

A Geometric Proof of the Isogonal Disc Theorem

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Abstract

Let P be an interior point of a triangle ABC , and let Q be its isogonal conjugate. We give a purely geometric proof that the incenter I lies in the disc with diameter PQ . Unless $P = I$, the incenter lies in the interior of this disc.

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1 Introduction

Two points P and Q are *isogonal conjugates* with respect to a triangle ABC if the lines AP and AQ are symmetric about the internal angle bisector at A , and similarly at B and C .

In a 2022 note, Lukarevski and Scott [3] conjectured that, for any interior point P of a triangle with isogonal conjugate Q , the incenter must lie inside the disc with diameter PQ . Rabinowitz [4] proved the conjecture using barycentric coordinates and extensive symbolic computation. This left open the question of whether the result admits a conceptual geometric proof. In a later paper [2], the authors wrote that “this conjecture seems to be as far from a purely synthetic geometric proof as ever.” We give such a proof here.

The key observation is that three natural circles through the incenter separate P from Q . Inversion about the incenter transforms these circles into the sidelines of a surprisingly natural acute triangle whose orthocenter is the original incenter. The desired disc containment then becomes an angle statement.

2 Three circles separating isogonal conjugates

We begin with two standard facts.

The following result is given in [1, §238].

Lemma 1. *Let P be interior to $\triangle ABC$, and let Q be its isogonal conjugate (Figure 1). Then*

$$\angle BPC + \angle BQC = 180^\circ + A.$$

The following result is well known [1, §289].

Lemma 2. *Let I be the incenter of $\triangle ABC$ (Figure 2). Then*

$$\angle BIC = 90^\circ + \frac{A}{2}.$$

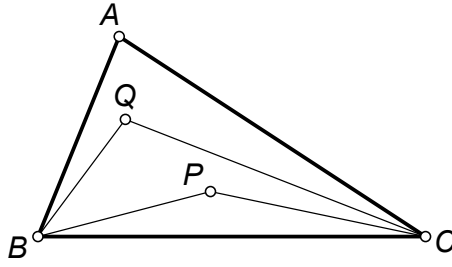


Figure 1: Isogonal conjugates P and Q .

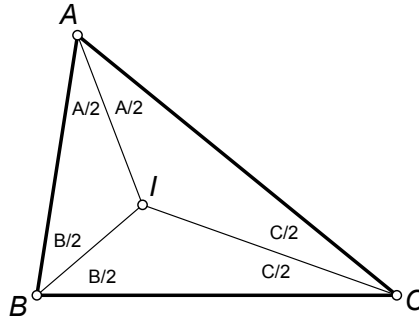


Figure 2: The incenter I .

Lemma 3. *Let P be interior to $\triangle ABC$, and let Q be its isogonal conjugate. Let I be the incenter of $\triangle ABC$, and let ω_A be the circle through B, I, C (Figure 3). Then P and Q lie in opposite closed regions determined by ω_A .*

Proof. The isogonal conjugate of an interior point is interior, so P and Q lie on the same side of BC as A . On that side of BC , the arc BIC of ω_A is the locus of points X for which

$$\angle BXC = \angle BIC.$$

For points inside ω_A the subtended angle is larger, and for points outside it is smaller. By Lemma 1,

$$\angle BPC + \angle BQC = 180^\circ + A,$$

and by Lemma 2,

$$2\angle BIC = 180^\circ + A.$$

Thus

$$(\angle BPC - \angle BIC) + (\angle BQC - \angle BIC) = 0.$$

Therefore the two differences have opposite signs, unless both are zero. Hence one of P, Q is in the closed disc bounded by ω_A and the other is in the closed exterior. $\square$

The same argument applies cyclically to the circles ω_B through C, I, A and ω_C through A, I, B .

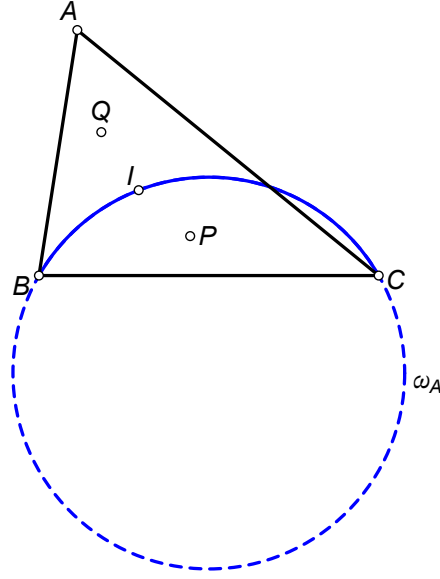


Figure 3: The circle ω_A separates P and Q .

3 Inversion at the incenter

The next lemma contains the geometric idea on which the proof rests.

Lemma 4. *Let I be the incenter of $\triangle ABC$. Let $\omega_A, \omega_B, \omega_C$ be the circles through BIC , CIA , and AIB , respectively. Consider an inversion centered at I , with arbitrary radius. Let A', B', C' be the inverses of A, B, C , respectively (Figure 4). Then:*

1. *the circles $\omega_A, \omega_B, \omega_C$ invert respectively to the lines $B'C'$, $C'A'$, and $A'B'$;*
2. *the point I is the orthocenter of $\triangle A'B'C'$, and this triangle is acute;*
3. *inversion preserves every ray from I .*

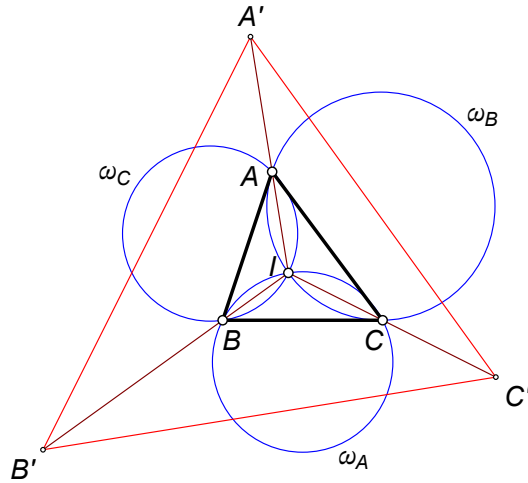


Figure 4: The three circles invert to the sidelines of $\triangle A'B'C'$.

Proof. Since ω_A passes through the center of inversion, its image is a line. Since it also contains B and C , that line is $B'C'$. The other two statements in part (1) follow cyclically.

Let t be the tangent to ω_A at I (Figure 5). Then

$$\angle(t, IB) = \angle ICB = \frac{C}{2}.$$

But

$$\angle AIB = 90^\circ + \frac{C}{2}$$

by Lemma 2. Hence $t \perp AI$.

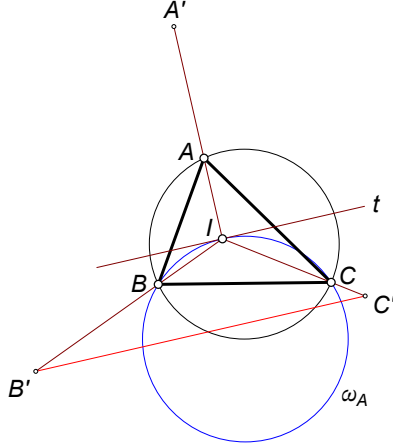


Figure 5: The tangent t to ω_A at I .

The image of a circle through the center of inversion is a line parallel to its tangent there, so $B'C' \perp AI$. Since A' lies on the ray IA , this gives $IA' \perp B'C'$. Cyclically,

$$IB' \perp C'A', \quad IC' \perp A'B',$$

so I is the orthocenter of $\triangle A'B'C'$.

Inversion fixes the rays from I , and therefore, by Lemma 2,

$$\angle B'IC' = \angle BIC = 90^\circ + \frac{A}{2},$$

with the two cyclic analogues. These three angles are all less than 180° and sum to 360° . Thus the three rays IA', IB', IC' are not contained in any closed half-plane through I , so I lies in the interior of $\triangle A'B'C'$. An interior orthocenter occurs precisely in an acute triangle. Hence $\triangle A'B'C'$ is acute, proving part (2).

Finally, inversion centered at I sends every point to another point on the same ray from I . This proves part (3). $\square$

4 A lemma for an acute triangle

Lemma 5. *Let $A_1A_2A_3$ be an acute triangle with orthocenter H , and let ℓ_i be the sideline opposite A_i . Suppose that, for each i , X and Y lie in opposite closed half-planes bounded by ℓ_i . Then*

$$\angle XHY > 90^\circ.$$

Proof. For each sideline ℓ_i , call the closed half-plane containing H the *near side* and the other closed half-plane the *far side*. Since ℓ_i weakly separates X and Y , we may classify X and Y oppositely with respect to ℓ_i : if one is on the near side, the other is on the far side. If a point lies on ℓ_i , it may be assigned to either closed half-plane. Thus the near/far patterns (N/F) of X and Y are complementary.

No point can lie on the far side of all three sidelines. Hence neither X nor Y can have the pattern (F, F, F) . Since their patterns are complementary, neither can have the pattern (N, N, N) . Therefore one of X, Y is on the far side of exactly one sideline and the other is on the far side of exactly two. After interchanging X and Y , we may suppose that X is on the far side of exactly one sideline. After relabelling the vertices, we may take this sideline to be ℓ_1 . Thus

$$X : (F, N, N), \quad Y : (N, F, F).$$

Consider first the ray HX . Since H and X lie on opposite sides of $\ell_1 = A_2A_3$, the segment HX meets ℓ_1 . On the other hand, H and X both lie on the near sides of ℓ_2 and ℓ_3 . These near half-planes are convex, so the point where HX meets ℓ_1 lies on the segment A_2A_3 . Consequently the ray HX lies in the closed angular sector bounded by the rays HA_2 and HA_3 .

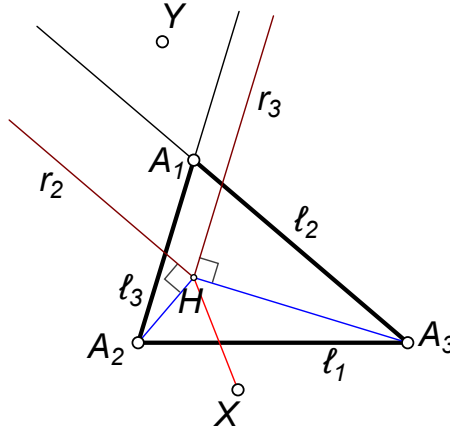


Figure 6: The two angular sectors used in Lemma 5.

Now consider Y . The far sides of $\ell_2 = A_1A_3$ and $\ell_3 = A_1A_2$ meet in the angle vertically opposite $\angle A_2A_1A_3$. Through H , draw rays r_2, r_3 parallel respectively to A_1A_3 and A_1A_2 , directed toward the A_1 -side (Figure 6). Because the triangle is acute, the altitude A_1H lies strictly inside $\angle A_2A_1A_3$. It follows that the entire vertical angle containing Y lies inside the open sector bounded by r_2 and r_3 . Hence the ray HY lies strictly inside that sector.

Finally, since H is the orthocenter,

$$HA_2 \perp A_1A_3, \quad HA_3 \perp A_1A_2.$$

Thus

$$HA_2 \perp r_2, \quad HA_3 \perp r_3.$$

The closed sector containing HX and the open sector containing HY are therefore separated at both ends by right angles. Accordingly every ray in the first sector makes an angle at least 90° with every ray in the second, and because HY lies strictly inside the second sector, equality cannot occur. Hence $\angle XHY > 90^\circ$. $\square$

5 The Isogonal Disc Theorem

Theorem 6. *Let P and Q be distinct interior points of $\triangle ABC$, each distinct from the incenter I . Let $\omega_A, \omega_B, \omega_C$ be the circles through BIC , CIA , and AIB , respectively. Suppose P and Q lie in opposite closed regions determined by each of these circles. Then the incenter I lies in the disc with diameter PQ .*

Proof. Let P' and Q' be the images of P and Q under an inversion with center I and arbitrary radius. Let A', B', C' be the images of A, B, C , respectively. Under inversion, the circles $\omega_A, \omega_B, \omega_C$ become the three sidelines of $\triangle A'B'C'$. Hence each sideline of $\triangle A'B'C'$ weakly separates P' from Q' .

By Lemma 4, the triangle $A'B'C'$ is acute and has orthocenter I . Lemma 5 therefore gives

$$\angle P'IQ' > 90^\circ.$$

Since inversion centered at I preserves rays from I ,

$$\angle PIQ = \angle P'IQ' > 90^\circ.$$

A point sees a segment under an angle greater than a right angle exactly when it lies inside the circle having that segment as diameter. Hence I lies in the interior of the disc with diameter PQ . $\square$

Theorem 7 (Isogonal Disc Theorem). *Let P be a point interior to $\triangle ABC$, and let Q be its isogonal conjugate. Then the incenter I lies in the disc with diameter PQ . If $P \neq I$, then I lies in the interior of that disc.*

Proof. If $P = I$, then $Q = I$, and the assertion is immediate. Assume $P \neq I$. Since isogonal conjugation is an involution and fixes I , we also have $Q \neq I$. By Lemma 3, P and Q lie in opposite closed regions determined by each of the circles $\omega_A, \omega_B, \omega_C$. The result now follows from Theorem 6. $\square$

Acknowledgments

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