

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^{\infty} \frac{\sqrt{x}}{x^3 + 1} dx$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\begin{aligned} \Omega &= \int_0^{\infty} \frac{\sqrt{x}}{x^3 + 1} dx \stackrel{x=y^2}{=} \int_0^{\infty} \frac{y \cdot 2y dy}{y^6 + 1} = 2 \int_0^{\infty} \frac{y^2}{1 + (y^3)^2} dy = \frac{2}{3} \int_0^{\infty} \frac{(y^3)'}{1 + (y^3)^2} dy = \\ &= \frac{2}{3} \arctan y^3 \Big|_0^3 = \frac{2}{3} (\arctan \infty - \arctan 0) = \frac{2}{3} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{3} \end{aligned}$$

Solution 2 by Daniel Sitaru – Romania

We use the formula:

$$\begin{aligned} \int_0^{\infty} \frac{x^m}{1 + x^n} dx &= \frac{\pi}{n \sin\left(\frac{n+1}{n}\pi\right)}; 0 < m+1 < n \\ \Omega &= \int_0^{\infty} \frac{\sqrt{x}}{x^3 + 1} dx = \int_0^{\infty} \frac{x^{\frac{1}{2}}}{1 + x^3} dx = \frac{\pi}{3 \sin\left(\frac{\frac{1}{2} + 1}{3} \cdot \pi\right)} = \frac{\pi}{3 \sin \frac{\pi}{2}} = \frac{\pi}{3 \cdot 1} = \frac{\pi}{3} \end{aligned}$$

Solution 3 by Daniel Sitaru – Romania

We use the formula:

$$\begin{aligned} \int_0^{\infty} \frac{x^{a-1}}{1 + x^b} dx &= \frac{\pi}{b} \csc\left(\frac{a\pi}{b}\right) \\ \Omega &= \int_0^{\infty} \frac{\sqrt{x}}{1 + x^3} dx = \int_0^{\infty} \frac{x^{\frac{3}{2}-1}}{1 + x^3} dx = \frac{\pi}{3} \csc\left(\frac{\frac{3}{2}\pi}{3}\right) = \frac{\pi}{3} \csc \frac{\pi}{2} = \frac{\pi}{3} \cdot \frac{1}{\sin \frac{\pi}{2}} = \frac{\pi}{3} \cdot \frac{1}{1} = \frac{\pi}{3} \end{aligned}$$