

# A refinement of the AM–GM–HM inequality, via permutations means

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*In this short note, refinements of classical means are presented: both of the AM-GM inequality and the GM-HM inequality. The expressions that achieve these refinements are combinatorial expressions that include factorials, permutations, weights, and they themselves are also means.*

**Keywords :** AM-GM-HM inequality, classical means, refinement, permutation

**2020 Mathematics Subject Classification :** 26D15

In [1] the following double inequality was published as a proposed problem :

## 1. Proposition

If  $a_k, p_k \in \mathbb{R}_{>0}$ , with  $\sum_{k=1}^n p_k = 1$  and  $(\forall) \sigma \in S_n$  ( $\sigma$  is a permutation of the set  $\{1, 2, \dots, n\}$ ), then holds the inequalities :

$$\frac{1}{n} \cdot \sum_{k=1}^n a_k \geq \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}} \geq \sqrt[n]{\prod_{k=1}^n a_k} \geq \frac{n!}{\sum_{\sigma \in S_n} \frac{1}{\prod_{k=1}^n a_k^{p_{\sigma(k)}}}} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}. \quad (1)$$

## Proof

For the first inequality, by using means inequality - in weighted form :

$$\prod_{k=1}^n x_k^{p_k} \leq \sum_{k=1}^n p_k x_k. \quad (2)$$

we have : 
$$\sum_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}} \stackrel{(2)}{\leq} \sum_{\sigma \in S_n} \sum_{k=1}^n p_{\sigma(k)} \cdot a_k \leq \sum_{k=1}^n \sum_{\sigma \in S_n} p_{\sigma(k)} \cdot a_k =$$

(by fixing  $a_k$ ,  $k = \overline{1, n}$ , the another numbers can be selected in  $(n-1)!$  times )

$$= (p_{\sigma(1)} + p_{\sigma(2)} + \dots + p_{\sigma(n)}) \cdot a_1 \cdot (n-1)! + \dots + (p_{\sigma(1)} + p_{\sigma(2)} + \dots + p_{\sigma(n)}) \cdot a_n \cdot (n-1)! =$$

$$= (n-1)! \cdot (a_1 + a_2 + \dots + a_n).$$

Dividing by  $n!$ , on obtain the first inequality .

Now , by using *means inequality* for  $n! = \text{Card}(S_n)$  numbers ,

$$\begin{aligned}
 \text{hold : } \sum_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}} &\geq n! \cdot \sqrt[n]{\prod_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}}} = \text{(again , by fixing} \\
 \text{un } a_k , k = \overline{1, n} , \text{ the another numbers can be selected in } (n-1)! \text{ times )} &= \\
 = n! \cdot \sqrt[n]{a_1^{(n-1)!} \cdot a_2^{(n-1)!} \cdot \dots \cdot a_n^{(n-1)!}} &= n! \cdot \sqrt[n]{\left(\prod_{k=1}^n a_k\right)^{(n-1)!}} = \\
 = n! \cdot \sqrt[n]{\prod_{k=1}^n a_k} , \text{ therefore the second inequality in the statement .} &
 \end{aligned}$$

For the another two inequalities from enonce , on make the substitution

$$a_k \rightarrow \frac{1}{a_k} \text{ in the first two inequalities .}$$

Other demonstrations of the inequality from the statement can be found in [2] – as solutions to the proposed problem.

Let us remind our readers the formulas for the *classical means* of  $n$  positive numbers.

Given the positive real numbers  $a_1, a_2, \dots, a_n$ , the following *classical means* are well known :

$$A_n[a_1, a_2, \dots, a_n] := \frac{1}{n} \cdot \sum_{k=1}^n a_k \quad (3)$$

(the *arithmetic mean* of the numbers  $a_1, a_2, \dots, a_n$ ) ,

$$G_n[a_1, a_2, \dots, a_n] := \sqrt[n]{\prod_{k=1}^n a_k} \quad (4)$$

(the *geometric mean* of the numbers  $a_1, a_2, \dots, a_n$ ) ,

$$H_n[a_1, a_2, \dots, a_n] := \frac{n}{\sum_{k=1}^n \frac{1}{a_k}} , \quad (5)$$

(the *harmonic mean* of the numbers  $a_1, a_2, \dots, a_n$ )

$$\text{as well as the inequality between them , } A_n[a] \geq G_n[a] \geq H_n[a] , \quad (6)$$

(where equality takes place if and only if  $a_1 = a_2 = \dots = a_n$ ) .

Inequality (6) is also called the *AM-GM-HM inequality* or just *the inequality of classical means* .

## 2. Remark

Through inequality (1), a *refinement of the classical means* was practically established, meaning we inserted new expressions between terms already existing in inequality (6).

In the most general sense, the *mean* of a set of numbers is a number lying between the *minimum* and *maximum* of the respective set of numbers.

More precisely for the numbers  $x_1, x_2, \dots, x_n \in \mathbb{I} \subset \mathbb{R}$ , we will call their *mean*, the function  $\mathcal{M} : \mathbb{I}^n \longrightarrow \mathbb{I}$ , for which we have,

$$\min \{x_1, x_2, \dots, x_n\} \leq \mathcal{M}(x_1, x_2, \dots, x_n) \leq \max \{x_1, x_2, \dots, x_n\} . \quad (7)$$

We can then state the following :

### 3. Corollary

If  $a_k, p_k \in \mathbb{R}_{>0}$ , with  $\sum_{k=1}^n p_k = 1$  and  $(\forall) \sigma \in S_n$  ( $\sigma$  is a permutation of the set  $\{1, 2, \dots, n\}$ ), then the values:

$$\mathcal{M}[a_1, a_2, \dots, a_n] := \frac{1}{n!} \cdot \sum_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}} , \quad (8)$$

$$\mathcal{N}[a_1, a_2, \dots, a_n] := \frac{n!}{\sum_{\sigma \in S_n} \prod_{k=1}^n a_k^{p_{\sigma(k)}}} , \quad (9)$$

they are *means* of positive real numbers  $a_1, a_2, \dots, a_n$ .

### Proof

Everything results from a few inequalities that are obvious for means and from the inequalities in *Proposition 1*:

$$\begin{aligned} \max \{a_1, a_2, \dots, a_n\} &\geq A_n[a_1, a_2, \dots, a_n] \geq \mathcal{M}[a_1, a_2, \dots, a_n] \geq G_n[a_1, a_2, \dots, a_n] \geq \\ &\geq \mathcal{N}[a_1, a_2, \dots, a_n] \geq H_n[a_1, a_2, \dots, a_n] \geq \min \{a_1, a_2, \dots, a_n\} . \end{aligned} \quad (10)$$

### 4. Remark

Through the sequence of inequalities (10), the new *permutations means*,  $\mathcal{M}[a_1, a_2, \dots, a_n]$  and  $\mathcal{N}[a_1, a_2, \dots, a_n]$ , are highlighted, showing us how they are positioned relative to the *classical means* and how they *refine* the *AM-GM-HM inequality*.

## References

- [1] Bullen P. S. & Mitrinović D. S. & Vasić P. M., „*Means and Their Inequalities*”, D. Reidel Publishing Company, Dordrecht/Boston, 1988.
- [2] Mărghidanu D., “*A refinement for the inequality of classical means*”, (*Proposed problem*), in « [Mathematical Inequalities](https://www.facebook.com/photo?fbid=3276670499058702&set=gm.2616143325340379) », 16 may, 2020, *on line* : <https://www.facebook.com/photo?fbid=3276670499058702&set=gm.2616143325340379>