

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^{\infty} \frac{\sqrt[3]{x}}{x^2 + 1} dx$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

We use the formula:

$$\begin{aligned} \int_0^{\infty} \frac{x^m}{x^n + 1} dx &= \frac{\pi}{n \sin\left(\frac{m+1}{n}\pi\right)}; 0 < m+1 < n \\ \Omega &= \int_0^{\infty} \frac{\sqrt[3]{x}}{x^2 + 1} dx = \int_0^{\infty} \frac{x^{\frac{1}{3}}}{x^2 + 1} dx = \frac{\pi}{2 \sin\left(\frac{\frac{1}{3} + 1}{2}\pi\right)} = \frac{\pi}{2 \sin\left(\frac{2\pi}{3}\right)} = \\ &= \frac{\pi}{2 \sin\left(\pi - \frac{\pi}{3}\right)} = \frac{\pi}{2 \sin\frac{\pi}{3}} = \frac{\pi}{2 \cdot \frac{\sqrt{3}}{2}} = \frac{\pi}{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} \end{aligned}$$

Solution 2 by Daniel Sitaru – Romania

We use the formula:

$$\begin{aligned} \int_0^{\infty} \frac{x^{a-1}}{1+x^b} dx &= \frac{\pi}{b} \csc\left(\frac{a\pi}{b}\right) \\ \Omega &= \int_0^{\infty} \frac{\sqrt[3]{x}}{1+x^2} dx = \int_0^{\infty} \frac{x^{\frac{1}{3}}}{1+x^2} dx = \int_0^{\infty} \frac{x^{\frac{4}{3}-1}}{1+x^2} dx = \frac{\pi}{2} \csc\left(\frac{\frac{4}{3}\pi}{2}\right) = \\ &= \frac{\pi}{2} \cdot \frac{1}{\sin\left(\frac{2\pi}{3}\right)} = \frac{\pi}{2 \sin\left(\pi - \frac{\pi}{3}\right)} = \frac{\pi}{2 \sin\frac{\pi}{3}} = \frac{\pi}{2 \cdot \frac{\sqrt{3}}{2}} = \frac{\pi}{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} \end{aligned}$$