2-b^2)^{\frac{3}{2}}}$$

$$\int_0^{\pi}\frac{dx}{(a+bcosx)^2}=.\frac{a\pi}{(a^2-b^2)^{\frac{3}{2}}}\ (B)$$

$$I''(a)=\frac{d}{da}\int_0^{\pi}\frac{dx}{(a+bcosx)^2}=\frac{d}{da}\left(.\frac{a\pi}{(a^2-b^2)^{\frac{3}{2}}}\right)$$

$$-\int_0^{\pi}\frac{2dx}{(a+bcosx)^3}=\pi\frac{(a^2-b^2)^{\frac{3}{2}}-a.\frac{3}{2}(a^2-b^2)^{\frac{1}{2}}.2a}{(a^2-b^2)^3}$$

$$\int_0^{\pi}\frac{dx}{(a+bcosx)^3}=\frac{\pi(2a^2+b^2)}{2(a^2-b^2)^{\frac{5}{2}}}\ (C)$$

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$$\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{5-(5-3cosx)}{3}\right)^2 =$$

$$= \frac{-16 + 10(5-3cosx) - (5-3cosx)^2}{9} (D)$$

$$\int_0^{\pi} \frac{\sin^2 x}{.(5-3cosx)^3} dx \overset{D}{=}$$

$$= \frac{1}{9}\left(-16\int_0^{\pi} \frac{dx}{(5-3cosx)^3} + 10\int_0^{\pi} \frac{dx}{(5-3cosx)^2} - \int_0^{\pi} \frac{dx}{5-3cosx}\right) =$$

$$= \frac{1}{9}\left(-16\frac{\pi(2\times 5^2+3^2)}{2(5^2-3^2)^{\frac{5}{2}}} + 10\frac{5\pi}{(5^2-3^2)^{\frac{3}{2}}} - \frac{\pi}{\sqrt{5^2-3^2}}\right) =$$

$$= \frac{1}{9}\left(-16\times\frac{59\pi}{2048} + 10\times\frac{5\pi}{64} - \frac{\pi}{4}\right) = \frac{\pi}{128}$$