

ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{\sqrt[3]{x^3 + y^3}} \leq \frac{1}{\sqrt[3]{2}} (b - a)(\ln b - \ln a)$$

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Solution by Tapas Das-India

$$\begin{aligned} x^3 + y^3 &= \frac{x^3}{(1)^2} + \frac{y^3}{(1)^2} \stackrel{\text{Radon}}{\geq} \frac{(x + y)^3}{4} \text{ then } \sqrt[3]{x^3 + y^3} \geq \frac{x + y}{2^{\frac{2}{3}}} \\ \int_a^b \int_a^b \frac{dxdy}{\sqrt[3]{x^3 + y^3}} &\leq 2^{\frac{2}{3}} \int_a^b \int_a^b \frac{dxdy}{x + y} \stackrel{AM-HM}{\leq} \frac{2^{\frac{2}{3}}}{4} \int_a^b \int_a^b \left(\frac{1}{x} + \frac{1}{y} \right) dxdy = \\ &= \frac{2^{\frac{2}{3}}}{4} \left(\int_a^b \int_a^b \left(\frac{1}{x} \right) dxdy + \int_a^b \int_a^b \left(\frac{1}{y} \right) dxdy \right) = \frac{2^{\frac{2}{3}}}{4} ((\ln x)_a^b (y)_a^b + (\ln y)_a^b (x)_a^b) = \\ &= 2 \times \frac{2^{\frac{2}{3}}}{4} (b - a)(\ln b - \ln a) = \frac{1}{\sqrt[3]{2}} (b - a)(\ln b - \ln a) \end{aligned}$$

Equality holds for $a = b$.