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Proposed by

Nguyen Hung Cuong-Vietnam, Dang Ngoc Minh-Vietnam

Kostantinos Geronikolas-Greece, Prashanna Khanal-India

Jenish Rijal-Nepal, Neculai Stanciu-Romania

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Bogdan Fuștei-Romania, Viorel Ovidiu Ignătescu-Romania

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Solutions by

Daniel Sitaru-Romania, Soumava Chakraborty-India

Mirsadix Muzefferov-Azerbaijan, Tapas Das-India

Adgozel Adgozalev-Azerbaijan

4201. In ΔABC the following relationship holds:

$$\frac{a}{r_a\sin^2 A}+\frac{b}{r_b\sin^2 B}+\frac{c}{r_c\sin^2 C}\ge\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{r_a\sin^2 A}+\frac{b}{r_b\sin^2 B}+\frac{c}{r_c\sin^2 C}=\sum_{cyc}\frac{a}{r_a\sin^2 A}=\sum_{cyc}\frac{a}{\frac{F}{s-a}\cdot\frac{a^2}{4R^2}}=$$

$$=\frac{4R^2}{F}\sum_{cyc}\frac{s-a}{a}=\frac{4R^2}{F}\left(s\sum\frac{1}{a}-3\right)=$$

$$=\frac{4R^2}{rs}\left(s\cdot\frac{ab+bc+ca}{abc}-3\right)=\frac{4R^2}{rs}\left(s\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$

$$=\frac{4R^2}{rs}\cdot\frac{s^2+r^2+4Rr-12Rr}{4Rr}=\frac{R}{r^2s}\cdot(s^2+r^2-8Rr)\ge$$

$$\overset{GERRETSEN}{\ge}\frac{R}{R^2S}(16Rr-5r^2+r^2-8Rr)=\frac{R}{r^2s}(8Rr-4r^2)=\frac{R(8R-4r)}{rs}\overset{EULER}{\ge}$$

$$\ge\frac{R(16r-4r)}{rs}\overset{MITRINOVIC}{\ge}\frac{\frac{2}{3\sqrt{3}}s\cdot 12r}{rs}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.

4202. In any Δ ABC the following relationship holds :

$$(n_a+n_b+n_c)(m_a+m_b+m_c)\ge 3(m_aw_a+m_bw_b+m_cw_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{4s(s-a)}{(b+c)^2}=\sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2}=3-\sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$=4s\sum_{cyc}\frac{1}{b+c}-4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2-\frac{2}{2s(s^2+2Rr+r^2)}.\sum_{cyc}(b+c)\right)$$

$$=\frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)}-\frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}+\frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc}\frac{4s(s-a)}{(b+c)^2}\boxed{\overset{(\bullet)}{=}}\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

$$\text{Now, }3\sum_{cyc}m_a w_a=6\sum_{cyc}\left(m_a.\frac{\sqrt{bcs(s-a)}}{b+c}\right)\overset{CBS}{\le}$$

$$3.\sqrt{\left(\sum_{cyc}\left(2\sum_{cyc}a^2-3a^2\right)bc\right)}.\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}}\overset{\text{via }(\bullet)}{=}$$

$$3.\sqrt{4(s^4-(4Rr+r^2)^2-6Rrs^2).\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{4(s^2+2Rr+r^2)^2}}\overset{?}{\le}$$

$$\sqrt{(9s^2-80Rr-2r^2)(4s^2-28Rr+29r^2)}$$

$$\overset{squaring}{\Leftrightarrow} 27s^8-(554Rr-163r^2)s^6+r^2(1320R^2-1244Rr+484r^2)s^4+$$
$$r^3(9552R^3-1604R^2r-2790Rr^2+299r^3)s^2+$$
$$r^4(8960R^4+480R^3r-6616R^2r^2-2388Rr^3-49r^4)\boxed{\overset{?}{\ge}\ (1)}\ 0\ \text{and}\ \because$$

$$27(s^2-16Rr+5r^2)^4\overset{Gerretsen}{\ge}0\ \therefore\ \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)}\overset{?}{\ge}27(s^2-16Rr+5r^2)^4$$
$$\Leftrightarrow(1174R-377r)s^6-r(40152R^2-24676Rr+3566r^2)s^4+$$
$$r^2(16(28245R^3-26021R^2r+7925Rr^2-825r^3)+12R^2r+10Rr^2-r^3)s^2-$$
$$r^3\begin{pmatrix}128(13754R^4-17284R^3r+8152R^2r^2-1669Rr^3+132r^4)+\\32R^3r-40R^2r^2+20Rr^3+28r^4\end{pmatrix}\boxed{\overset{?}{\ge}\ (2)}\ 0$$

$$\text{and}\ \because\ (1174R-377r)(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\ge}0\ \therefore\ \text{in order to prove (2),}$$

$$\text{it suffices to prove : LHS of (2)}\overset{?}{\ge}(1174R-377r)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow(16200R^2-11030Rr+2089r^2)s^4-$$
$$\begin{pmatrix}64(7027R^3-6824R^2r+2222Rr^2-236r^3)-\\16R^3+4R^2r-8Rr^2+30r^3\end{pmatrix}s^2+$$
$$r^2\begin{pmatrix}128(23814R^4-30000R^3r+14164R^2r^2-3012Rr^3+236r^4)-\\32R^3r+72R^2r^2-2Rr^3-7r^4\end{pmatrix}\boxed{\overset{?}{\ge}\ (3)}\ 0\ \text{and}\ \because$$

$$(16200R^2-11030Rr+2089r^2)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0$$
$$\therefore\ \text{in order to prove (3), it suffices to prove :}$$
$$\text{LHS of (3)}\overset{?}{\ge}(16200R^2-11030Rr+2089r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (17172R^3-19557R^2r+8737Rr^2-1454r^3)s^2 \overset{?}{\underset{④}{\geq}} r\begin{pmatrix}64(4293R^4-6155R^3r+3482R^2r^2-877Rr^3+86r^4)+\\ 8R^3r+32R^2r^2-15Rr^3+2r^4\end{pmatrix}$$

$$\text{Now, } (17172R^3-19557R^2r+8737Rr^2-1454r^3)s^2 \overset{\text{Rouche + Euler}}{\geq} \left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq} \text{RHS of } ④$$

$$\Leftrightarrow 6156t^4-17172t^3+11159t^2-3164t+572 \overset{?}{\geq} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(6156t^3-4860t^2+1439t-286) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ④ \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore 3\sum_{cyc} m_a w_a \leq \sqrt{(9s^2-80Rr-2r^2)(4s^2-28Rr+29r^2)}$$

$$\overset{\substack{\text{Mohamed Amine Ben Ajiba} \\ \text{and} \\ \text{Chu-Yang}}}{\leq} \left(\sum_{cyc} n_a\right)\left(\sum_{cyc} m_a\right)$$

$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 74;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)$ and so,

$$(n_a+n_b+n_c)(m_a+m_b+m_c) \geq 3(m_a w_a+m_b w_b+m_c w_c)\ \forall\ ABC,$$
$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4203. In any Δ ABC holds :

$$\sqrt{\sum_{cyc}\frac{bc}{a^2}+\frac{31}{4}-\frac{8r}{R}} \geq \sum_{cyc}\frac{w_a}{a} \geq \sqrt{\sum_{cyc}\frac{bc}{a^2}+\frac{103}{36}+\frac{16r}{9R}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(a^2\cos^2\frac{B-C}{2}\right) = \frac{1}{4Rrs}.\sum_{cyc}\left(a(b+c)^2(-s^2+sa+bc)\right)$$

$$= \frac{1}{4Rrs}.\begin{pmatrix}-s^2(2s(s^2+4Rr+r^2)+12Rrs)+2s((s^2+4Rr+r^2)^2-8Rrs^2)+\\ 8Rrs(2(s^2-4Rr-r^2)+s^2+4Rr+r^2)\end{pmatrix}$$

$$= \frac{(2R+r)s^2+r^2(4R+r)}{2R} \Rightarrow 2\sum_{cyc}\frac{w_b w_c}{bc} = 2.\sum_{cyc}\frac{\frac{2ca}{c+a}\cos\frac{B}{2}.\frac{2ab}{a+b}\cos\frac{C}{2}}{bc}$$

$$=\frac{4\sum_{cyc}\frac{a^2\cos^2\frac{B-C}{2}}{\cos\frac{B-C}{2}}}{s^2+2Rr+r^2}\overset{0<\cos\frac{B-C}{2}\le 1}{\geq}\frac{4\sum_{cyc}\left(a^2\cos^2\frac{B-C}{2}\right)}{s^2+2Rr+r^2}=\frac{4.\frac{(2R+r)s^2+r^2(4R+r)}{2R}}{s^2+2Rr+r^2}$$

$$\Rightarrow 2\sum_{cyc}\frac{w_bw_c}{bc}\geq\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}\therefore\left(\sum_{cyc}\frac{w_a}{a}\right)^2\geq\sum_{cyc}\frac{bc}{a^2}-$$

$$\frac{\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)}{4s^2(s^2+2Rr+r^2)^2}+\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}.\begin{pmatrix}4Rrs.2s(s^2-6Rr-3r^2)+\\2(s^2+4Rr+r^2).4Rrs.2s+(s^2+4Rr+r^2)^3\end{pmatrix}+$$

$$\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}\overset{?}{\geq}\sum_{cyc}\frac{bc}{a^2}+\frac{103}{36}+\frac{16r}{9R}\Leftrightarrow(32R+8r)s^6-$$

$$r(448R^2-87Rr-16r^2)s^4-r^2(988R^3+236R^2r-46Rr^2-8r^3)s^2-$$

$$9Rr^3(4R+r)^3\boxed{\overset{?}{\geq}\ (1)}\ 0\ and\because P=(32R+8r)(s^2-16Rr+5r^2)^3+$$

$$r(1088R^2-9Rr-104r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore\text{in order to prove (1),}$$

$$\text{it suffices to prove : LHS of (1)}\overset{?}{\geq}P\Leftrightarrow(2313R^3-547R^2r-438Rr^2+112r^3)s^2$$

$$\boxed{\overset{?}{\geq}\ (2)}\ r(37008R^4-21460R^3r-1389R^2r^2+2706Rr^3-400r^4)$$

$$\text{LHS of (2)}\overset{\text{Gerretsen}}{\geq}(2313R^3-547R^2r-438Rr^2+112r^3)(16Rr-5r^2)\overset{?}{\geq}$$

$$\text{RHS of (2)}\Leftrightarrow 1143t^3-2884t^2+1276t-160\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(1143t^2-598t+80)\overset{?}{\geq}0\to\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$w_a\geq h_a\therefore\left(\sum_{cyc}\frac{w_a}{a}\right)^2\leq\sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{(b+c)^2}+\frac{2.16Rr^2s^2}{(s^2+2Rr+r^2)(4Rrs)}.\sum_{cyc}\frac{a^2}{2rs}$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{8Rrs^2(s^2-6Rr-3r^2)+(s^2+4Rr+r^2).16Rrs^2+(s^2+4Rr+r^2)^3}{4s^2(s^2+2Rr+r^2)^2}+$$

$$\frac{8(s^2-4Rr-r^2)}{s^2+2Rr+r^2}\overset{?}{\leq}\sum_{cyc}\frac{bc}{a^2}+\frac{31}{4}-\frac{8r}{R}\Leftrightarrow-32s^6+(224R^2-63Rr-64r^2)s^4+$$

$$r(444R^3+204R^2r-62Rr^2-32r^3)s^2+Rr^2(4R+r)^3\boxed{\overset{?}{\geq}\ (3)}\ 0$$

$$\text{Now, }Q=-32s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{\text{Double-Rouche}}{\geq}0$$

$\therefore$ in order to prove ③, it suffices to prove : LHS of ③ $\overset{?}{\geq}$ Q $\Leftrightarrow (96R - 703r)s^4 +$

$r(2492R^2 + 1740Rr + 322r^2)s^2 + r^2(4R + r)^3 \boxed{\overset{?}{\underset{④}{\geq}}}$ 0 and it's trivially true if :

$96R - 703r \geq 0$ and when : $96R - 703r < 0$, then via $(*)$, $Q' =$
$(96R - 703r)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \geq 0 \therefore$ to prove ④,

it suffices to prove : LHS of ④ $\overset{?}{\geq} Q' \Leftrightarrow (12R^3 + 50R^2r - 391Rr^2 + 54r^3)s^2$

$\boxed{\overset{?}{\underset{⑤}{\geq}}} r(192R^4 - 1264R^3r - 1020R^2r^2 - 261Rr^3 - 22r^4)$

$\boxed{\textit{Case 1}}$ $12R^3 + 50R^2r - 391Rr^2 + 54r^3 \geq 0$ and then : LHS of ⑤ $\overset{\text{Gerretsen}}{\geq}$
$(12R^3 + 50R^2r - 391Rr^2 + 54r^3)(16Rr - 5r^2) \overset{?}{\geq}$ RHS of ⑤
$\Leftrightarrow 1002t^3 - 2743t^2 + 1540t - 124 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$
$\Leftrightarrow (t - 2)(1002t^2 - 739t + 62) \overset{?}{\geq} 0 \to$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ⑤ is true

$\boxed{\textit{Case 2}}$ $12R^3 + 50R^2r - 391Rr^2 + 54r^3 < 0$ and then : LHS of ⑤ $\overset{\text{Gerretsen}}{\geq}$
$(12R^3 + 50R^2r - 391Rr^2 + 54r^3)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of ⑤
$\Leftrightarrow 24t^5 + 28t^4 - 32t^3 - 89t^2 - 348t + 92 \overset{?}{\geq} 0 \Leftrightarrow$
$(t - 2)(24t^4 + 76t^3 + 120t^2 + 151t - 46) \overset{?}{\geq} 0 \to$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ⑤ is true

$$\therefore \sqrt{\sum_{\text{cyc}} \frac{bc}{a^2} + \frac{31}{4} - \frac{8r}{R}} \geq \sum_{\text{cyc}} \frac{w_a}{a} \geq \sqrt{\sum_{\text{cyc}} \frac{bc}{a^2} + \frac{103}{36} + \frac{16r}{9R}}, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

4204. In any Δ ABC the following relationship holds :

$$\sum_{\text{cyc}} \frac{bc}{a^2} - \frac{r}{R} - \frac{1}{4} \geq \sum_{\text{cyc}} \frac{w_a^2}{a^2} \geq \sum_{\text{cyc}} \frac{bc}{a^2} + \frac{r}{9R} - \frac{29}{36}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} \frac{w_a^2}{a^2} = \sum_{\text{cyc}} \frac{4bc.s(s-a)}{a^2(b+c)^2} = \sum_{\text{cyc}} \frac{bc((b+c)^2 - a^2)}{a^2(b+c)^2} = \sum_{\text{cyc}} \frac{bc}{a^2} - \sum_{\text{cyc}} \frac{bc}{(b+c)^2}$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}\cdot\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}\cdot\left(\begin{array}{c}4Rrs.2s(s^2-6Rr-3r^2)+\\2(s^2+4Rr+r^2).4Rrs.2s+(s^2+4Rr+r^2)^3\end{array}\right)$$

$$\overset{?}{\leq}\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}\Leftrightarrow -4s^6+(32R^2-15Rr-8r^2)s^4+$$

$$r(60R^3-4R^2r-14Rr^2-4r^3)s^2+Rr^2(4R+r)^3\ \boxed{\underset{(1)}{\overset{?}{\geq}}}\ 0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ *and* $s^2-(m+n)\leq 0$, where $m=$ $2R^2+10Rr-r^2$ & $n=2(R-2r)\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(*)}{\leq}0$$

$$\Rightarrow P=-4s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\overset{?}{\geq}$ P

$$\Leftrightarrow(16R-95r)s^4+r(316R^2+188Rr+34r^2)s^2+r^2(4R+r)^3\ \boxed{\underset{(2)}{\overset{?}{\geq}}}\ 0$$

and it's trivi*ally* true if : $16R-95r\geq 0$ *and* when : $16R-95r<0$, *then via* $(*)$, $Q=(16R-95r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0\therefore$ to prove (2),

it suffices to prove : LHS of (2) $\overset{?}{\geq}$ Q $\Leftrightarrow$

$$(4R^3+16R^2r-109Rr^2+14r^3)s^2\ \boxed{\underset{(3)}{\overset{?}{\geq}}}\ r\left(\begin{array}{c}64R^4-336R^3r-276R^2r^2-\\71Rr^3-6r^4\end{array}\right)$$

$\boxed{\textit{Case 1}}$ $4R^3+16R^2r-109Rr^2+14r^3\geq 0$ *and* then : LHS of (3) $\overset{\text{Gerretsen}}{\geq}$

$$(4R^3+16R^2r-109Rr^2+14r^3)(16Rr-5r^2)\overset{?}{\geq}\text{RHS of (3)}$$

$$\Leftrightarrow 143t^3-387t^2+210t-16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(143t^2-101t+8)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow$ (3) is true

$\boxed{\text{Case 2}}$ $4R^3+16R^2r-109Rr^2+14r^3<0$ and then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$$(4R^3+16R^2r-109Rr^2+14r^3)(4R^2+4Rr+3r^2)\overset{?}{\geq}\text{RHS of ③}$$

$$\Leftrightarrow 2t^5+2t^4-3t^3-7t^2-25t+6\overset{?}{\geq}0\Leftrightarrow(t-2)(2t^4+6t^3+9t^2+11t-3)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow \text{③ is true}\therefore\text{combining both cases,}$$

$$\text{③}\Rightarrow\text{②}\Rightarrow\text{①is true}\ \forall\,\Delta\,ABC\therefore\sum_{cyc}\frac{w_a^2}{a^2}\leq\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}$$

$$\text{Again,}\sum_{cyc}\frac{w_a^2}{a^2}=\sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{(b+c)^2}\overset{AM-GM}{\geq}\sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{4bc}=\sum_{cyc}\frac{bc}{a^2}-\frac{3}{4}$$

$$\overset{?}{\geq}\sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36}\Leftrightarrow\frac{1}{18}\overset{?}{\geq}\frac{r}{9R}\Leftrightarrow R\overset{?}{\geq}2r\rightarrow\text{true via Euler}$$

$$\therefore\sum_{cyc}\frac{w_a^2}{a^2}\geq\sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36}\ \text{and so,}$$

$$\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}\geq\sum_{cyc}\frac{w_a^2}{a^2}\geq\sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36}\ \forall\,ABC,$$

$$''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

4205. In any $\Delta\,ABC$ holds :

$$w_aw_b+w_bw_c+w_cw_a\geq\sqrt{3(g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}w_bw_c\right)^2=\sum_{cyc}w_b^2w_c^2+2w_aw_bw_c\left(\sum_{cyc}w_a\right)\geq$$

$$\frac{256R^2r^4s^4}{(s^2+2Rr+r^2)^2}\cdot\frac{(2R+r)s^2+r^2(4R+r)}{8Rr^2s^2}+\frac{32Rr^2s^2}{s^2+2Rr+r^2}\left(\sum_{cyc}h_a\right)$$

$$\left(\because\sum_{cyc}\frac{1}{w_a^2}=\frac{(2R+r)s^2+r^2(4R+r)}{8Rr^2s^2};\text{Reference : Solution to Inequality in Triangle}\atop\text{by Dang Ngoc Minh}-258;\text{published at www.ssmrmh.ro}\right)$$

$$=\frac{32Rr^2s^2\left((2R+r)s^2+r^2(4R+r)\right)+16r^2s^2(s^2+2Rr+r^2)(s^2+4Rr+r^2)}{(s^2+2Rr+r^2)^2}\overset{?}{\geq}3\sum_{cyc}g_b^2g_c^2$$

$$= \frac{3r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R}$$

$\left(\begin{array}{c}\textbf{Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba} - 69;\\ \textbf{published at www.ssmrmh.ro}\end{array}\right)$

$$\Leftrightarrow -(2R+18r)s^6+(16R^3+128R^2r-31Rr^2-30r^3)s^4 -$$
$$r(192R^4-376R^3r-308R^2r^2-40Rr^3+6r^4)s^2 -$$
$$r^2\left(192R^5-96R^4r-420R^3r^2-276R^2r^3-69Rr^4-6r^5\right)\boxed{\overset{?}{\underset{①}{\geq}}}\,0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m =$
$2R^2+10Rr-r^2$ & $n = 2(R-2r)\sqrt{R^2-2Rr}$ $\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(*)}{\leq} 0$$
$$\Rightarrow P = -(2R+18r)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $\overset{?}{\geq}$ P

$$\Leftrightarrow (8R^3+16R^2r-387Rr^2+6r^3)s^4 -$$
$$r(64R^4-1624R^3r-1196R^2r^2-258Rr^3-12r^4)s^2 -$$
$$r^2\left(192R^5-96R^4r-420R^3r^2-276R^2r^3-69Rr^4-6r^5\right)\boxed{\overset{?}{\underset{②}{\geq}}}\,0$$

$\boxed{\textit{Case 1}}$ $8R^3+16R^2r-387Rr^2+6r^3\geq 0$ and then, since $Q =$
$$(8R^3+16R^2r-387Rr^2+6r^3)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\overset{?}{\geq}$ Q

$$\Leftrightarrow (48R^4+514R^3r-2837R^2r^2+1080Rr^3-12r^4)s^2 \boxed{\overset{?}{\underset{③}{\geq}}}$$
$$r\left(560R^5+680R^4r-25463R^3r^2+15895R^2r^3-2676Rr^4+36r^5\right)$$

Now, with $t = \frac{R}{r}$, $8t^3+16t^2-387t+6\geq 0 \Rightarrow (t-6)(8t^2+64t-3)\geq 12 > 0$
$$\Rightarrow t > 6 \Rightarrow 48t^4+514t^3-2837t^2+1080t-12$$
$$= (t-6)(48t^3+802t^2+1975t+12930)+77568 > 0 \therefore \text{LHS of ③} \overset{\text{Gerretsen}}{\geq}$$
$$(48R^4+514R^3r-2837R^2r^2+1080Rr^3-12r^4)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of ③}$$
$$\Leftrightarrow 208t^5+7304t^4-22499t^3+15570t^2-2916t+24 \overset{?}{\geq} 0 \Leftrightarrow$$
$$(t-2)(208t^4+7720t^3-7059t^2+1452t-12) \overset{?}{\geq} 0; \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \text{ is true}$$

$\boxed{\textit{Case 2}}$ $8R^3+16R^2r-387Rr^2+6r^3 < 0$ and then, since $T =$
$$(8R^3+16R^2r-387Rr^2+6r^3)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3) \overset{\text{via } (*)}{\geq} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\overset{?}{\geq}$ T

$$\Leftrightarrow (8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)s^2 \boxed{\overset{?}{\underset{④}{\geq}}}$$

$$r(128R^5+400R^4r-6000R^3r^2-4603R^2r^3-1154Rr^4-96r^5)$$

$\boxed{\text{Case }(2a)}$ $8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4 \geq 0$; then : $LHS_{④} \overset{\text{Gerretsen}}{\geq}$

$$(8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)(16Rr-5r^2) \overset{?}{\geq} RHS_{④}$$

$$\Leftrightarrow 25t^4+915t^3-2760t^2+1744t-168 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)(25t^3+965t^2-830t+84) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ④ \text{ is true}$$

$\boxed{\text{Case }(2b)}$ $8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4 < 0$; then : $LHS_{④} \overset{\text{Gerretsen}}{\geq}$

$$(8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)(4R^2+4Rr+3r^2) \overset{?}{\geq} RHS_{④}$$

$$\Leftrightarrow 8t^6+16t^5+41t^4-13t^3-128t^2-652t+240 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)(8t^5+32t^4+105t^3+197t^2+266t-120) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow ④ \text{ is true} \therefore \text{combining cases } (2a) \text{ and } (2b), ④ \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\forall \Delta\, ABC \text{ and so, } w_a w_b + w_b w_c + w_c w_a \geq \sqrt{3(g_a^2 g_b^2 + g_b^2 g_c^2 + g_c^2 g_a^2)}\ \forall\, ABC,$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4206. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{bc}{m_a^2} \leq \frac{2R}{r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \frac{bc}{m_a^2} \overset{Tereshin}{\leq} \sum_{cyc} \frac{16R^2bc}{(b^2+c^2)^2} \overset{AM-GM}{\leq}$$

$$\sum_{cyc} \frac{16R^2bc}{4b^2c^2} = 4R^2.\sum_{cyc} \frac{1}{bc} = 4R^2.\frac{a+b+c}{abc} = 4R^2.\frac{2p}{4prR} = \frac{2R}{r}$$

$$Equality\ \ holds\ \ for:\ \ a=b=c$$

4207. In ΔABC the following relationship holds:

$$\frac{1}{r} \leq \frac{r_a}{h_a^2} + \frac{r_b}{h_b^2} + \frac{r_c}{h_c^2} \leq \frac{27R^3(R-r)}{8r^3s^2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{r_a}{h_a^2}=\sum_{cyc}\frac{s.\tan\left(\frac{A}{2}\right)}{\frac{4F^2}{a^2}}=\frac{s}{4F^2}\sum_{cyc}a^2\tan\left(\frac{A}{2}\right)=\frac{s}{4F^2}\sum_{cyc}a\cdot a\tan\left(\frac{A}{2}\right)=$$

$$=\frac{s}{4F^2}\sum_{cyc}a(r_a-r)=\frac{s}{4F^2}\left(\sum_{cyc}ar_a-\sum_{cyc}ar\right)=$$

$$=\frac{s}{4F^2}(2s(2R-r)-2sr)=\frac{s}{4F^2}(4sR-4sr)=$$

$$=\frac{s^2(R-r)}{F^2}\overset{Mitrinovici}{\leq}\frac{\frac{27R^2}{4}(R-r)}{S^2r^2}=\frac{27R^2(R-r)}{4S^2r^2}\overset{Euler}{\leq}\frac{27R^3(R-r)}{8S^2r^3}\quad (RHS)$$

$$\frac{r_a}{h_a^2}+\frac{r_b}{h_b^2}+\frac{r_c}{h_c^2}=\frac{\left(\frac{1}{h_a}\right)^2}{\frac{1}{r_a}}+\frac{\left(\frac{1}{h_b}\right)^2}{\frac{1}{r_b}}+\frac{\left(\frac{1}{h_c}\right)^2}{\frac{1}{r_c}}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}\frac{1}{r_a}}=\frac{\frac{1}{r^2}}{\frac{1}{r}}=\frac{1}{r}\quad (LHS)$$

$$Equality\ \ holds\ \ for:\quad a=b=c$$

4208. **In ΔABC the following relationship holds:**

$$\frac{a}{h_b^2}+\frac{b}{h_c^2}+\frac{c}{h_c^2}\geq\frac{4\sqrt{3}}{3R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_b^2}+\frac{b}{h_c^2}+\frac{c}{h_a^2}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{abc}{(h_ah_bh_c)^2}}=3\sqrt[3]{\frac{abc}{\left(\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\right)^2}}=3\sqrt[3]{\frac{(abc)^3}{64F^6}}=$$

$$=3\cdot\frac{abc}{4F^2}=3\cdot\frac{4RF}{4F^2}=\frac{3R}{F}=\frac{3R}{rs}\overset{MITRINOVIC}{\geq}\frac{3R}{r\cdot\frac{3\sqrt{3}}{2}R}=\frac{2}{r\sqrt{3}}\geq\frac{4\sqrt{3}}{3R}\Leftrightarrow$$

$$\Leftrightarrow 6R\geq 12r\Leftrightarrow R\geq 2r\quad \text{(Euler)}$$

Equality holds for $a=b=c$.

4209. In ΔABC the following relationship holds:

$$\frac{a}{h_b^3}+\frac{b}{h_c^3}+\frac{c}{h_a^3}\ge\frac{8\sqrt{3}}{9R^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_b^3}+\frac{b}{h_c^3}+\frac{c}{h_c^3}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{abc}{(h_ah_bh_c)^3}}=\frac{3}{h_ah_bh_c}\cdot\sqrt[3]{4Rrs}\overset{EULER}{\ge}\frac{3}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\sqrt[3]{8r^2s}\ge$$

$$\overset{MITRINOVIC}{\ge}\frac{3abc}{8F^3}\cdot 2\sqrt[3]{r^2\cdot 3\sqrt{3}r}=\frac{3abc}{4F^3}\cdot\sqrt{3}\cdot r=\frac{3\sqrt{3}\cdot 4RFr}{4F^3}=\frac{3\sqrt{3}Rr}{F^2}=$$

$$=\frac{3\sqrt{3}Rr}{r^2s^2}=\frac{3\sqrt{3}R}{rs^2}\ge\frac{8\sqrt{3}}{9R^2}\Leftrightarrow 27\sqrt{3}R^3\ge 8\sqrt{3}rs^2$$

$$27R^3\ge 8rs^2$$

$$27R^3=27R^2\cdot 2r\ge 8rs^2\Leftrightarrow 27R^2\ge 4s^2\Leftrightarrow s^2\le\frac{27R^2}{4}\Leftrightarrow s\le\frac{3\sqrt{3}}{2}R\quad\text{(MITRINOVIC)}$$

Equality holds for $a=b=c$.

4210. In ΔABC the following relationship holds:

$$192\sqrt{3}r^3\le(a+b)(b+c)(c+a)\le 24\sqrt{3}R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

We will use the known result:

$$(a+b)(b+c)(c+a)=2s(s^2+r^2+2Rr)$$

We must prove that:

$$192\sqrt{3}r^3\le 2s(s^2+r^2+2Rr)\le 24\sqrt{3}R^3$$

$$2s(s^2+r^2+2Rr)\overset{MITRINOVIC-GERRETSEN}{\le}2\cdot\frac{3\sqrt{3}R}{2}(4R^2+4Rr+3r^2+r^2+2Rr)=$$

$$=3\sqrt{3}R(4R^2+6Rr+4r^2)\le 3\sqrt{3}R\left(4R^2+6R\cdot\frac{R}{2}+4\cdot\frac{R^2}{4}\right)=$$

$$=3\sqrt{3}R(4R^2+4R^2)=24\sqrt{3}R^3$$

$$2s(s^2+r^2+2Rr)\overset{MITRINOVIC-EULER}{\geq}2\cdot 3\sqrt{3}r(27r^2+r^2+2\cdot 2r\cdot r)=$$

$$=6\sqrt{3}r(28r^2+4r^2)=6\sqrt{3}r\cdot 32r^2=192\sqrt{3}r^3$$

Equality holds for $a=b=c$.

4211. In ΔABC the following relationship holds:

$$\frac{a}{r_a\cos^2\frac{A}{2}}+\frac{b}{r_b\cos^2\frac{B}{2}}+\frac{c}{r_c\cos^2\frac{C}{2}}\geq\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{r_a\cos^2\frac{A}{2}}+\frac{b}{r_b\cos^2\frac{B}{2}}+\frac{c}{r_c\cos^2\frac{C}{2}}=\sum_{cyc}\frac{a}{r_a\cos^2\frac{A}{2}}=$$

$$=\sum_{cyc}\frac{a}{\frac{F}{s-a}\cdot\frac{s(s-a)}{bc}}=\sum_{cyc}\frac{abc}{Fs}=\frac{3abc}{Fs}=\frac{3\cdot 4RF}{Fs}=\frac{12R}{s}\geq\frac{12\cdot\frac{2}{3\sqrt{3}}s}{s}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.

4212. In ΔABC the following relationship holds:

$$(h_a)^{\cos(B)+\cos(C)}.(h_b)^{\cos(A)+\cos(C)}.(h_c)^{\cos(B)+\cos(A)}\leq\left(\frac{3R}{2}\right)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \Delta ABC\ wlog:\ a\leq b\leq c,\quad h_a\geq h_b\geq h_c$$

$$\rightarrow\begin{cases}\cos(A)+\cos(B)\geq\cos(A)+\cos(C)\geq\cos(C)+\cos(B)\\ \cos(A)\geq\cos(B)\geq\cos(C)\end{cases}$$

$$(h_a)^{\cos(B)+\cos(C)}.(h_b)^{\cos(A)+\cos(C)}.(h_c)^{\cos(B)+\cos(A)}\overset{Weighted\ AM-GM}{\leq}$$

$$\left(\frac{h_a(\cos(B)+\cos(C))+h_b(\cos(A)+\cos(C))+h_c(\cos(B)+\cos(A))}{2(\cos(A)+\cos(B)+\cos(C))}\right)^{2\sum\cos(A)}\overset{Chebyshev}{\leq}$$

$$\leq\left(\frac{\frac{1}{3}(h_a+h_b+h_c)(2(\cos(A)+\cos(B)+\cos(C)))}{2(\cos(A)+\cos(B)+\cos(C))}\right)^{2\sum\cos(A)}=$$

$$=\left(\frac{1}{3}(h_a+h_b+h_c)\right)^{2\sum\cos(A)}\le\left(\frac{1}{3}\cdot\frac{9R}{2}\right)^{2\sum\cos(A)}\overset{Jensen}{\lesssim}\left(\frac{3R}{2}\right)^{3}$$

Equality holds if $a=b=c$

4213. In any Δ ABC and $\forall$ m, n $\ge$ 1, the following relationship holds :

$$\sum_{cyc}\frac{a^n+b^n}{h_c^m}\le\sum_{cyc}\frac{a^n+b^n}{r_c^m}$$

Proposed by Prashanna Khanal, Jenish Rijal-Nepal

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then : $b^n+c^n\le c^n+a^n\le a^n+b^n$ $(\because n\ge 1>0)$ and $\frac{1}{h_a^m}\ge\frac{1}{h_b^m}\ge\frac{1}{h_c^m}$ $(\because m\ge 1>0)$ and also, $\frac{1}{r_a^m}\le\frac{1}{r_b^m}\le\frac{1}{r_c^m}$

$$\left(\because\frac{1}{r_a}=\frac{1}{s\tan\frac{A}{2}}\le\frac{1}{r_b}=\frac{1}{s\tan\frac{B}{2}}\le\frac{1}{r_c}=\frac{1}{s\tan\frac{C}{2}}\text{ and }\because m\ge 1>0\right)\text{ and so,}$$

$$\sum_{cyc}\frac{b^n+c^n}{h_a^m}\overset{Chebyshev}{\le}\frac{1}{3}\left(\sum_{cyc}(b^n+c^n)\right)\left(\sum_{cyc}\frac{1}{h_a^m}\right)\to(1)\text{ and also,}$$

$$\sum_{cyc}\frac{b^n+c^n}{r_a^m}\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum_{cyc}(b^n+c^n)\right)\left(\sum_{cyc}\frac{1}{r_a^m}\right)\to(2)$$

$\therefore$ (1) and (2) $\Rightarrow$ it suffices to prove : $\sum_{cyc}\frac{1}{h_a^m}\overset{?}{\le}\sum_{cyc}\frac{1}{r_a^m}\Leftrightarrow 2^m.\sum_{cyc}(s-a)^m\overset{?}{\ge}\sum_{cyc}a^m$

$$\Leftrightarrow 2^m.\sum_{cyc}x^m\boxed{\overset{?}{\underset{(*)}{\ge}}}\sum_{cyc}(y+z)^m$$

$(x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x, c=x+y)$

We note that : for $m=1$, LHS of $(*)=2\sum_{cyc}x$ and RHS of $(*)=$

$\sum_{cyc}(y+z)=2\sum_{cyc}x=$ LHS of $(*)$ and when : $m>1$, we have :

via Holder's Inequality, $y^m+z^m\ge\frac{1}{2^{m-1}}.(y+z)^m$ and analogs

$$\Rightarrow\sum_{cyc}(y^m+z^m)\ge\frac{1}{2^{m-1}}.\sum_{cyc}(y+z)^m\Rightarrow 2^m.\sum_{cyc}x^m\ge\sum_{cyc}(y+z)^m$$

$\Rightarrow(*)$ is true and so, $\sum_{cyc}\frac{a^n+b^n}{h_c^m}\le\sum_{cyc}\frac{a^n+b^n}{r_c^m}$ $\forall\,\Delta$ ABC and $\forall$ m, n $\ge$ 1,

$'' = ''$ iff Δ ABC is equilateral (QED)

4214. In ΔABC the following relationship holds:

$$\frac{3r^2}{2R^2} \le \sum \frac{rr_a}{b^2+c^2} \le \frac{2R-r}{8r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \frac{rr_a}{b^2+c^2} \overset{AM-GM}{\le} \sum_{cyc} \frac{rr_a}{2bc} = \frac{r}{2}\sum_{cyc} \frac{r_a}{bc} =$$

$$= \frac{r}{2}\left(\frac{ar_a+br_b+cr_c}{abc}\right) = \frac{r.2p(2R-r)}{2.4pr.R} = \frac{2R-r}{4R} \overset{Euler}{\le} \frac{2R-r}{8r} \quad (RHS)$$

$$\sum \frac{rr_a}{b^2+c^2} = r\sum_{cyc} \frac{r_a}{b^2+c^2}$$

$$In\ \Delta ABC\ wlog:\ \begin{cases} a \le b \le c \to r_a \le r_b \le r_c \\ b^2+c^2 \ge a^2+c^2 \ge b^2+a^2 \to \dfrac{1}{b^2+c^2} \le \dfrac{1}{a^2+c^2} \le \dfrac{1}{b^2+a^2} \end{cases}$$

$$\sum_{cyc} \frac{r_a}{b^2+c^2} \overset{Chebyshev}{\ge} \frac{1}{3}\sum_{cyc} r_a \cdot \sum_{cyc} \frac{1}{b^2+c^2} == \frac{1}{3}(4R+r).\sum_{cyc} \frac{1^2}{b^2+c^2} \overset{Bergstrom}{\ge}$$

$$\ge \frac{4R+r}{3}\cdot\frac{(1+1+1)^2}{2(b^2+c^2+a^2)} = \frac{4R+r}{3}\cdot\frac{9}{2(b^2+c^2+a^2)} \overset{Euler\ Leibniz}{\ge} \frac{9r}{3}\cdot\frac{9}{2.9R^2} = \frac{3r}{2R^2}$$

$$\sum \frac{rr_a}{b^2+c^2} \ge \frac{3r^2}{2R^2} \quad (LHS)$$

$$Equality\ holds\ for:\ \ a = b = c$$

4215. In any Δ ABC the following relationship holds:

$$\sum_{cyc} \frac{m_b - m_c}{(m_a+m_b)(2\sin^2 A + 2\sin^2 B - \sin^2 C)} \ge 0$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{?}{\geq} \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = 3+\sum_{cyc}\frac{x+y}{z}$$

$$= 3+\sum_{cyc}\frac{y}{x}+\sum_{cyc}\frac{x}{y} \Leftrightarrow \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \underset{①}{\overset{?}{\geq}} 3+\sum_{cyc}\frac{x}{y}$$

$$\text{Now, } \sum_{cyc}\frac{x^2}{y^2} \geq \frac{1}{3}\left(\sum_{cyc}\frac{x}{y}\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3.\sum_{cyc}\frac{x}{y} \Rightarrow \sum_{cyc}\frac{x^2}{y^2} \geq \sum_{cyc}\frac{x}{y} \text{ and } \because \sum_{cyc}\frac{y}{x} \overset{AM-GM}{\geq} 3$$

$$\therefore \sum_{cyc}\frac{x^2}{y^2}+\sum_{cyc}\frac{y}{x} \geq \sum_{cyc}\frac{x}{y}+3 \Rightarrow ① \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{x}{y}\right)^2 \geq \frac{1}{xyz}\left(3xyz+\sum_{cyc}(xy(x+y))\right) = \frac{1}{xyz}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) \text{ and with}$$

$$x \equiv a, y \equiv b, z \equiv c, \text{ we get : } \left(\sum_{cyc}\frac{a}{b}\right)^2 \geq \frac{1}{abc}.\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) \rightarrow (m) \text{ and now,}$$

$$\text{we seek to prove : } \sum_{cyc}\frac{b-c}{c^2(a+b)} \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc}\frac{(a^2b^4-a^2b^2c^2)(c+a)}{a^2b^2c^2(a+b)(b+c)(c+a)} \overset{?}{\geq} 0$$

$$\Leftrightarrow abc.\sum_{cyc}ab^3+\sum_{cyc}a^3b^4 \boxed{\underset{(*)}{\overset{?}{\geq}}} 2a^2b^2c^2.\left(\sum_{cyc}a\right)$$

$$\text{Now, } \sum_{cyc}a^3b^4 = a^2b^2c^2.\sum_{cyc}\frac{ab^2}{c^2} = a^2b^2c^2.\sum_{cyc}\frac{\left(\frac{b}{c}\right)^2}{\frac{1}{a}} \overset{Bergstrom}{\geq} a^2b^2c^2.\frac{abc.\left(\sum_{cyc}\frac{a}{b}\right)^2}{\sum_{cyc}ab}$$

$$\overset{via\ (m)}{\geq} a^2b^2c^2.\frac{abc.\frac{1}{abc}.\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)}{\sum_{cyc}ab} \therefore \sum_{cyc}a^3b^4 \geq a^2b^2c^2.\left(\sum_{cyc}a\right) \rightarrow (i)$$

$$\text{and again, } abc.\sum_{cyc}ab^3 = a^2b^2c^2.\sum_{cyc}\frac{b^2}{c} \overset{Bergstrom}{\geq} a^2b^2c^2.\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}a}$$

$$\therefore abc.\sum_{cyc}ab^3 \geq a^2b^2c^2.\left(\sum_{cyc}a\right) \rightarrow (ii) \text{ and so, } (i)+(ii) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{b-c}{c^2(a+b)} \geq 0 \ \& \text{ implementing it on a triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3},$$

$$\text{we arrive at}: \sum_{cyc}\frac{\frac{2}{3}\cdot(m_b-m_c)}{\frac{2}{27}\cdot(4m_c^2)(m_a+m_b)}\geq 0\Rightarrow\sum_{cyc}\frac{m_b-m_c}{(m_a+m_b)(2a^2+2b^2-c^2)}\geq 0$$

$$\Rightarrow\sum_{cyc}\frac{m_b-m_c}{(m_a+m_b)\left(4R^2(2\sin^2 A+2\sin^2 B-\sin^2 C)\right)}\text{ and so,}$$

$$\sum_{cyc}\frac{m_b-m_c}{(m_a+m_b)(2\sin^2 A+2\sin^2 B-\sin^2 C)}\geq 0\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4216. **In any Δ ABC the following relationship holds :**

$$\sum_{cyc}\sin\frac{A}{2}\leq\sqrt{2+\frac{R}{8r}}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2=\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=\frac{2R-r}{2R}+\left(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(\sum_{cyc}\operatorname{cosec}\frac{A}{2}\right)$$

$$=\frac{2R-r}{2R}+\frac{r}{2R}\cdot\sum_{cyc}\sqrt{\frac{bc(s-a)}{r^2s}}\overset{CBS}{\leq}\frac{2R-r}{2R}+\frac{1}{2R.\sqrt{s}}\cdot\sqrt{\sum_{cyc}ab}\cdot\sqrt{\sum_{cyc}(s-a)}$$

$$=\frac{2R-r}{2R}+\frac{1}{2R.\sqrt{s}}\cdot\sqrt{s^2+4Rr+r^2}.\sqrt{s}\overset{Gerretsen}{\leq}\frac{2R-r}{2R}+\frac{1}{2R}\cdot\sqrt{4R^2+8Rr+4r^2}$$

$$=\frac{2R-r}{2R}+\frac{2R+2r}{2R}=\frac{4R+r}{2R}\overset{?}{\leq}2+\frac{R}{8r}\Leftrightarrow R(R+16r)\overset{?}{\geq}4r(4R+r)\Leftrightarrow R^2\overset{?}{\geq}4r^2$$

$$\rightarrow\text{true via Euler}\therefore\sum_{cyc}\sin\frac{A}{2}\leq\sqrt{2+\frac{R}{8r}}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

Solution 2 by Adgozel Adgozalov-Azerbaijan

In any triangle *ABC*, the following identity for half-angles is well-known:

$$\sum \sin\frac{A}{2} = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R}$$

cyc

Therefore, the inequality is equivalent to:

$$\left(1+\frac{r}{R}\right)^2 \le 2 + \frac{R}{8r}$$

Let $x = \frac{R}{r}$**. By Euler's inequality,** $x \ge 2$**. Substituting** x **gives:**

$$\left(1+\frac{1}{x}\right)^2 \le 2 + \frac{x}{8}$$

$$1 + \frac{2}{x} + \frac{1}{x^2} \le 2 + \frac{x}{8} \iff \frac{x}{8} + 1 - \frac{2}{x} - \frac{1}{x^2} \ge 0$$

Multiplying by $8x^2 > 0$**:**

$$x^3 + 8x^2 - 16x - 8 \ge 0$$

Factoring the polynomial:

$$(x-2)(x^2+10x+4) \ge 0$$

Since $x = \frac{R}{r} \ge 2$ **we have** $x-2 \ge 0$ **and** $x^2+10x+4 > 0$**, so the inequality holds directly.**
Equality holds if and only if $x = 2 \iff R = 2r$**, which corresponds to an equilateral triangle.**

4217. In any acute Δ ABC the following relationship holds :

$$\frac{a\tan A}{r_a} + \frac{b\tan B}{r_b} + \frac{c\tan C}{r_c} \ge 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For an acute } \Delta \text{ ABC}, \sum_{\text{cyc}} \tan A \overset{\text{Jensen}}{\ge} 3\sqrt{3} \Rightarrow \prod_{\text{cyc}} \tan A \ge 3\sqrt{3}$$

$$\therefore \text{LHS} \overset{\text{AM-GM}}{\ge} 3.\sqrt[3]{\frac{4Rrs.3\sqrt{3}}{rs^2}} \overset{\text{Mitrinovic}}{\ge} 3.\sqrt[3]{4.\frac{2}{3\sqrt{3}}.3\sqrt{3}} = 6 \ \forall \text{ acute ABC},$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4218.

In any Δ ABC the following relationship holds :

$$\frac{a\cot\frac{A}{2}}{r_a}+\frac{b\cot\frac{B}{2}}{r_b}+\frac{c\cot\frac{C}{2}}{r_c}\ge 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{\text{AM-GM}}{\ge}3.\sqrt[3]{\frac{4Rrs.s^3}{(r_ar_br_c)^2}}=3.\sqrt[3]{\frac{4Rrs^4}{r^2s^4}}\overset{\text{Euler}}{\ge}3.\sqrt[3]{\frac{8r}{r}}=6\ \forall\ \text{ABC},$$
$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

4219. **Prove that in any ΔABC holds:**

$$\sin^6\frac{A}{2}+\sin^6\frac{B}{2}+\sin^6\frac{C}{2}\ge\frac{3}{64}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Adgozel Adgozelov – Azerbaijan

Let A,B,C be the internal angles of ΔABC. Using the fundamental trigonometric identity:

$$\sum\sin^2\frac{A}{2}=1-2\prod\sin\frac{A}{2}$$

By Euler's Inequality ($R\ge 2r$):

$$\prod\sin\frac{A}{2}=\frac{r}{4R}\overset{\text{Euler}}{\le}\frac{1}{8}$$

Therefore:

$$\sum\sin^2\frac{A}{2}=1-2\prod\sin\frac{A}{2}\overset{\text{Euler}}{\ge}1-2\left(\frac{1}{8}\right)=\frac{3}{4}$$

Now, applying the Power Mean Inequality:

$$\sum_{cyc}\sin^6\frac{A}{2}\ge 3\cdot\left(\frac{\sum_{cyc}\sin^2\frac{A}{2}}{3}\right)^3\ge 3\cdot\frac{1}{27}\cdot\left(\frac{3}{4}\right)^3=\frac{3}{64}$$

Equality holds if and only if $A = B = C = 60^\circ$ (ΔABC is equilateral).

4220. In ΔABC the following relationship holds:

$$\frac{a}{r_b^2}+\frac{b}{r_c^2}+\frac{c}{r_a^2}\ge\frac{4\sqrt{3}}{3R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{r_b^2}+\frac{b}{r_c^2}+\frac{c}{r_a^2}\ge 3\sqrt[3]{\frac{abc}{(r_ar_br_c)^2}}=3\sqrt[3]{\frac{abc}{\left(\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\right)^2}}=3\sqrt[3]{\frac{abc}{\left(\frac{F^3s}{F^2}\right)^2}}=$$

$$=3\sqrt[3]{\frac{4RF}{(Fs)^2}}=3\sqrt[3]{\frac{4RF}{F^2s^2}}=3\sqrt[3]{\frac{4R}{Fs^2}}=3\sqrt[3]{\frac{4R}{rs^3}}=\frac{3}{s}\sqrt[3]{\frac{4R}{r}}\overset{EULER}{\ge}\frac{3}{s}\sqrt[3]{\frac{8r}{r}}=$$

$$=\frac{3}{s}\cdot\sqrt[3]{8}\overset{MITRINOVIC}{\ge}\frac{3}{\frac{3\sqrt{3}R}{2}}\cdot 2=\frac{4}{\sqrt{3}R}=\frac{4\sqrt{3}}{3R}$$

Equality holds for $a=b=c$.

4221. In ΔABC the following relationship holds:

$$\frac{a^2}{w_bw_c}+\frac{b^2}{w_cw_a}+\frac{c^2}{w_aw_b}\ge 4$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^2}{w_bw_c}+\frac{b^2}{w_cw_a}+\frac{c^2}{w_aw_a}\ge 3\sqrt[3]{\left(\frac{abc}{w_aw_bw_c}\right)^2}\ge 3\sqrt[3]{\left(\frac{abc}{\sqrt{s(s-a)}\cdot\sqrt{s(s-b)}\cdot\sqrt{s(s-c)}}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{abc}{sF}\right)^2}=3\sqrt[3]{\left(\frac{4RF}{sF}\right)^2}=3\sqrt[3]{\left(\frac{4R}{s}\right)^2}\overset{MITRINOVIC}{\ge}3\sqrt[3]{\left(\frac{4R}{\frac{3\sqrt{3}}{2}R}\right)^2}=3\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2}=$$

$$=3\sqrt[3]{\frac{64}{27}}=3\cdot\frac{4}{3}=4.$$ Equality holds for $a=b=c$.

4222. In ΔABC the following relationship holds:

$$\frac{a^2}{r_ar_b}+\frac{b^2}{r_br_c}+\frac{c^2}{r_cr_a}\geq 4$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^2}{r_ar_b}+\frac{b^2}{r_br_c}+\frac{c^2}{r_cr_a}\geq 3\sqrt[3]{\left(\frac{abc}{r_ar_br_c}\right)^2}=3\sqrt[3]{\left(\frac{4RF}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2}=$$

$$=3\sqrt[3]{\left(\frac{4RF}{\frac{F^3\cdot s}{s(s-a)(s-b)(s-c)}}\right)^2}=3\sqrt[3]{\left(\frac{4RF\cdot F^2}{F^3s}\right)^2}=3\sqrt[3]{\left(\frac{4R}{s}\right)^2}\overset{MITRINOVIC}{\geq}$$

$$\geq 3\sqrt[3]{\left(\frac{4\cdot\frac{2}{3\sqrt{3}}s}{s}\right)^2}=3\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2}=3\sqrt[3]{\frac{64}{27}}=3\cdot\frac{4}{3}=4$$

Equality holds for $a=b=c$.

4223. In ΔABC the following relationship holds:

$$\frac{r_a^2}{h_bh_c}+\frac{r_b^2}{h_ch_a}+\frac{r_c^2}{h_ah_b}\geq 3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a^2}{h_bh_c}+\frac{r_b^2}{h_ch_a}+\frac{r_c^2}{h_ah_b}\geq 3\sqrt[3]{\left(\frac{r_ar_br_c}{h_ah_bh_c}\right)^2}=3\sqrt[3]{\left(\frac{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\right)^2}=$$

$$=\frac{3}{4}\sqrt[3]{\left(\frac{abc}{(s-a)(s-b)(s-c)}\right)^2}=\frac{3}{4}\sqrt[3]{\left(\frac{abcs}{s(s-a)(s-b)(s-c)}\right)^2}=\frac{3}{4}\sqrt[3]{\left(\frac{4RFs}{F^2}\right)^2}=$$

$$=\frac{3}{4}\sqrt[3]{\left(\frac{4Rs}{rs}\right)^2}=\frac{3}{4}\sqrt[3]{\left(\frac{4R}{r}\right)^2}\overset{EULER}{\geq}\frac{3}{4}\sqrt[3]{\left(\frac{8r}{r}\right)^2}=\frac{3}{4}\sqrt[3]{64}=\frac{3}{4}\cdot 4=3$$

Equality holds for $a=b=c$.

4224. In ΔABC the following relationship holds:

$$\frac{a}{w_b}+\frac{b}{w_c}+\frac{c}{w_a}\geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{w_b}+\frac{b}{w_c}+\frac{c}{w_a}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{abc}{w_aw_bw_c}}\geq 3\sqrt[3]{\frac{4RF}{\sqrt{s(s-a)}\cdot\sqrt{s(s-b)}\cdot\sqrt{s(s-c)}}}=$$

$$=3\sqrt[3]{\frac{4RF}{s\cdot F}}=3\sqrt[3]{\frac{4R}{s}}\overset{MITRINOVIC}{\geq}3\sqrt[3]{\frac{4\cdot\frac{2}{3\sqrt{3}}s}{s}}=3\sqrt[3]{\frac{8}{3\sqrt{3}}}=3\sqrt[3]{\left(\frac{2}{\sqrt{3}}\right)^2}=$$

$$=3\cdot\frac{2}{\sqrt{3}}=\frac{6}{\sqrt{3}}=\frac{6\sqrt{3}}{3}=2\sqrt{3}$$

Equality holds for $a=b=c$.

4225. In ΔABC the following relationship holds:

$$\frac{a}{h_a\sin^2\frac{A}{2}}+\frac{b}{h_b\sin^2\frac{B}{2}}+\frac{c}{h_c\sin^2\frac{C}{2}}\geq 8\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\sin^2\frac{A}{2}}+\frac{b}{h_b\sin^2\frac{B}{2}}+\frac{c}{h_c\sin^2\frac{C}{2}}=\sum_{cyc}\frac{a}{h_a\cdot\sin^2\frac{A}{2}}=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\frac{(s-b)(s-c)}{bc}}=$$

$$=\frac{abc}{2F}\sum_{cyc}\frac{a}{(s-b)(s-c)}=\frac{4RF}{2F}\cdot\frac{1}{(s-a)(s-b)(s-c)}\sum_{cyc}a(s-a)=$$

$$=\frac{2Rs}{s(s-a)(s-b)(s-c)}\cdot 2r(4R+r)\overset{EULER}{\geq}\frac{2Rs}{F^2}\cdot 2r(8r+r)=\frac{4Rrs\cdot 9r}{F\cdot rs}=$$

$$=\frac{36Rr}{F}=\frac{36Rr}{rs}=\frac{36R}{s}\overset{MITRINOVIC}{\geq}\frac{36R}{\frac{3\sqrt{3}}{2}R}=\frac{72}{3\sqrt{3}}=\frac{24}{\sqrt{3}}=\frac{24\sqrt{3}}{3}=8\sqrt{3}$$

Equality holds for $a=b=c$.

4226. **In ΔABC the following relationship holds:**

$$\frac{a}{h_a\sin^2 A}+\frac{b}{h_b\sin^2 B}+\frac{c}{h_c\sin^2 C}\geq\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{h_a\sin^2 A}+\frac{b}{h_b\sin^2 B}+\frac{c}{h_c\sin^2 C}=\sum_{cyc}\frac{a}{h_a\sin^2 A}=\sum\frac{a}{\frac{2F}{a}\cdot\left(\frac{a}{2R}\right)^2}=\sum\frac{a\cdot 4R^2}{2F\cdot a}=$$

$$=\frac{4R^2}{2F}\cdot 3=\frac{12R^2}{2F}=\frac{6R^2}{rs}\overset{EULER}{\geq}\frac{6R\cdot 2r}{rs}=\frac{12R}{s}\overset{MITRINOVIC}{\geq}\frac{12R}{\frac{3\sqrt{3}R}{2}}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.

4227. **In any Δ ABC the following relationship holds :**

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}\leq 2\cos\frac{A-B}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}=2\sin\frac{3A+3B}{4}\cos\frac{3A-3B}{4}\leq 2\cos\frac{3A-3B}{4}$$

$$\left(\because 0<\sin\frac{3A+3B}{4}\leq 1\ as\ 0<\frac{3A+3B}{4}<\frac{3\pi}{4}\right)\overset{?}{\leq}2\cos\frac{A-B}{2}=2\cos^2\frac{A-B}{4}-1$$

$$\Leftrightarrow 4\cos^3\theta-3\cos\theta\overset{?}{\leq}2\cos^2\theta-1\ \left(\theta=\frac{A-B}{4}\right)$$

$$\Leftrightarrow (t-1)(2t^2-1+2t^2+2t) \overset{?}{\leq} 0\ (t=\cos\theta) \to \text{true} \because -\frac{\pi}{4} < \theta = \frac{A-B}{4} < \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{2}} < t \leq 1 \Rightarrow 2t^2-1 > 0 \therefore \sin\frac{3A}{2} + \sin\frac{3B}{2} \leq 2\cos\frac{A-B}{2}\ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \cos\frac{A-B}{4} = 1 \wedge \sin\frac{3A+3B}{4} = 1 \Rightarrow \text{iff } A = B \text{ and subsequently iff}$$

$$\sin\frac{3A}{2} = 1 \text{ and} \because 0 < \frac{3A}{2} < \frac{3\pi}{2} \therefore \sin\frac{3A}{2} = 1 \text{ iff } \frac{3A}{2} = \frac{\pi}{2} \Rightarrow \text{iff } A = \frac{\pi}{3} \text{ and so,}$$

$$\text{in conclusion}, '' = '' \text{ iff } A = B = \frac{\pi}{3} \text{(QED)}$$

4228. In any acute Δ ABC with $\tan A + \tan C = 2\tan B$

the following relationship holds : $\cos A + \cos C \leq \frac{3\sqrt{2}}{4}$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\tan A + \tan C = 2\tan B \Rightarrow \frac{\sin A}{\cos A} + \frac{\sin C}{\cos C} = 2\tan B \Rightarrow \frac{\sin(A+C)}{\cos A\cos C} = 2\tan B$$

$$\Rightarrow \frac{\sin B}{\cos A\cos C} = \frac{2\sin B}{\cos B} \Rightarrow 2\cos A\cos C = \cos B \Rightarrow -\cos B + \cos(A-C) = \cos B$$

$$\Rightarrow 2\cos^2\frac{A-C}{2} = 1 + 2\cos B \text{ and so}, (\cos A + \cos C)^2 = \left(2\sin^2\frac{B}{2}\right)(1+2\cos B)$$

$$= (1-\cos B)(1+2\cos B) = \frac{9}{8} - \left(\frac{1}{8} - \cos B + 2\cos^2 B\right) = \frac{9}{8} - \frac{(4\cos B - 1)^2}{8} \leq \frac{9}{8}$$

$$\Rightarrow \boxed{\cos A + \cos C \leq \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}} \left(\because 0 < A, C < \frac{\pi}{2} \Rightarrow \cos A + \cos C > 0\right)$$

$$\text{and } '' = '' \text{ iff } \boxed{\cos B = \frac{1}{4}} \text{ and consequently, iff } \cos(A-C) = \frac{1}{2},$$

$$\text{that is iff } (A-C) = \frac{\pi}{3} \text{ or } (C-A) = \frac{\pi}{3}$$

When $A - C = \frac{\pi}{3}$, then : $\tan A + \tan C = 2\tan B \Rightarrow \tan C + \frac{\sqrt{3}+\tan C}{1-\sqrt{3}\tan C} = 2\sqrt{15}$

and solving this quadratic equation in $''\tan C''$, we get : $\boxed{\tan C = \frac{3\sqrt{5}-6}{\sqrt{3}}}$

$\left(\because C < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}\right)$ and consequently, via $\tan A + \tan C = 2\sqrt{15}$, we get :

$$\boxed{\tan A = \frac{3\sqrt{5}+6}{\sqrt{3}}} \text{ and similarly, if } C - A = \frac{\pi}{3}, \text{ we get :}$$

$$\tan C = \frac{3\sqrt{5}+6}{\sqrt{3}} \text{ and } \tan A = \frac{3\sqrt{5}-6}{\sqrt{3}} \text{ and so, } \cos A + \cos C \le \frac{3\sqrt{2}}{4}$$

$$\forall \text{ acute } \Delta\, ABC \text{ with } \tan A + \tan C = 2\tan B,$$

$$" = " \text{ iff } \left\{\begin{matrix} A = \tan^{-1}\frac{3\sqrt{5}+6}{\sqrt{3}} \\ B = \cos^{-1}\frac{1}{4} \\ C = \tan^{-1}\frac{3\sqrt{5}-6}{\sqrt{3}} \end{matrix}\right\} \text{ or } \left\{\begin{matrix} A = \tan^{-1}\frac{3\sqrt{5}-6}{\sqrt{3}} \\ B = \cos^{-1}\frac{1}{4} \\ C = \tan^{-1}\frac{3\sqrt{5}+6}{\sqrt{3}} \end{matrix}\right\} \text{ (QED)}$$

4229. In ΔABC the following relationship holds:

$$\frac{m_a m_b}{r_c^2} + \frac{m_b m_c}{r_a^2} + \frac{m_c m_a}{r_b^2} \ge 3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a m_b}{r_c^2} + \frac{m_b m_c}{r_a^2} + \frac{m_c m_a}{r_b^2} \overset{AM-GM}{\ge} 3 \cdot \sqrt[3]{\left(\frac{m_a m_b m_c}{r_a r_b r_c}\right)^2} \ge 3 \cdot \sqrt[3]{\left(\frac{r_a r_b r_c}{r_a r_b r_c}\right)^2} = 3$$

Equality holds for $a = b = c$.

4230. In ΔABC the following relationship holds:

$$abc \le 3\sqrt{3}R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$abc = 4RF = 4Rrs \overset{EULER}{\le} 4R \cdot \frac{R}{2} s =$$

$$= 2R^2 \cdot s \overset{MITRINOVIC}{\le} 2R^2 \cdot \frac{3\sqrt{3}}{2} R = 3\sqrt{3}R^3$$

Equality holds for $a = b = c$.

4231. In ΔABC the following relationship holds:

$$\frac{GA\cdot GB}{r_c^2}+\frac{GB\cdot GC}{r_a^2}+\frac{GC\cdot GA}{r_b^2}\ge\frac{4}{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{GA\cdot GB}{r_c^2}+\frac{GB\cdot GC}{r_a^2}+\frac{GC\cdot GA}{r_b^2}\ge$$

$$\ge 3\cdot\sqrt[3]{\left(\frac{GA\cdot GB\cdot GC}{r_ar_br_c}\right)^2}=3\sqrt[3]{\left(\frac{\frac{2}{3}m_a\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c}{r_ar_br_c}\right)^2}=$$

$$=3\cdot\left(\frac{2}{3}\right)^2\cdot\sqrt[3]{\left(\frac{m_am_bm_c}{r_ar_br_c}\right)^2}=3\cdot\frac{4}{9}\cdot\sqrt[3]{\left(\frac{r_ar_br_c}{r_ar_br_c}\right)^2}=\frac{4}{3}$$

Equality holds for $a=b=c$.

4232. In ΔABC the following relationship holds:

$$\frac{GA}{r_b}+\frac{GB}{r_c}+\frac{GC}{r_a}\ge 2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{GA}{r_b}+\frac{GB}{r_c}+\frac{GC}{r_a}\ge 3\sqrt[3]{\frac{GA\cdot GB\cdot GC}{r_ar_br_c}}=$$

$$=3\sqrt[3]{\frac{\frac{2}{3}m_a\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c}{r_ar_br_c}}=3\cdot\frac{2}{3}\sqrt[3]{\frac{m_am_bm_c}{r_ar_br_c}}\ge 2\sqrt[3]{\frac{r_ar_br_c}{r_ar_br_c}}=2$$

Equality holds for $a=b=c$.

4233. In any Δ ABC the following relationship holds :

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\le\frac{3(r_a+r_b+r_c)}{a+b+c}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a}{a}=\sum_{cyc}\frac{s\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=\frac{s}{4R}\cdot\sum_{cyc}\sec^2\frac{A}{2}=\frac{s}{4R}\cdot\frac{s^2+(4R+r)^2}{s^2}$$

$$\overset{Doucet}{\le}\frac{1}{4R}\cdot\frac{\frac{(4R+r)^2}{3}+(4R+r)^2}{s}\overset{Euler}{\le}\frac{(4R+r)(9R)}{3R(a+b+c)}=\frac{3(r_a+r_b+r_c)}{a+b+c},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

4234. **In any Δ ABC such that $r_a\neq r_b\neq$**

r_c, the following relationship holds :

$$2s^2<\frac{r_a^4}{(r_a-r_b)(r_a-r_c)}+\frac{r_b^4}{(r_b-r_c)(r_b-r_a)}+\frac{r_c^4}{(r_c-r_a)(r_c-r_b)}<\frac{27}{2}(3R^2-8r^2)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x\neq y\neq z,\ \sum_{cyc}\frac{x^4}{(x-y)(x-z)}$$

$$=\frac{1}{(x-y)(y-z)(z-x)}\cdot\left(-x^4(y-z)-y^4(z-x)-z^4(x-y)\right)$$

$$\therefore\sum_{cyc}\frac{x^4}{(x-y)(x-z)}=-\frac{x^4(y-z)+y^4(z-x)+z^4(x-y)}{(x-y)(y-z)(z-x)}\to (1)$$

$$\text{Now, } x^4(y-z)+y^4(z-x)+z^4(x-y)=$$

$$x^4(y-z)-x(y-z)(y^3+y^2z+yz^2+z^3)+yz(y-z)(y^2+yz+z^2)$$

$$=(y-z)\left(x(x^3-y^3)-y^2z(x-y)-yz^2(x-y)-z^3(x-y)\right)$$

$$=(y-z)(x-y)(x^3+x^2y+xy^2-y^2z-yz^2-z^3)$$

$$=(y-z)(x-y)\left((x-z)(x^2+xz+z^2)+y(x-z)(x+z)+y^2(x-z)\right)$$

$$\therefore \sum_{cyc} x^4(y-z) = -(x-y)(y-z)(z-x)\left(\sum_{cyc} x^2 + \sum_{cyc} xy\right) \to (2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \sum_{cyc} \frac{x^4}{(x-y)(x-z)} = \sum_{cyc} x^2 + \sum_{cyc} xy \to (3) \text{ and via}$$

$$x \equiv r_a, y \equiv r_b, z \equiv r_c, \text{ we get :}$$

$$\frac{r_a^4}{(r_a-r_b)(r_a-r_c)} + \frac{r_b^4}{(r_b-r_c)(r_b-r_a)} + \frac{r_c^4}{(r_c-r_a)(r_c-r_b)} = \sum_{cyc} r_a^2 + \sum_{cyc} r_a r_b$$

$$= (4R+r)^2 - s^2 \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{<} 16R^2 + 4R^2 + r^2 - 27r^2 = \frac{81R^2}{2} - \frac{41R^2}{2} - 26r^2$$

$$\overset{\text{Euler}}{<} \frac{81R^2}{2} - \frac{41.4r^2}{2} - 26r^2 = \frac{81R^2}{2} - 108r^2 = \frac{27}{2}(3R^2 - 8r^2)$$

$$(\text{equality doesn't occur because } r_a \neq r_b \neq r_c \Rightarrow a \neq b \neq c) \text{ and also,}$$

$$\sum_{cyc} \frac{r_a^4}{(r_a-r_b)(r_a-r_c)} = (4R+r)^2 - s^2 \overset{\text{Doucet or Trucht}}{>} 3s^2 - s^2 = 2s^2 \text{ and so,}$$

$$2s^2 < \frac{r_a^4}{(r_a-r_b)(r_a-r_c)} + \frac{r_b^4}{(r_b-r_c)(r_b-r_a)} + \frac{r_c^4}{(r_c-r_a)(r_c-r_b)} < \frac{27}{2}(3R^2 - 8r^2) \ \forall \Delta ABC \ (QED)$$

4235.

In any ΔABC such that $a \neq b \neq c$, the following relationship holds :

$$72r^2 < \frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} < 18R^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{x^4}{(x-y)(x-z)}$$

$$= \frac{1}{(x-y)(y-z)(z-x)} . \left(-x^4(y-z) - y^4(z-x) - z^4(x-y)\right)$$

$$\therefore \sum_{cyc}\frac{x^4}{(x-y)(x-z)} = -\frac{x^4(y-z)+y^4(z-x)+z^4(x-y)}{(x-y)(y-z)(z-x)} \to (1)$$

$$\text{Now, } x^4(y-z)+y^4(z-x)+z^4(x-y) =$$
$$x^4(y-z)-x(y-z)(y^3+y^2z+yz^2+z^3)+yz(y-z)(y^2+yz+z^2)$$
$$=(y-z)\left(x(x^3-y^3)-y^2z(x-y)-yz^2(x-y)-z^3(x-y)\right)$$
$$=(y-z)(x-y)(x^3+x^2y+xy^2-y^2z-yz^2-z^3)$$
$$=(y-z)(x-y)\left((x-z)(x^2+xz+z^2)+y(x-z)(x+z)+y^2(x-z)\right)$$
$$\therefore \sum_{cyc}x^4(y-z) = -(x-y)(y-z)(z-x)\left(\sum_{cyc}x^2+\sum_{cyc}xy\right) \to (2)$$
$$\therefore (1) \text{ and } (2) \Rightarrow \sum_{cyc}\frac{x^4}{(x-y)(x-z)} = \sum_{cyc}x^2+\sum_{cyc}xy \to (3) \text{ and via}$$
$$x \equiv a, y \equiv b, z \equiv c, \text{ we get :}$$
$$\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-c)(b-a)}+\frac{c^4}{(c-a)(c-b)} = \sum_{cyc}a^2+\sum_{cyc}ab <$$
$$2\sum_{cyc}a^2 \overset{\text{Leibnitz}}{<} 18R^2 \text{ and also, } \sum_{cyc}\frac{a^4}{(a-b)(a-c)} = \sum_{cyc}a^2+\sum_{cyc}ab > 2\sum_{cyc}ab$$
$$\overset{\text{Gordon}}{>} 8\sqrt{3}.rs \overset{\text{Mitrinovic}}{>} 8\sqrt{3}.r.3\sqrt{3}.r = 72r^2 \left(\begin{matrix}\text{equality doesn't occur because}\\ a \neq b \neq c\end{matrix}\right)$$
$$\text{and so, } 72r^2 < \frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-c)(b-a)}+\frac{c^4}{(c-a)(c-b)} < 18R^2$$
$$\forall \Delta\, ABC \text{ (QED)}$$

4236. In ΔABC the following relationship holds:

$$\frac{\sqrt{3}r^2}{R^3} \le \frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(a+c)(b+c)} \le \frac{\sqrt{3}R^2}{32r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(a+c)(b+c)} =$$
$$\frac{a}{(2p-c)(2p-b)}+\frac{b}{(2p-a)(2p-c)}+\frac{c}{(2p-a)(2p-b)} =$$
$$\frac{a(2p-a)+b(2p-b)+c(2p-c)}{(2p-a)(2p-c)(2p-b)} = \frac{4p^2-(a^2+b^2+c^2)}{(a+b)(b+c)(a+c)} \overset{AM-GM}{\le}$$
$$\frac{4p^2-2(p^2-r^2-4Rr)}{8abc} = \frac{2p^2+2r^2+8Rr}{32prR} \overset{\text{Gerretsen,Mitrinovici,Euler}}{\le}$$

$$\frac{8R^2+8Rr+6r^2+2r^2+8Rr}{32.3\sqrt{3}r.2r.r}=\frac{8(R^2+2rR+r^2)}{192\sqrt{3}r^3}\le$$

$$\frac{(R+r)^2}{24\sqrt{3}r^3}\le\frac{3R^2}{32\sqrt{3}r^3}=\frac{3\sqrt{3}R^2}{3.32r^3}=\frac{\sqrt{3}R^2}{32r^3}\ (RHS)$$

$$\sum_{cyc}\frac{a}{(a+b)(a+c)}=\frac{4p^2-(a^2+b^2+c^2)}{(a+b)(b+c)(a+c)}=$$

$$\frac{2p^2+2r^2+8Rr}{(a+b)(b+c)(a+c)}\overset{AM-GM}{\ge}\frac{2p^2+2r^2+8Rr}{\left(\frac{(a+b)+(a+c)+(b+c)}{3}\right)^3}=$$

$$\frac{54(p^2+r^2+4Rr)}{64p^3}\overset{Gerretsen,Mitrinovici}{\ge}\frac{27(12Rr+3r^2+r^2+4Rr)}{32.\frac{81\sqrt{3}R^3}{8}}=$$

$$\frac{16Rr+4r^2}{12\sqrt{3}R^3}\overset{Euler}{\ge}\frac{36r^2}{12\sqrt{3}R^3}=\frac{\sqrt{3}r^2}{R^3}\ (LHS)$$

$$Equality\ holds\ for:\quad a=b=c$$

4237.

In any Δ ABC such that $a\neq b\neq c$, the following relationship holds:

$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}<\frac{3\sqrt{3}R}{2r}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}$$

$$=\frac{-rs}{(a-b)(b-c)(c-a)r^2s}\cdot\sum_{cyc}\left((b+c)(b-c)(s-b)(s-c)\right)$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\sum_{cyc}\left((b^2-c^2)(-s^2+sa+bc)\right)$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\begin{pmatrix}-s^2\sum_{cyc}(b^2-c^2)+s\sum_{cyc}\left(a(b^2-c^2)\right)+\\ \sum_{cyc}\left(bc(b^2-c^2)\right)\end{pmatrix}$$

$$\therefore\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}$$

$$=\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\left(s\sum_{cyc}\left(a(b^2-c^2)\right)+\sum_{cyc}\left(bc(b^2-c^2)\right)\right)\to ①$$

$$\text{Now,}\sum_{cyc}\left(a(b^2-c^2)\right)=ab(b-a)+c^2(b-a)-c(b+a)(b-a)$$

$$=(b-a)\left(b(a-c)-c(a-c)\right)=(a-b)(b-c)(c-a)=\sum_{cyc}\left(a(b^2-c^2)\right)\to ②$$

$$\sum_{cyc}\left(bc(b^2-c^2)\right)=bc(b-c)(b+c)-a(b-c)(b^2+bc+c^2)+a^3(b-c)$$

$$=(b-c)\left(b^2(c-a)+bc(c-a)-a(c-a)(c+a)\right)$$

$$=(b-c)(c-a)\left((b+a)(b-a)+c(b-a)\right)=-2s(a-b)(b-c)(c-a)$$

$$=\sum_{cyc}\left(bc(b^2-c^2)\right)\to ③\text{ and so, }①,②\text{ and }③\Rightarrow LHS=$$

$$\frac{-1}{(a-b)(b-c)(c-a)r}\cdot\left(s(a-b)(b-c)(c-a)-2s(a-b)(b-c)(c-a)\right)$$

$$=\frac{s}{r}\overset{\text{Mitrinovic}}{<}\frac{3\sqrt{3}R}{2r};\text{ equality doesn't occur because }a\neq b\neq c\text{ and so,}$$

$$\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}<\frac{3\sqrt{3}R}{2r}\ \forall\,\Delta\,ABC\ (QED)$$

4238. In any $\Delta\,ABC$ such that $a\neq b\neq c$, the following relationship holds :

$$\frac{1}{27R^6}<\frac{1}{(a-b)(a-c)a^2bc}+\frac{1}{(b-c)(b-a)b^2ca}+\frac{1}{(c-a)(c-b)c^2ab}<\frac{1}{1728r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For }x\neq y\neq z,\sum_{cyc}\frac{1}{(x-y)(x-z)x^2yz}$$

$$=\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}\cdot\left(-yz(y-z)-zx(z-x)-xy(x-y)\right)$$

$$=-\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}\cdot\left(yz(y-z)-x(y-z)(y+z)+x^2(y-z)\right)$$

$$=-\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}\cdot(y-z)\left(y(z-x)-x(z-x)\right)$$

$$=\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}\cdot(y-z)(z-x)(x-y)$$

$$\therefore \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz} = \frac{1}{x^2y^2z^2} \text{ and via } x \equiv a, y \equiv b, z \equiv c,$$

$$\frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ab} = \frac{1}{a^2b^2c^2}$$

$$= \frac{1}{16R^2r^2s^2} \text{ and } \because \text{ via Mitrinovic and Euler,}$$

$$\frac{1}{16R^2.\frac{R^2}{4}.\frac{27R^2}{4}} < \frac{1}{16R^2r^2s^2} < \frac{1}{64r^2.r^2.27r^2}$$

$$(\text{equality doesn't occur because } a \neq b \neq c) \therefore \frac{1}{27R^6} <$$

$$\frac{1}{(a-b)(a-c)a^2bc} + \frac{1}{(b-c)(b-a)b^2ca} + \frac{1}{(c-a)(c-b)c^2ab} < \frac{1}{1728r^6} \text{ (QED)}$$

4239.

In any Δ ABC such that $r_a \neq r_b \neq r_c$, the following relationship holds :

$$\frac{64}{729R^6} < \frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c} + \frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a} + \frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b} < \frac{1}{729r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x \neq y \neq z, \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz}$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(-yz(y-z) - zx(z-x) - xy(x-y)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.\left(yz(y-z) - x(y-z)(y+z) + x^2(y-z)\right)$$

$$= -\frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)\left(y(z-x) - x(z-x)\right)$$

$$= \frac{1}{(x-y)(y-z)(z-x)x^2y^2z^2}.(y-z)(z-x)(x-y)$$

$$\therefore \sum_{cyc} \frac{1}{(x-y)(x-z)x^2yz} = \frac{1}{x^2y^2z^2} \text{ and via } x \equiv r_a, y \equiv r_b, z \equiv r_c,$$

$$\frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c}+\frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a}+\frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b}$$

$$=\frac{1}{r_a^2r_b^2r_c^2}=\frac{1}{r^2s^4}\ \textit{and} \because \text{via Mitrinovic and Euler},\frac{1}{\frac{R^2}{4}\cdot\frac{729R^4}{16}}<\frac{1}{r^2s^4}<\frac{1}{r^2.729r^4}$$

$$(\text{equality doesn't occur because } r_a\neq r_b\neq r_c\Rightarrow a\neq b\neq c)$$

$$\therefore\frac{64}{729R^6}<\frac{1}{(r_a-r_b)(r_a-r_c)r_a^2r_br_c}+\frac{1}{(r_b-r_c)(r_b-r_a)r_b^2r_cr_a}+$$

$$\frac{1}{(r_c-r_a)(r_c-r_b)r_c^2r_ar_b}<\frac{1}{729r^6}\ (\text{QED})$$

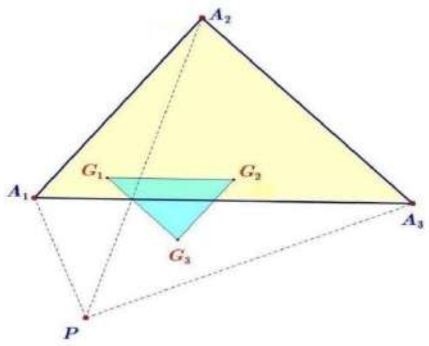
4240.

Let P,A_1,A_2,A_3 are any points on a plane and G_1,G_2,G_3 are the centroids of PA_1A_2,PA_2A_3,PA_3A_1 respectively. Prove that :

1) $G_1G_2\parallel A_1A_3, G_2G_3\parallel A_1A_2$ and $G_3G_1\parallel A_2A_3$

2) $[\Delta A_1A_2A_3]=9[\Delta G_1G_2G_3]$

Proposed by Jafar Nikpour-Iran



Solution by Mirsadix Muzefferov-Azerbaijan

Let's determine the coordinates of the centers of gravity G_1,G_2,G_3

$$G_1\left(\frac{x_1+x_2+x_4}{3};\frac{y_1+y_2+y_4}{3}\right),$$

$$G_2\left(\frac{x_2+x_3+x_4}{3};\frac{y_2+y_3+y_4}{3}\right), G_3\left(\frac{x_1+x_3+x_4}{3};\frac{y_1+y_3+y_4}{3}\right)$$

$$\overrightarrow{G_1G_2}=\left(\frac{x_3-x_1}{3};\frac{y_3-y_1}{3}\right), \overrightarrow{A_1A_3}=(x_3-x_1;y_3-y_1)\Rightarrow G_1G_2\parallel A_1A_3$$

$$\overrightarrow{G_2G_3}=\left(\frac{x_1-x_2}{3};\frac{y_1-y_2}{3}\right), \overrightarrow{A_1A_2}=(x_2-x_1;y_2-y_1)\Rightarrow G_2G_3\parallel A_1A_2$$

$$\overrightarrow{G_3G_1}=\left(\frac{x_2-x_3}{3};\frac{y_2-y_3}{3}\right), \overrightarrow{A_2A_3}=(x_3-x_2;y_3-y_2)\Rightarrow G_3G_1\parallel A_2A_3$$

$$\left.\begin{array}{l} G_1G_2\parallel A_1A_3\\ G_2G_3\parallel A_1A_2\\ G_3G_1\parallel A_2A_3\end{array}\right\}\Rightarrow \Delta G_1G_2G_3\sim\Delta A_1A_2A_3\Rightarrow k=3 \text{ here } k-\text{ is the ratio of similitude}$$

$$\text{Then } [\Delta A_1A_2A_3]\div[\Delta G_1G_2G_3]=k^2=9.$$

4241. *In a non – equilateral ΔABC with $\frac{2a}{R}=\frac{3\sqrt{3}b}{s}=\frac{s}{r}$ prove that:*

$$R=\frac{14}{3}r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\text{Let } \frac{2a}{R}=\frac{3\sqrt{3}b}{s}=\frac{s}{r}=k \text{ then } a=\frac{kR}{2}, b=\frac{ks}{3\sqrt{3}}, c=kr$$

$$abc=4Rrs \text{ or}, \frac{kR}{2}\cdot\frac{ks}{3\sqrt{3}}\cdot kr=4Rrs\Rightarrow k=2\sqrt{3}$$

$$a+b+c=2s \text{ or}, \frac{kR}{2}+\frac{ks}{3\sqrt{3}}+kr=2s \text{ or } \sqrt{3}R+\frac{2s}{3}+2\sqrt{3}r\overset{k=2\sqrt{3}}{=}2s$$

$$\frac{4s}{3}=\sqrt{3}(R+2r) \text{ or}, \frac{s}{r}=\frac{3\sqrt{3}}{4}\left(\frac{R}{r}+2\right)\ (1)$$

$$ab=\frac{2\sqrt{3}}{3}Rs, bc=\frac{4\sqrt{3}}{3}rs, ca=6Rr$$

$$ab+bc+ca=s^2+r^2+4Rr \text{ or}, \frac{2\sqrt{3}}{3}Rs+\frac{4\sqrt{3}}{3}rs+6Rr=s^2+r^2+4Rr$$

$$or,\frac{2\sqrt{3}}{3}xy+\frac{4\sqrt{3}}{3}y+6x$$

$$=y^2+1+4x,\left(x=\frac{R}{r},y=\frac{s}{r}\overset{(1)}{=}\frac{3\sqrt{3}}{4}\left(\frac{R}{r}+2\right)=\frac{3\sqrt{3}}{4}(x+2)\right)$$

$$\frac{2\sqrt{3}}{3}x.\frac{3\sqrt{3}}{4}(x+2)+\frac{4\sqrt{3}}{3}.\frac{3\sqrt{3}}{4}(x+2)+6x=\left(\frac{3\sqrt{3}}{4}(x+2)\right)^2+1+4x$$

$$\frac{3}{2}x(x+2)+3(x+2)+6x=\frac{27}{16}(x+2)^2+1+4x$$

$$\frac{3}{2}x^2+12x+6=\frac{27}{16}x^2+\frac{43x}{4}+\frac{31}{4}\ \ or\ \ 3x^2-20x+28=0$$

$$(3x-14)(x-2)=0$$

$$x=\frac{14}{3}\left(as\ x=\frac{R}{r}\neq 2, since\ \ triangle\ is\ \ non-equilateral\right)\ \ or\ R=\frac{14}{3}r\,.$$

4242. **In any Δ ABC with P $\rightarrow$ an interior point,**

the following relationship holds :

$$\sum_{\text{cyc}}\frac{\text{PA}^2}{a}\geq 2\sqrt{3}\text{r}$$

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Soumava Chaktraborty-Kolkata-India

Via Murray S. Klamkin (1975), if x,y,z are real numbers, then :

$$(x+\text{y}+\text{z})(x.\text{PA}^2+\text{y}.\text{PB}^2+\text{z}.\text{PC}^2)\geq \text{yz}a^2+\text{z}x\text{b}^2+x\text{yc}^2$$

$$\text{and via } x\equiv\frac{1}{a},\text{y}\equiv\frac{1}{\text{b}},\text{z}\equiv\frac{1}{\text{c}},\text{we get}:\left(\sum_{\text{cyc}}\frac{1}{a}\right)\left(\sum_{\text{cyc}}\frac{\text{PA}^2}{a}\right)\geq\sum_{\text{cyc}}\frac{a^2}{\text{bc}}$$

$$\Rightarrow\sum_{\text{cyc}}\frac{\text{PA}^2}{a}\geq\frac{\sum_{\text{cyc}}a^3}{\sum_{\text{cyc}}ab}\overset{\text{Chebyshev}}{\geq}\frac{\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a\right)}{3\sum_{\text{cyc}}ab}\geq\frac{\left(\sum_{\text{cyc}}ab\right)\left(\sum_{\text{cyc}}a\right)}{3\sum_{\text{cyc}}ab}$$

$$\overset{\text{Mitrinovic}}{\geq}\frac{6\sqrt{3}\text{r}}{3}\ \text{and so,}:\sum_{\text{cyc}}\frac{\text{PA}^2}{a}\geq 2\sqrt{3}\text{r}\ \ \forall\,\Delta\,\text{ABC},$$

$$''=''\ \text{iff}\ \Delta\ \text{ABC is equilateral (QED)}$$

4243. **In ΔABC the following relationship holds:**

$$\frac{3R}{r}-2\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3\sqrt[3]{abc}}{a+b+c}\geq 4$$

Proposed by Alex Szoros-Romania

Solution by Tapas Das-India

$$Lemma: \forall\, x,y,z>0,\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{(x+y+z)}{\sqrt[3]{xyz}}$$

$$Proof: 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)=\left(\frac{x}{y}+\frac{x}{y}+\frac{y}{z}\right)+\left(\frac{y}{z}+\frac{y}{z}+\frac{z}{x}\right)+\left(\frac{z}{x}+\frac{z}{x}+\frac{x}{y}\right)\overset{AM-GM}{\ge}$$

$$\ge\sum\frac{3x}{\sqrt[3]{xyz}}=\frac{3(x+y+z)}{\sqrt[3]{xyz}}\ \ or\ \ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{(x+y+z)}{\sqrt[3]{xyz}}\ \ (proof\ complete)$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3\sqrt[3]{abc}}{a+b+c}=\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{3\sqrt[3]{abc}}{a+b+c}\ge$$

$$\overset{\substack{AM-GM\\ Lemma}}{\ge}\frac{2}{3}\times 3+\frac{1}{3}\cdot\frac{a+b+c}{\sqrt[3]{abc}}+\frac{3\sqrt[3]{abc}}{a+b+c}\overset{AM-GM}{\ge}2+2\sqrt{\frac{1}{3}\cdot\frac{a+b+c}{\sqrt[3]{abc}}\cdot\frac{3\sqrt[3]{abc}}{a+b+c}}=4$$

$$\frac{3\sqrt[3]{abc}}{a+b+c}\overset{AM-GM}{\le}\frac{a+b+c}{a+b+c}=1\ \ \&\ \ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{CBS}{\le}$$

$$\le\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}\overset{Leibniz,Steining}{\le}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$We\ need\ to\ show: 1+\frac{3R}{2r}\le\frac{3R}{r}-2\ or,\frac{3R}{2r}\ge 3\ or, R\ge 2r\ true\ by\ Euler.$$

Equality holds for an equilateral triangle.

4244. In $a$ny Δ ABC the following relationship ho$l$ds :

$$\sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r}\ge\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\le}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{Bogdan\ Fustei}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\le}\sqrt{\frac{R}{2r}}$$

$$\overset{\text{squaring}}{\Leftrightarrow} R^2(4R+r)^2+6r^2(4R+r)^2 \boxed{\underset{(*)}{\overset{?}{\geq}}} 6r(3R-r)s^2$$

$$\text{Now}, 6r(3R-r)s^2 \overset{\text{Rouche}}{\leq} 6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq} (R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2) \overset{?}{\geq} 6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\text{and} \because R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{it suffices to prove following squaring}:$$

$$R^2(16R^2+4Rr-63r^2)^2 \overset{?}{\geq} 144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right) \overset{?}{\geq} 0 \to \text{true}$$

$$(\text{strict inequality}) \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r}-\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}} \geq$$

$$\frac{\sum_{cyc} n_a}{4r}+\frac{3s}{4r}-\sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}} \overset{\text{Bogdan Fustei}}{=} \sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a}{a-\frac{an_a}{s}}$$

$$=\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a(s+n_a)}{a(s-n_a)(s+n_a)} \overset{\text{Bogdan Fustei}}{=} \sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{r_a(s+n_a)}{a.2h_ar_a}$$

$$=\sum_{cyc}\frac{n_a+s}{4r}-\sum_{cyc}\frac{s+n_a}{4r}=0 \text{ and so},$$

$$\sqrt{\frac{R}{2r}\left(\frac{R}{r}+\frac{1}{4}\right)}+\frac{3s}{4r} \geq \sum_{cyc}\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}} \ \forall \text{ ABC},$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4245. In any Δ ABC the following relationship holds:

$$\frac{p_a\cdot\sqrt{\frac{w_a}{g_a}}+p_b\cdot\sqrt{\frac{w_b}{g_b}}+p_c\cdot\sqrt{\frac{w_c}{g_c}}}{r_a+r_b+r_c} \leq \left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_aw_bw_c}{h_ah_bh_c}\right)^{\frac{1}{3}}\cdot\frac{((h_a-r)(h_b-r)(h_c-r))^{\frac{1}{3}}}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{R}{2r}\right)\cdot\left(\frac{w_a w_b w_c}{h_a h_b h_c}\right)^2\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^2}{64r^6}=$$

$$=\frac{\left(\frac{R}{2r}\right)}{64r^6}\cdot\left(\frac{32Rr^2s^3}{(a+b)(b+c)(c+a)}\cdot\frac{R}{2r^2s^2}\cdot\frac{r^3(a+b)(b+c)(c+a)}{4Rrs}\right)^2=\frac{R^3}{8r^3}$$

$$\therefore \text{RHS}=\sqrt{\frac{R}{2r}}\to (1)\ \textit{and}\ \because p_a\cdot\sqrt{\frac{w_a}{g_a}}\le n_a\ \textit{and analogs}$$

(Reference : Solution to Inequality in Triangle – 118 by Bogdan Fustei; publshed at www.ssmrmh.ro)

$$\therefore\frac{\sum_{cyc}\left(p_a\cdot\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}\le\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\le}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\le}\text{RHS}\overset{\text{via }(1)}{=}\sqrt{\frac{R}{2r}}$$

$$\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2+6r^2(4R+r)^2\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{\text{Rouche}}{\le}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\ge}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\textit{and}\because R-2r\overset{\text{Euler}}{\ge}0\ \therefore \text{it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\ge}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\ge}0\to\text{true}$$

$$(\text{strict inequality})\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{p_a\cdot\sqrt{\frac{w_a}{g_a}}+p_b\cdot\sqrt{\frac{w_b}{g_b}}+p_c\cdot\sqrt{\frac{w_c}{g_c}}}{r_a+r_b+r_c}\le\left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_a w_b w_c}{h_a h_b h_c}\right)^{\frac{1}{3}}\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^{\frac{1}{3}}}{2r}$$

$\forall$ ABC, " = " iff Δ ABC is equilateral (QED)

4246. In any Δ ABC the following relationship holds :

$$\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\le\left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_a w_b w_c}{h_a h_b h_c}\right)^{\frac{1}{3}}\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^{\frac{1}{3}}}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{R}{2r}\right)\cdot\left(\frac{w_a w_b w_c}{h_a h_b h_c}\right)^2\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^2}{64r^6}=$$

$$=\frac{\left(\frac{R}{2r}\right)}{64r^6}\cdot\left(\frac{32Rr^2s^3}{(a+b)(b+c)(c+a)}\cdot\frac{R}{2r^2s^2}\cdot\frac{r^3(a+b)(b+c)(c+a)}{4Rrs}\right)^2=\frac{R^3}{8r^3}$$

$$\therefore \text{RHS}=\sqrt{\frac{R}{2r}}\to (1)\ \textit{and}\ \frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\leq}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\text{RHS}\overset{\text{via }(1)}{=}\sqrt{\frac{R}{2r}}$$

$$\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2+6r^2(4R+r)^2\boxed{\underset{(*)}{\overset{?}{\geq}}}6r(3R-r)s^2$$

$$\text{Now},6r(3R-r)s^2\overset{\text{Rouche}}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\textit{and}\because R-2r\overset{\text{Euler}}{\geq}0\ \therefore \text{it suffices to prove following squaring}:$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\geq}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\geq}0\to\text{true}$$

$$(\text{strict inequality})\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{n_a+n_b+n_c}{r_a+r_b+r_c}\leq\left(\frac{R}{2r}\right)^{\frac{1}{6}}\cdot\left(\frac{w_a w_b w_c}{h_a h_b h_c}\right)^{\frac{1}{3}}\cdot\frac{\left((h_a-r)(h_b-r)(h_c-r)\right)^{\frac{1}{3}}}{2r}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4247. **In any Δ ABC the following relationship holds :**

$$\sum_{cyc}\left(\frac{r_a-r}{w_a}.\sqrt{\frac{h_a}{r_a}}\right)\geq\frac{\sum_{cyc}n_a+\sum_{cyc}\left(p_a.\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)=\sum_{cyc}\left(\frac{rs.a(b+c)}{s(s-a).2bc\cos\frac{A}{2}}\cdot\sqrt{\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}.s\tan\frac{A}{2}}}\right)$$

$$=\sqrt{\frac{r}{2R}}\cdot\sum_{cyc}\frac{ra(b+c)}{(s-a)bc.2\cos\frac{A}{2}\sin\frac{A}{2}}=\sqrt{\frac{r}{2R}}\cdot\sum_{cyc}\frac{2Rr.a(b+c)}{(s-a)abc}$$

$$=\sqrt{\frac{r}{2R}}\cdot\frac{2Rr}{4Rrs}\cdot\sum_{cyc}\frac{a(s+s-a)}{s-a}=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}\cdot\left(s\sum_{cyc}\frac{a-s+s}{s-a}+2s\right)$$

$$=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}\cdot\left(s\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+2s\right)=\sqrt{\frac{r}{2R}}\cdot\frac{1}{2s}.s.\frac{4R}{r}$$

$$\Rightarrow\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)=\sqrt{\frac{2R}{r}}\to (1)\ \textit{and again}, p_a.\sqrt{\frac{w_a}{g_a}}\le n_a\ \textit{and analogs}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle} - 118\text{ by Bogdan Fustei;}\\ \text{publshed at www.ssmrmh.ro}\end{matrix}\right)$$

$$\Rightarrow\frac{\sum_{cyc}n_a+\sum_{cyc}\left(p_a.\sqrt{\frac{w_a}{g_a}}\right)}{r_a+r_b+r_c}\le\frac{2\sum_{cyc}n_a}{4R+r}\overset{CBS}{\le}\frac{2.\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{2.\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}=\frac{2.\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\le}\sum_{cyc}\left(\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}\right)$$

$$\overset{\text{via }(1)}{=}\sqrt{\frac{2R}{r}}\overset{\text{squaring}}{\Leftrightarrow}R^2(4R+r)^2+6r^2(4R+r)^2\ \boxed{\overset{?}{\ge}_{(*)}}\ 6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{\text{Rouche}}{\le}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\ge}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\textit{and}\because R-2r\overset{\text{Euler}}{\ge}0\ \therefore\text{ it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\ge}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\ge}0\to\text{true}$$

$$(\text{strict inequality}) \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \left(\frac{r_a - r}{w_a} \cdot \sqrt{\frac{h_a}{r_a}} \right) \geq \frac{\sum_{\text{cyc}} n_a + \sum_{\text{cyc}} \left(p_a \cdot \sqrt{\frac{w_a}{g_a}} \right)}{r_a + r_b + r_c} \ \forall \text{ ABC},$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4248. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}} \geq \frac{r_a + r_b + r_c + l_a + l_b + l_c + n_a + n_b + n_c}{h_a + h_b + h_c + m_a + m_b + m_c + r_a + r_b + r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} n_a \overset{\text{CBS}}{\leq} \sqrt{3 \sum_{\text{cyc}} n_a^2} \overset{\text{Bogdan Fustei}}{=} \sqrt{3 \sum_{\text{cyc}} \left(s^2 - \frac{r}{R} \cdot s^2 \cdot \sec^2 \frac{A}{2} \right)}$$

$$= \sqrt{3 \left(\frac{(3R - r)s^2 - r(4R + r)^2}{R} \right)} \therefore s\sqrt{3} + n_a + n_b + n_c \overset{\text{CBS}}{\leq} \sqrt{2} \cdot \sqrt{3s^2 + \left(\sum_{\text{cyc}} n_a \right)^2}$$

$$\leq \sqrt{2} \cdot \sqrt{3s^2 + 3 \left(\frac{(3R - r)s^2 - r(4R + r)^2}{R} \right)} \text{ and so, it suffices to prove :}$$

$$\frac{R}{2r} \overset{?}{\geq} \frac{6s^2 + 6\left(\frac{(3R - r)s^2 - r(4R + r)^2}{R} \right)}{\left(\frac{s^2 + 4Rr + r^2}{2R} + 4R + r \right)^2} \Leftrightarrow s^4 +$$

$$(16R^2 - 180Rr + 50r^2)s^2 + 64R^4 + 96R^3r + 820R^2r^2 + 396Rr^3 + 49r^4 \boxed{\overset{?}{\geq}_{(1)}} 0$$

$$\text{and} \because (s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (4R^2 - 37Rr + 10r^2)s^2 +$$

$$16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \boxed{\overset{?}{\geq}_{(2)}} 0 \text{ and it's trivially true if :}$$

$$4R^2 - 37Rr + 10r^2 \geq 0 \text{ and when : } 4R^2 - 37Rr + 10r^2 < 0, \text{ then :}$$

$$\text{LHS of (2)} \overset{\text{Blundon-Gerretsen}}{\geq}$$

$$(4R^2-37Rr+10r^2).\frac{R(4R+r)^2}{4R-2r}+16R^4+24R^3r+141R^2r^2+139Rr^3+6r^4\overset{?}{\geq}0$$

$$\Leftrightarrow 128t^5-496t^4+384t^3+317t^2-244t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2+10t(t-2)+(t^2-4)+3)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (2)\Rightarrow (1)\text{ is true }\forall\,\Delta\,ABC\therefore\sqrt{\frac{R}{2r}}\geq\frac{s\sqrt{3}+n_a+n_b+n_c}{h_a+h_b+h_c+r_a+r_b+r_c}\ \forall\,ABC\to\text{(i)}$$

$$\text{and now, }(b+c)^2\overset{?}{\geq}32Rr.\cos^2\frac{A}{2}=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0\Leftrightarrow(b+c-2a)^2\overset{?}{\geq}0$$

$$\to\text{true}\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow m_a\overset{\text{Lascu}}{\geq}\frac{b+c}{2}.\cos\frac{A}{2}\geq\sqrt{8Rr}.\cos^2\frac{A}{2}$$

$$\text{and analogs}\Rightarrow\sum_{\text{cyc}}m_a\geq\sqrt{8Rr}.\frac{4R+r}{2R}=\sqrt{\frac{2r}{R}}.(4R+r)$$

$$\therefore\sqrt{\frac{R}{2r}}.(m_a+m_b+m_c)\geq r_a+r_b+r_c\to\text{(ii) and finally,}$$

$$l_a+l_b+l_c\overset{\text{AM-GM}}{\leq}\sum_{\text{cyc}}\sqrt{s(s-a)}\overset{\text{CBS}}{\leq}\sqrt{3s}.\sqrt{\sum_{\text{cyc}}(s-a)}\Rightarrow l_a+l_b+l_c\leq s\sqrt{3}\to\text{(iii)}$$

$$\therefore\text{(iii)}\Rightarrow\frac{r_a+r_b+r_c+l_a+l_b+l_c+n_a+n_b+n_c}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}\leq$$

$$\frac{(s\sqrt{3}+n_a+n_b+n_c)+r_a+r_b+r_c}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}\overset{\text{via (i)}}{\leq}$$

$$\frac{\left(\sqrt{\frac{R}{2r}}.(h_a+h_b+h_c+r_a+r_b+r_c)\right)+(r_a+r_b+r_c)}{h_a+h_b+h_c+r_a+r_b+r_c+m_a+m_b+m_c}\overset{?}{\leq}\sqrt{\frac{R}{2r}}$$

$$\Leftrightarrow\sqrt{\frac{R}{2r}}.(m_a+m_b+m_c)\overset{?}{\geq}r_a+r_b+r_c\to\text{true via (ii) and so,}$$

$$\sqrt{\frac{R}{2r}}\geq\frac{r_a+r_b+r_c+l_a+l_b+l_c+n_a+n_b+n_c}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}\ \forall\,\Delta\,ABC,$$

$$''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4249. **In any Δ ABC the following relationship holds :**

$$\sqrt{\frac{2R}{r}} \geq \frac{s\sqrt{3}+l_a+l_b+l_c+2(n_a+n_b+n_c+r_a+r_b+r_c)}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{Bogdan\ Fustei}{=} \sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)} \therefore s\sqrt{3}+n_a+n_b+n_c \overset{CBS}{\leq} \sqrt{2}.\sqrt{3s^2+\left(\sum_{cyc} n_a\right)^2}$$

$$\leq \sqrt{2}.\sqrt{3s^2+3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)} \text{ and so, it suffices to prove :}$$

$$\frac{R}{2r} \overset{?}{\geq} \frac{6s^2+6\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}{\left(\frac{s^2+4Rr+r^2}{2R}+4R+r\right)^2} \Leftrightarrow s^4+$$

$$(16R^2-180Rr+50r^2)s^2+64R^4+96R^3r+820R^2r^2+396Rr^3+49r^4 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0$$

$$\text{and} \because (s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2-16Rr+5r^2)^2 \Leftrightarrow (4R^2-37Rr+10r^2)s^2+$$

$$16R^4+24R^3r+141R^2r^2+139Rr^3+6r^4 \boxed{\overset{?}{\underset{(2)}{\geq}}} 0 \text{ and it's trivially true if :}$$

$$4R^2-37Rr+10r^2 \geq 0 \text{ and when : } 4R^2-37Rr+10r^2 < 0, \text{ then :}$$

$$\text{LHS of (2)} \overset{Blundon-Gerretsen}{\geq}$$

$$(4R^2-37Rr+10r^2).\frac{R(4R+r)^2}{4R-2r}+16R^4+24R^3r+141R^2r^2+139Rr^3+6r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 128t^5-496t^4+384t^3+317t^2-244t-12 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2+10t(t-2)+(t^2-4)+3) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta \text{ ABC} \therefore \sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+n_a+n_b+n_c}{h_a+h_b+h_c+r_a+r_b+r_c} \ \forall \text{ ABC} \rightarrow \text{(i)}$$

$$and\ now, (b+c)^2 \overset{?}{\geq} 32Rr.\cos^2\frac{A}{2} = 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0$$

$$\to \text{true} \therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \Rightarrow m_a \overset{Lascu}{\geq} \frac{b+c}{2}.\cos\frac{A}{2} \geq \sqrt{8Rr}.\cos^2\frac{A}{2}$$

$$and\ analogs \Rightarrow \sum_{cyc} m_a \geq \sqrt{8Rr}.\frac{4R+r}{2R} = \sqrt{\frac{2r}{R}}.(4R+r)$$

$$\therefore \sqrt{\frac{R}{2r}}.(m_a+m_b+m_c) \geq r_a+r_b+r_c \to (ii)\ and\ finally,$$

$$l_a+l_b+l_c \overset{AM-GM}{\leq} \sum_{cyc}\sqrt{s(s-a)} \overset{CBS}{\leq} \sqrt{3s}.\sqrt{\sum_{cyc}(s-a)} \Rightarrow l_a+l_b+l_c \leq s\sqrt{3} \to (iii)$$

$$\therefore (iii) \Rightarrow \frac{s\sqrt{3}+l_a+l_b+l_c+2(n_a+n_b+n_c+r_a+r_b+r_c)}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c} \leq$$

$$\frac{2\left(\left(s\sqrt{3}+n_a+n_b+n_c\right)+r_a+r_b+r_c\right)}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c} \overset{via\ (i)}{\leq}$$

$$\frac{2\left(\sqrt{\frac{R}{2r}}.(h_a+h_b+h_c+r_a+r_b+r_c)\right)+2(r_a+r_b+r_c)}{h_a+h_b+h_c+r_a+r_b+r_c+m_a+m_b+m_c} \overset{?}{\leq} \sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow \sqrt{\frac{2R}{r}}.(m_a+m_b+m_c) \overset{?}{\geq} 2(r_a+r_b+r_c) \to \text{true via (ii) and so,}$$

$$\sqrt{\frac{2R}{r}} \geq \frac{s\sqrt{3}+l_a+l_b+l_c+2(n_a+n_b+n_c+r_a+r_b+r_c)}{h_a+h_b+h_c+m_a+m_b+m_c+r_a+r_b+r_c}\ \forall\,\Delta\,ABC,$$

$$'' = ''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

4250. In any $\Delta\,ABC$ the following relationship holds :

$$\sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3}+n_a+n_b+n_c}{h_a+h_b+h_c+r_a+r_b+r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{Bogdan\ Fustei}{=} \sqrt{3\sum_{cyc}\left(s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \therefore s\sqrt{3} + n_a + n_b + n_c \overset{CBS}{\leq} \sqrt{2}.\sqrt{3s^2 + \left(\sum_{cyc} n_a\right)^2}$$

$$\leq \sqrt{2}.\sqrt{3s^2 + 3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \text{ and so, it suffices to prove :}$$

$$\frac{R}{2r} \overset{?}{\geq} \frac{6s^2 + 6\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)}{\left(\frac{s^2+4Rr+r^2}{2R} + 4R + r\right)^2} \Leftrightarrow s^4 +$$

$$(16R^2 - 180Rr + 50r^2)s^2 + 64R^4 + 96R^3r + 820R^2r^2 + 396Rr^3 + 49r^4 \overset{?}{\underset{(1)}{\geq}} 0$$

$$\text{and } \because (s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{ in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (4R^2 - 37Rr + 10r^2)s^2 +$$

$$16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \overset{?}{\underset{(2)}{\geq}} 0 \text{ and it's trivially true if :}$$

$$4R^2 - 37Rr + 10r^2 \geq 0 \text{ and when : } 4R^2 - 37Rr + 10r^2 < 0, \text{ then :}$$

$$\text{LHS of (2)} \overset{Blundon-Gerretsen}{\geq}$$

$$(4R^2 - 37Rr + 10r^2).\frac{R(4R+r)^2}{4R-2r} + 16R^4 + 24R^3r + 141R^2r^2 + 139Rr^3 + 6r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 128t^5 - 496t^4 + 384t^3 + 317t^2 - 244t - 12 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+3)(21t^2 + 10t(t-2) + (t^2-4) + 3) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta ABC \therefore \sqrt{\frac{R}{2r}} \geq \frac{s\sqrt{3} + n_a + n_b + n_c}{h_a + h_b + h_c + r_a + r_b + r_c} \ \forall ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4251. In any Δ ABC the following relationship holds :

$$\frac{R}{2r} \geq \frac{s\sqrt{3}.(n_a + n_b + n_c)}{(r_a + r_b + r_c)(h_a + h_b + h_c)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} n_a \overset{CBS}{\leq} \sqrt{3\sum_{cyc} n_a^2} \overset{Bogdan\ Fustei}{=} \sqrt{3\sum_{cyc}\left(s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{3\left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)} \text{ and so,}$$

$$\text{it suffices to prove : } \frac{R^2}{4r^2} \overset{?}{\geq} \frac{3s^2\left(3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)\right)}{(4R+r)^2(s^2+4Rr+r^2)^2}.4R^2$$

$$\Leftrightarrow (16R^3 + 8R^2r - 431Rr^2 + 144r^3)s^4 +$$

$$r(128R^4 + 96R^3r + 2328R^2r^2 + 1154Rr^3 + 144r^4)s^2 + Rr^2(4R+r)^4 \boxed{\overset{?}{\geq} 1} \ 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ *and* $s^2 - (m+n) \leq 0$, where $m =$ $2R^2 + 10Rr - r^2$ & $n = 2(R-2r)\sqrt{R^2 - 2Rr} \therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(*)}{\leq} 0$$

1 is trivi*ally* true if : $16R^3 + 8R^2r - 431Rr^2 + 144r^3 \geq 0$ *and* when : $16R^3 + 8R^2r - 431Rr^2 + 144r^3 < 0$, then vi*a* $(*)$, in order to prove 1,

it suffices to prove : LHS of 1 $\overset{?}{\geq}$

$$(16R^3 + 8R^2r - 431Rr^2 + 144r^3)(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3)$$

$$\Leftrightarrow \left(16R^5 + 120R^4r - 375R^3r^2 - 1433R^2r^3 + 1224Rr^4 - 36r^5\right)s^2 \boxed{\overset{?}{\geq} 2}$$

$$r\left(256R^6 + 256R^5r - 6816R^4r^2 - 2864R^3r^3 + 433R^2r^4 + 324Rr^5 + 36r^6\right)$$

Case 1 $X = 16R^5 + 120R^4r - 375R^3r^2 - 1433R^2r^3 + 1224Rr^4 - 36r^5 \geq 0$

and then : LHS of 2 $\overset{Gerretsen}{\geq} X(16Rr - 5r^2) \overset{?}{\geq}$ RHS of 2

$$\Leftrightarrow 176t^5 + 24t^4 - 2021t^3 + 2924t^2 - 780t + 16 \overset{?}{\geq} 0 \ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(176t^2(t^2-4) + 376t^2(t-2) + 187t^2 + 364t + 4(t-2)\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

Case 2 $X < 0$ *and* then : LHS of 2 $\overset{Blundon-Gerretsen}{\geq} X.\frac{R(4R+r)^2}{4R-2r} \overset{?}{\geq}$ RHS of 2

$$\Leftrightarrow 256t^8 + 1024t^7 - 5536t^6 + 1968t^5 + 5569t^4 + 323t^3 + 506t^2 + 468t +$$

$$72 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(4t+1)^2(16t^4 + 120t^3 + 36t^2 + 5t(t-2) + t + 18) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{Euler}{\geq} 2 \Rightarrow$ 2 is true $\therefore$ combining both c*a*ses, 2 $\Rightarrow$ 1 is true $\forall\ \Delta$ ABC

$$\therefore \frac{R}{2r} \geq \frac{s\sqrt{3}.(n_a + n_b + n_c)}{(r_a + r_b + r_c)(h_a + h_b + h_c)} \ \forall \text{ ABC}, '' = '' \text{ iff } \Delta \text{ ABC is equil}\textit{ateral} \text{ (QED)}$$

4252. In any Δ ABC the following relationship holds :

$$\frac{2g_b g_c}{h_a} \le \frac{w_a \cdot \sqrt{4r^2+(b-c)^2}}{n_a-\sqrt{4r^2+(b-c)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$
$c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set "S" of relations : $y^2+z^2 = x^2(m^2-2n)$,
$y^3+z^3 = x^3(m^3-3nn)$, and now, $g_b g_c \overset{?}{\le} r_a w_a \overset{\text{squaring and via Bogdan Fustei}}{\Leftrightarrow}$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\le} \frac{s(s-b)(s-c)}{s-a}\cdot\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\ge} (4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$
$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$
$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2 \boxed{\underset{(1)}{\overset{?}{\ge}}} 0 \text{ and via}$$

set of relations "S", to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\boxed{\underset{(2)}{\overset{?}{\ge}}}0$$

and it's trivially true when : $\Omega_2 \ge 0$ and when : $\Omega_2 < 0$, then :
$3m^3-5m^2-32m-28 = (m-5)(3m^2+10m+18)+62 < 0$
$\Rightarrow (m-5)(3m^2+10m+18) < -62 < 0 \Rightarrow m < 5 \to$ (i)
Now, LHS of (2) is a quadratic polynomial in "n" with discriminant, $\boxed{\delta} =$
$\sigma_2^2-4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3-75m^2+40m+196)$ and when :
$9m^3-75m^2+40m+196 \le 0$, then $\delta \le 0 \Rightarrow$ (2) is trivially true and when :
$9m^3-75m^2+40m+196 > 0$, then : $9m^3-75m^2+40m+196$

$$= \left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3} > 0$$
$$\Rightarrow \left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7) > \frac{14}{3} > 0 \text{ and } \because m \overset{\text{via (i)}}{\le} 5$$
$$\Rightarrow 3m(m-5)-3m-\frac{86}{3} < 0 \therefore m < \frac{7}{3} \to \text{(ii) and so, when : } \delta > 0,$$

$$\text{then it suffices to prove}: 2\Omega_1.n \overset{?}{\le} -\Omega_2 - \sqrt{\delta} \Leftrightarrow \text{and} \because n \overset{AM-GM}{\le} \frac{m^2}{4}$$

$$\therefore \text{suffices to prove}: \Omega_1.\frac{m^2}{2} \overset{?}{\le} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\ge} 2.\sqrt{\delta}$$

$$\text{and} \because 3m^2-2m-14 = \left(m+\frac{5}{3}\right)(3m-7) - \frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{it suffices to prove}:$$

$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\ge} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\ge} 0} \therefore g_b g_c \le r_a w_a$$

$$\Rightarrow h_a.\frac{w_a.\sqrt{4r^2+(b-c)^2}}{n_a - \sqrt{4r^2+(b-c)^2}} \overset{\text{Bogdan Fustei}}{=} \frac{2rs}{a}.\frac{w_a}{\frac{n_a}{\left(\frac{an_a}{s}\right)} - 1} = \frac{2rs}{a}.\frac{aw_a}{s-a} = \frac{2rs}{s-a}.w_a$$

$$= 2r_a w_a \ge 2g_b g_c \text{ and so}, \frac{2g_b g_c}{h_a} \le \frac{w_a.\sqrt{4r^2+(b-c)^2}}{n_a - \sqrt{4r^2+(b-c)^2}} \ \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4253. In any Δ ABC the following relationship holds :

$$\frac{g_b g_c}{r} \le \frac{n_a w_a}{n_a - \sqrt{4r^2+(b-c)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$ $c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2+z^2 = x^2(m^2-2n),$ $y^3+z^3 = x^3(m^3-3nn)$, and now, $g_b g_c \overset{?}{\le} r_a w_a \overset{\substack{\text{squaring} \\ \text{and via Bogdan Fustei}}}{\Leftrightarrow}$

$$\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\le} \frac{s(s-b)(s-c)}{s-a}.\frac{4bc}{(b+c)^2}.s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\ge} (4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$

$$\Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4.yz - 32x^3.yz(y+z) -$$

$$5x^2.yz(y+z)^2 + 3x.yz(y^3+z^3) + 13x.y^2z^2(y+z) + 3y^2z^2(y+z)^2 \boxed{\overset{?}{\underset{(1)}{\ge}}} 0 \text{ and via}$$

set of relations "S", to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\boxed{\overset{?}{\geq}\ ②}\ 0$$

$$\text{and it's trivially true when}: \Omega_2 \geq 0 \text{ and when}: \Omega_2 < 0, \text{then}:$$
$$3m^3-5m^2-32m-28=(m-5)(3m^2+10m+18)+62<0$$
$$\Rightarrow (m-5)(3m^2+10m+18)<-62<0 \Rightarrow m<5 \rightarrow (i)$$
$$\text{Now, LHS of ② is a quadratic polynomial in "n" with discriminant, } \boxed{\delta} =$$
$$\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196) \text{ and when}:$$
$$9m^3-75m^2+40m+196 \leq 0, \text{then } \delta \leq 0 \Rightarrow ② \text{ is trivially true and when}:$$
$$9m^3-75m^2+40m+196>0, \text{then}: 9m^3-75m^2+40m+196$$
$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$
$$\Rightarrow \left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0 \text{ and } \because m \overset{\text{via (i)}}{\leq} 5$$
$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0 \therefore m<\frac{7}{3} \rightarrow (ii) \text{ and so, when}: \delta>0,$$
$$\text{then it suffices to prove}: 2\Omega_1.n \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow \text{and} \because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$$
$$\therefore \text{suffices to prove}: \Omega_1.\frac{m^2}{2} \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$
$$\text{and} \because 3m^2-2m-14=\left(m+\frac{5}{3}\right)(3m-7)-\frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{it suffices to prove}:$$
$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$
$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0} \therefore g_b g_c \leq r_a w_a$$
$$\Rightarrow r.\frac{n_a w_a}{n_a-\sqrt{4r^2+(b-c)^2}} \overset{\text{Bogdan Fustei}}{=} r.\frac{n_a w_a}{n_a-\frac{an_a}{s}}=\frac{rs}{s-a}.w_a=r_a w_a \geq g_b g_c$$
$$\text{and so}, \frac{g_b g_c}{r} \leq \frac{n_a w_a}{n_a-\sqrt{4r^2+(b-c)^2}} \ \forall \Delta ABC,$$
$$"=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4254. In ΔABC the following relationship holds:

$$\frac{2r_a}{h_b+h_c}+\frac{m_b+m_c}{2r_a} \leq \frac{R}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\textit{We will show}: a^2-a(b+c)+4bc>0 \textit{ in any } \Delta ABC \ (1)$$
$$\textit{Proof}: \textit{we know that } a>|b-c| \textit{ then } a^2>(b-c)^2$$
$$a^2-a(b+c)+4bc>(b-c)^2-a(b+c)+4bc=b^2+2bc+c^2-a(b+c)$$
$$=(b+c)^2-a(b+c)=(b+c)(b+c-a)>0$$
$$\textit{as in } \Delta ABC, b+c>a (\textit{proof complete})$$

$$\frac{m_b+m_c}{2r_a}\overset{Panaitopol}{\leq}\frac{R}{4r}\left(\frac{h_b+h_c}{r_a}\right)$$

$$we\ will\ show:\ \frac{2r_a}{h_b+h_c}+\frac{m_b+m_c}{2r_a}\leq\frac{R}{r}\ \ or,\frac{2r_a}{h_b+h_c}+\frac{R}{4r}\left(\frac{h_b+h_c}{r_a}\right)\leq\frac{R}{r}\ \ (2)$$

$$let\ x=\frac{2r_a}{h_b+h_c}=\frac{\frac{2F}{s-a}}{\frac{2F}{b}+\frac{2F}{c}}=\frac{bc}{(b+c)(s-a)}\ then\ \frac{h_b+h_c}{r_a}=\frac{2}{x}$$

$$from\ (2)we\ get\ ,x+\frac{R}{4r}\left(\frac{2}{x}\right)\leq\frac{R}{r}\ or,\frac{2r}{R}x^2-2x+1\leq 0\ (3)$$

$$\frac{r}{R}=\frac{4F^2}{abcs}=\frac{4(s-a)(s-b)(s-c)}{abc}$$

$$putting\ the\ value\ of\ x\ and\ \frac{r}{R}\ in(3)we\ get$$

$$2\left(\frac{4(s-a)(s-b)(s-c)}{abc}\right)\left(\frac{bc}{(b+c)(s-a)}\right)^2-2\frac{bc}{(b+c)(s-a)}+1\leq 0$$

$$or,8bc(s-b)(s-c)-2abc(b+c)+a(b+c)^2(s-a)\leq 0$$

$$or,2bc(a^2-(b-c)^2)-2abc(b+c)+a(b+c)^2(s-a)\leq 0$$

$$or,4bc(a^2-(b-c)^2)-4abc(b+c)+a(b+c)^2(2s-2a)\leq 0(Multiply\ by\ 2)$$

$$or,4bc(a^2-(b-c)^2)-4abc(b+c)+a(b+c)^2(b+c-a)\leq 0$$

$$or,a^2(4bc-(b+c)^2)+\left((b+c)^3-4bc(b+c)\right)a-4bc(b-c)^2\leq 0$$

$$or,-a^2(b-c)^2+a(b+c)(b-c)^2-4bc(b-c)^2\leq 0$$

$$or,-(b-c)^2(a^2-a(b+c)+4bc)\leq 0\ true\ by\ (1)$$

Equality holds for b=c

4255. In any Δ ABC the following relationship holds :

$$m_a\leq\sqrt{\frac{R+2r}{16r}}.(r_b+r_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a,y=s-b,z=s-c$ and then : $a=y+z,b=z+x,$ $c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n),$ $y^3+z^3=x^3(m^3-3nn),y^4+z^4=x^4((m^2-2n)^2-2n^2),$

$$y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right),and\ now,m_a\overset{?}{\leq}\sqrt{\frac{R+2r}{16r}}.(r_b+r_c)$$

$$\Leftrightarrow (b-c)^2+4s(s-a)\overset{?}{\le}\left(\frac{R}{4r}+\frac{1}{2}\right).\frac{16R^2s^2(s-a)^2a^2}{16R^2.s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y-z)^2+4x(x+y+z)\overset{?}{\le}\frac{(y+z)(z+x)(x+y)+8xyz}{16xyz}.\frac{x(x+y+z)(y+z)^2}{yz}$$

$$\Leftrightarrow x^3(y+z)^3+2x^2(y^4+z^4)+16x^2.yz(y^2+z^2)-36x^2.y^2z^2+x(y^5+z^5)+$$
$$14x.yz(y^3+z^3)-27x.y^2z^2(y+z)+yz(y^4+z^4)-12y^2z^2(y^2+z^2)+38y^3z^3$$
$\boxed{\overset{?}{\underset{(1)}{\ge}}}$ 0 and via set of relations "S", in order to prove (1), it suffices to prove,

following simplification : $m^5+m^4n+2m^4+9m^3n+m^3+8m^2n-64mn^2-$
$64n^2+16n^2(4n-m^2)\overset{?}{\ge}0$ and $\because 16n^2(4n-m^2)\ge 4nm^2(4n-m^2)$
$\therefore$ it suffices to prove :

$$m^5+m^4n+2m^4+9m^3n+m^3+8m^2n-64mn^2-64n^2+4nm^2(4n-m^2)\overset{?}{\ge}0$$

$$\Leftrightarrow \overbrace{(16m^2-64m-64)}^{\Omega_1}n^2-\overbrace{(3m^4-9m^3-8m^2)}^{\Omega_2}n+\overbrace{m^3(m+1)^2}^{\Omega_3}\boxed{\overset{?}{\underset{(2)}{\ge}}}0$$

$\boxed{\text{Case 1}}$ $m^2-4m-4>0$ and now, LHS of (2) is a quadratic polynomial in "n" with discriminant, $\delta=\Omega_2^2-4\Omega_1.\Omega_3=$
$m^3(9m^5-118m^4+161m^3+848m^2+832m+256)$ and when $\delta\le 0$, then (2) is trivially true and when : $\delta>0$, *then it suffices to prove* :

$$2\Omega_1.n\overset{?}{\le}\Omega_2-\sqrt{\delta} \text{ and } \because n\overset{AM-GM}{\le}\frac{m^2}{4}\therefore \text{ it suffices to prove : } m^2.\Omega_1\overset{?}{\le}2\Omega_2-2\sqrt{\delta}$$

$$\Leftrightarrow 3m^4-9m^3-16m^2+32m+32\overset{?}{\ge}\sqrt{\delta} \text{ and } \because m^2-4m>4>0\therefore\boxed{m>4}$$
$$\Rightarrow 3m^4-9m^3-16m^2+32m+32=$$
$(m-4)(3m^3+2m^2+m(m-4)+16)+96>0\therefore$ it suffices to prove :

$$(3m^4-9m^3-16m^2+32m+32)^2\overset{?}{\ge}m^3\begin{pmatrix}9m^5-118m^4+161m^3+\\848m^2+832m+256\end{pmatrix}$$

$$\Leftrightarrow\boxed{64(m+1)(m^2-4m-4)\left(4m^4+m^3+11m^2+(m-4)(m+4)\right)\overset{?}{\ge}0}$$

$\to$ true (strict inequality) $\Rightarrow$ (2) $\Rightarrow$ (1) is true

$\boxed{\text{Case 2}}$ $m^2-4m-4<0$ and then, (2) can be re-written as :

$$\overbrace{(64+64m-16m^2)}^{-\Omega_1}n^2+\overbrace{(3m^4-9m^3-8m^2)}^{\Omega_2}n-\overbrace{m^3(m+1)^2}^{\Omega_3}\boxed{\overset{?}{\underset{(3)}{\le}}}0$$

Discriminant of LHS of (3) is again, $\delta=\Omega_2^2-4(-\Omega_1).(-\Omega_3)$
$$=m^3(9m^5-118m^4+161m^3+848m^2+832m+256)$$
$$=\frac{(m^3(9m-44)-30m^3)(9m-44)}{9}-\left(\frac{1807m^2}{81}+\frac{10820m}{729}\right)(9m-44)+$$
$$178m+\frac{686m}{729}+256>0\ \left(\because m^2-4m-4<0\Rightarrow 0<m<2+2\sqrt{2}<\frac{44}{9}\right)$$

$$\therefore \text{ it suffices to prove : } 2(-\Omega_1).n \underbrace{\overset{?}{\leq}}_{(*)} -\Omega_2+\sqrt{\delta} \text{ AND } 2(-\Omega_1).n \underbrace{\overset{?}{\geq}}_{(**)} -\Omega_2-\sqrt{\delta}$$

$$\text{Now, } 2(-\Omega_1).n \leq \frac{(-\Omega_1).m^2}{2} \overset{?}{\leq} -\Omega_2+\sqrt{\delta} \Leftrightarrow m^2(24+23m-5m^2) \overset{?}{\leq} \sqrt{\delta} \text{ and}$$

$$\because 24+23m-5m^2=(5-m)(5m+2)+14>0 \left(\because m<2+2\sqrt{2}<5\right)$$

$$\therefore \text{ it suffices to prove :}$$

$$m^3\left(9m^5-118m^4+161m^3+848m^2+832m+256\right) \overset{?}{\geq} m^4(24+23m-5m^2)^2$$

$$\Leftrightarrow \boxed{16m^3(m+1)(m-2)^2(4+4m-m^2) \overset{?}{\geq} 0} \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\text{and } 2(-\Omega_1).n+\sqrt{\delta}>\sqrt{\delta}=\sqrt{\Omega_2^2+4(-\Omega_1).\Omega_3} \overset{-\Omega_1>0}{>} \sqrt{\Omega_2^2}=|\Omega_2| \geq -\Omega_2$$

$$\Rightarrow (**) \text{ is true and so, } ③ \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore m_a \leq \sqrt{\frac{R+2r}{16r}}.(r_b+r_c)\ \forall\ \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4256. In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a\right)\left(\sum_{cyc} r_a\right) \leq \prod_{cyc}(r_a+r_b+h_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}\left(4R\cos^2\frac{A}{2}+h_a\right)=64R^3.\frac{s^2}{16R^2}+\frac{2r^2s^2}{R}+$$

$$16R^2.\sum_{cyc}\left(\frac{s(s-b)}{ca}.\frac{s(s-c)}{ab}.\frac{bc}{2R}.\frac{b^2c^2}{b^2c^2}\right)+4R.\sum_{cyc}\left(\frac{s(s-a)}{bc}.\frac{ca}{2R}.\frac{ab}{2R}\right)$$

$$=4Rs^2+\frac{2r^2s^2}{R}+$$

$$\frac{1}{2Rr^2}\begin{pmatrix}-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+(s^2+4Rr+r^2)^3- \\ 24Rrs^2(s^2+2Rr+r^2)+4Rrs^2(s^2+4Rr+r^2)\end{pmatrix}+$$

$$\frac{s}{R}\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\right) \overset{?}{\geq}$$

$$\frac{(s^2+4Rr+r^2)(4R+r).\sqrt{s^2+21Rr+12r^2}}{2R} \overset{\text{squaring}}{\Leftrightarrow} s^8+27r^2s^6+$$

$$(64R^4-272R^3r-120R^2r^2-5Rr^3+184r^4)s^4+$$

$$r\left(1024R^5-1664R^4r-1440R^3r^2-296R^2r^3-2Rr^4+3r^5\right)s^2-$$

$$r^2\begin{pmatrix}4096R^6+768R^5r-4608R^4r^2-3808R^3r^3-1248R^2r^4-\\189Rr^5-11r^6\end{pmatrix}\boxed{\overset{?}{\underset{①}{\geq}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ *and* $s^2-(m+n)\leq 0$, where $m=$
$2R^2+10Rr-r^2\ \&\ n=2(R-2r)\sqrt{R^2-2Rr}\ \therefore\ \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$
$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$
$\therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①}\overset{?}{\geq}(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)^2\Leftrightarrow$$
$$(8R^2+40Rr+23r^2)s^6+(48R^4-560R^3r-600R^2r^2+51Rr^3+178r^4)s^4+$$
$$r(1536R^5+1280R^4r+320R^3r^2-10Rr^4-r^5)s^2-$$
$$r^3(5376R^5+8448R^4r+5088R^3r^2+1488R^2r^3+213Rr^4+12r^5)\boxed{\overset{?}{\underset{②}{\geq}}}0\ \textit{and}\ \because$$

$$(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove ②,}$$
$$\text{it suffices to prove : LHS of ②}\overset{?}{\geq}(8R^2+40Rr+23r^2)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow(48R^4-176R^3r+1200R^2r^2+555Rr^3-167r^4)s^4+$$
$$r(1536R^5-4864R^4r-26560R^3r^2+936R^2r^3+8030Rr^4-1726r^5)s^2+$$
$$r^3\begin{pmatrix}64(428R^5+1948R^4r-858R^3r^2-669R^2r^3+350Rr^4-45r^5)\\+32R^3r^2+8R^2r^3-13Rr^4-7r^5\end{pmatrix}\boxed{\overset{?}{\underset{③}{\geq}}}0\ \textit{and}\ \because$$
$$P=(48R^4-176R^3r+1200R^2r^2+555Rr^3-167r^4)(s^2-16Rr+5r^2)^2$$
$$\overset{\text{Gerretsen}}{\geq}0\ \therefore\ \text{in order to prove ③, it suffices to prove : LHS of ③}\overset{?}{\geq}P$$
$$\Leftrightarrow(384R^5-1372R^4r+1700R^3r^2+837R^2r^3-358Rr^4-7r^5)s^2\boxed{\overset{?}{\underset{④}{\geq}}}$$
$$r(1536R^6-10016R^5r+26486R^4r^2+70R^3r^3-7343R^2r^4+2276Rr^5-161r^6)$$
$$\textit{and finally}, \text{LHS of ④}\overset{\text{Gerretsen}}{\geq}$$
$$\begin{pmatrix}384R^5-1372R^4r+1700R^3r^2+837R^2r^3-\\358Rr^4-7r^5\end{pmatrix}(16Rr-5r^2)\overset{?}{\geq}\text{RHS}_{④}$$
$$\Leftrightarrow 2304t^6-6928t^5+3787t^4+2411t^3-1285t^2-299t+98\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)(2304t^5-2320t^4-853t^3+705t^2+125t-49)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$$\Rightarrow ④\Rightarrow ③\Rightarrow ②\Rightarrow ①\ \text{is true}$$
$$\therefore\prod_{\text{cyc}}(r_a+r_b+h_c)\geq\frac{(s^2+4Rr+r^2)(4R+r).\sqrt{s^2+21Rr+12r^2}}{2R}$$
$$\overset{\text{Dang Ngoc Minh}}{\geq}\left(\sum_{\text{cyc}}h_a\right)\left(\sum_{\text{cyc}}r_a\right)\left(\sum_{\text{cyc}}w_a\right)$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 90;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right) \text{ and so,}$$

$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a\right)\left(\sum_{cyc} r_a\right) \le \prod_{cyc}(r_a + r_b + h_c)\ \forall\ ABC,$$
$$'' = ''\text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4257. In Δ ABC holds :

$$\frac{3\sum_{cyc}(23a^4 - 14b^2c^2)}{16\sum_{cyc} a^2} \ge \left(\sum_{cyc}\frac{m_a^2}{m_b + m_c}\right)^2 \ge \frac{3\sum_{cyc}(8a^4 + b^2c^2)}{16\sum_{cyc} a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{b+c} = \frac{(\sum_{cyc} a^2)(\sum_{cyc} ab) + 2\sum_{cyc} a^2b^2 - 16r^2s^2}{2s(s^2 + 2Rr + r^2)}$$

$$\Rightarrow \left(4\sum_{cyc} a^2\right)\left(\sum_{cyc}\frac{a^2}{b+c}\right)^2$$

$$= \frac{8(s^2 + 4Rr + r^2)(s^4 - (4Rr + r^2)^2 + (s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8r^2s^2)^2}{s^2(s^2 + 2Rr + r^2)^2} \overset{?}{\ge}$$

$$17\sum_{cyc} a^2b^2 - 128r^2s^2 = 17((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 128r^2s^2$$

$$\Leftrightarrow 15s^8 - (316Rr + 164r^2)s^6 + r^2(1740R^2 - 2236Rr - 634r^2)s^4 -$$
$$r^3(2592R^3 + 3752R^2r + 1748Rr^2 + 228r^3)s^2 -$$
$$r^4(1088R^4 + 1632R^3r + 884R^2r^2 + 204Rr^3 + 17r^4) \boxed{\overset{?}{\ge}_{(1)}} 0 \text{ and}$$

$$\because 15(s^2 - 16Rr + 5r^2)^4 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\ge} 15(s^2 - 16Rr + 5r^2)^4$$
$$\Leftrightarrow (161R - 116r)s^6 - r(5325R^2 - 4159Rr + 404r^2)s^4 +$$
$$r^2(32(1899R^3 - 1830R^2r + 549Rr^2 - 60r^3) + 24R^3 + 22R^2r - 5Rr^2 - 12r^3)s^2$$
$$-r^3\left(\begin{matrix}64(3844R^4 - 4794R^3r + 2254R^2r^2 - 468Rr^3 + 36r^4) +\\ 16R^4 + 24R^3r - 35R^2r^2 + 3Rr^3 + 44r^4\end{matrix}\right) \boxed{\overset{?}{\ge}_{(2)}} 0 \text{ and } \because$$

$$P = (161R - 116r)(s^2 - 16Rr + 5r^2)^3 +$$
$$r(2403R^2 - 3824Rr + 1336r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{to prove (2),}$$

suffices to prove : LHS of ② $\overset{?}{\geq}$ P $\Leftrightarrow (1755R^3 - 4821R^2r + 3850Rr^2 - 824r^3)s^2$ $\boxed{\overset{?}{\underset{③}{\geq}}}$ $r\left(\begin{matrix}32(788R^4 - 22535R^3r + 2029R^2r^2 - 703Rr^3 + 83r^4) + \\ 2R^4 + 15R^3r + 11R^2r^2 - 2Rr^3\end{matrix}\right)$; Finally, LHS of ③

$$\overset{\text{Rouche+Euler}}{\geq} (1755R^3 - 4821R^2r + 3850Rr^2 - 824r^3)\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}\right)$$

$\overset{?}{\geq} \text{RHS}_{③} \Leftrightarrow 2862t^5 - 14913t^4 + 26241t^3 - 17210t^2 + 2876t + 184 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$

$\Leftrightarrow (t-2)\left((t-2)(2862t^3 - 3465t^2 + 933t + 382) + 672\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$

$\therefore \left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \geq \frac{17\sum_{\text{cyc}} a^2b^2 - 128r^2s^2}{4\sum_{\text{cyc}} a^2}$ & implementing it on Δ XYZ with sides

$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ & area $\frac{F}{3}$, we get : $\frac{4}{9}\left(\sum_{\text{cyc}} \frac{m_a^2}{m_b + m_c}\right)^2$

$$\geq \frac{17\sum_{\text{cyc}} a^2b^2 - 128F^2}{12\sum_{\text{cyc}} a^2} \left(\because \overbrace{\sum_{\text{cyc}} m_a^2m_b^2 = \frac{9\sum_{\text{cyc}} a^2b^2}{16} \text{ and } \sum_{\text{cyc}} m_a^2 = \frac{3\sum_{\text{cyc}} a^2}{4}}^{\sigma}\right)$$

$$= \frac{17\sum_{\text{cyc}} a^2b^2 - 8\left(2\sum_{\text{cyc}} a^2b^2 - \sum_{\text{cyc}} a^4\right)}{12\sum_{\text{cyc}} a^2} \Rightarrow \left(\sum_{\text{cyc}} \frac{m_a^2}{m_b + m_c}\right)^2 \geq$$

$\frac{3\sum_{\text{cyc}}(8a^4 + b^2c^2)}{16\sum_{\text{cyc}} a^2}$ & also, $\frac{2\sum_{\text{cyc}} a^2b^2 - 23F^2}{\sum_{\text{cyc}} a^2} \overset{?}{\geq} \frac{1}{4}\left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \Leftrightarrow (16R - r)s^6 -$

$r(120R^2 + 196Rr + 64r^2)s^4 + r^2(192R^3 + 212R^2r + 68Rr^2 + 3r^3)s^2 +$

$r^3(128R^4 + 192R^3r + 104R^2r^2 + 24Rr^3 + 2r^4)$ $\boxed{\overset{?}{\underset{④}{\geq}}}$ 0 and $\because$ Q =

$(16R - r)(s^2 - 16Rr + 5r^2)^3 + r(648R^2 - 484Rr - 49r^2)(s^2 - 16Rr + 5r^2)^2$

$\overset{\text{Gerretsen}}{\geq}$ 0 ∴ in order to prove ④, it suffices to prove : LHS of ④ $\overset{?}{\geq}$ Q

$\Leftrightarrow \left(\begin{matrix}2160R^3 - 3327R^2r + 415Rr^2 + \\ 142r^3\end{matrix}\right)s^2$ $\boxed{\overset{?}{\underset{⑤}{\geq}}}$ $r\left(\begin{matrix}25056R^4 - 40560R^3r + \\ 14488R^2r^2 - 271Rr^3 - 338r^4\end{matrix}\right)$

Finally, $\text{LHS}_{⑤} \overset{\text{Gerretsen}}{\geq} (2160R^3 - 3327R^2r + 415Rr^2 + 142r^3)(16Rr - 5r^2)$

$\overset{?}{\geq}$ RHS of ⑤ $\Leftrightarrow 3168t^4 - 7824t^3 + 2929t^2 + 156t - 124 \overset{?}{\geq} 0 \Leftrightarrow$

$(t-2)(3168t^3 - 1488t^2 - 47t + 62) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{1}{4}\left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \leq$

$$\frac{2\sum_{cyc} a^2b^2 - 23F^2}{\sum_{cyc} a^2} \text{ \& invoking on } \Delta \text{ XYZ using } ''\sigma'' \text{ \& } 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4 = 16F^2,$$

$$\left(\sum_{cyc} \frac{m_a^2}{m_b + m_c}\right)^2 \leq \frac{3\sum_{cyc}(23a^4 - 14b^2c^2)}{16\sum_{cyc} a^2}, '' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4258. **In any Δ ABC the following relationship holds :**

$$r_b + r_c \geq max\left\{\frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c; \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c; \frac{h_b}{h_c}.r_b + \frac{h_c}{h_b}.r_c\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c \overset{?}{\geq} \frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c \overset{\text{squaring}}{\Leftrightarrow} r_b^2\left(1 - \frac{m_b^2}{m_c^2}\right) \overset{?}{\geq} r_c^2\left(\frac{m_c^2}{m_b^2} - 1\right)$$

$$\Leftrightarrow \frac{(m_c^2 - m_b^2)(r_b^2 m_b^2 - r_c^2 m_c^2)}{m_b^2 m_c^2} \overset{?}{\geq} 0$$

WLOG we may assume : b ≥ c and then : $m_c \geq m_b$ and so, it remains to prove :

$$(s-c)^2\left(2\sum_{cyc} a^2 - 3b^2\right) \overset{?}{\geq} (s-b)^2\left(2\sum_{cyc} a^2 - 3c^2\right)$$

$$\Leftrightarrow \left(2\sum_{cyc} a^2\right)(b-c)a \overset{?}{\geq} 3\big(b(s-c) + c(s-b)\big).s(b-c) \text{ and } \because b \geq c$$

$$\therefore \text{ it remains to prove :}$$

$$2\big((y+z)^2 + (z+x)^2 + (x+y)^2\big)(y+z) \overset{?}{\geq} 3(x+y+z)\big(z(z+x) + y(x+y)\big)$$

$$\begin{pmatrix} \text{where } x = s-a, y = s-b, z = s-c \text{ and then :} \\ a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z \end{pmatrix}$$

$$\Leftrightarrow 3xy(x+y) + 2x(y+z)^2 + 2xyz + 3(y^3+z^3) + 7yz(y+z) \overset{?}{\geq} 0$$

$$\rightarrow \text{ true } \because x,y,z > 0 \text{ (strict inequality) } \therefore \boxed{r_b + r_c \geq \frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c}$$

$$\text{Again, } r_b + r_c \overset{?}{\geq} \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c \overset{\text{squaring}}{\Leftrightarrow} r_b^2\left(1 - \frac{w_b^2}{w_c^2}\right) \overset{?}{\geq} r_c^2\left(\frac{w_c^2}{w_b^2} - 1\right)$$

$$\Leftrightarrow \frac{(w_c^2 - w_b^2)(r_b^2 w_b^2 - r_c^2 w_c^2)}{w_b^2 w_c^2} \overset{?}{\geq} 0$$

WLOG we may assume : b ≥ c and then : $w_c \geq w_b$ and so, it remains to prove :

$$\frac{s}{s-b}\cdot\frac{4ca}{(c+a)^2}\overset{?}{\geq}\frac{s}{s-c}\cdot\frac{4ab}{(a+b)^2}\ and\ \because (a+b)^2\geq(c+a)^2$$

$$\therefore \text{it suffices to prove}: c(s-c)\overset{?}{\geq}b(s-b)\Leftrightarrow(b+c)(b-c)\overset{?}{\geq}s(b-c)$$

$$\Leftrightarrow(2s-a-s)(b-c)\overset{?}{\geq}0\Leftrightarrow(s-a)(b-c)\overset{?}{\geq}0\to \text{true}$$

$$\therefore \boxed{r_b+r_c\geq\frac{w_b}{w_c}.r_b+\frac{w_c}{w_b}.r_c}\ and\ finally, r_b+r_c\overset{?}{\geq}\frac{h_b}{h_c}.r_b+\frac{h_c}{h_b}.r_c$$

$$\overset{\text{squaring}}{\Leftrightarrow} r_b^2\left(1-\frac{h_b^2}{h_c^2}\right)\overset{?}{\geq}r_c^2\left(\frac{h_c^2}{h_b^2}-1\right)\Leftrightarrow\frac{(h_c^2-h_b^2)(r_b^2h_b^2-r_c^2h_c^2)}{h_b^2h_c^2}\overset{?}{\geq}0$$

WLOG we may assume : $b\geq c$ and then : $h_c\geq h_b$ and so, it remains to prove :

$$r_bh_b\overset{?}{\geq}r_ch_c\Leftrightarrow\frac{r.s^2}{2R}.\sec^2\frac{B}{2}\overset{?}{\geq}\frac{r.s^2}{2R}.\sec^2\frac{C}{2}\to \text{true}\because f(x)=\sec\frac{x}{2}\text{ is }\uparrow\text{ on }(0,\pi)\ and$$

$$\because b\geq c\therefore\boxed{r_b+r_c\geq\frac{h_b}{h_c}.r_b+\frac{h_c}{h_b}.r_c}\ and\ so, from\ the\ boxed\ relations,$$

$$it's\ clear\ that: r_b+r_c\geq max\left\{\frac{m_b}{m_c}.r_b+\frac{m_c}{m_b}.r_c;\frac{w_b}{w_c}.r_b+\frac{w_c}{w_b}.r_c;\frac{h_b}{h_c}.r_b+\frac{h_c}{h_b}.r_c\right\}$$

$\forall$ ABC, " = " iff $b=c$ (QED)

4259. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_a}{w_a}\geq\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}\geq\sum_{cyc}\frac{m_a}{h_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}=\frac{4R}{2rs}.\sum_{cyc}\left(\frac{s(s-a)}{bc}.\frac{bc}{b+c}\right)$$

$$=\frac{2R}{r.2s(s^2+2Rr+r^2)}.\sum_{cyc}\left((s-a)\left(a^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{R\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)+s(s^2+4Rr+r^2)\right)}{r.s(s^2+2Rr+r^2)}$$

$$=\frac{R(s^2+8Rr+5r^2)}{r(s^2+2Rr+r^2)}\overset{(i)}{=}\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}$$

$$\text{Now, }\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2=\frac{1}{4r^2s^2}\left(\sum_{cyc}\left(\frac{a^2}{4}\left(2\sum_{cyc}a^2-3a^2\right)\right)+\sum_{cyc}\left(bc.\frac{2a^2+bc}{2}\right)\right)$$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh; published at www.ssmrmh.ro)

$$=\frac{1}{4r^2s^2}\cdot\left(2(s^2-4Rr-r^2)^2-(s^2+4Rr+r^2)+16Rrs^2+12r^2s^2+8Rrs^2\right)$$

$$\overset{?}{\le}\left(\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}\right)^2\overset{via\ (i)}{=}\frac{R^2(s^2+8Rr+5r^2)^2}{r^2(s^2+2Rr+r^2)^2}\Leftrightarrow$$

$$-s^8+(4R^2-4Rr-8r^2)s^6+r(64R^3+20R^2r-36Rr^2-14r^3)s^4+$$
$$r^2\overbrace{(256R^4+256R^3r+12R^2r^2-44Rr^3-8r^4)}^{M}s^2-$$
$$r^4\overbrace{(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)}^{N}\boxed{\overset{?}{\underset{(1)}{\ge}}}0$$

Now, Double – Rouche $\Rightarrow$ in order to prove (1), it suffices to prove :

$$\text{LHS of (1)}\overset{?}{\ge}-s^4\overbrace{(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)}^{T}$$

$$\Leftrightarrow-(24R+6r)s^6+(128R^3+68R^2r-24Rr^2-13r^3)s^4+rMs^2-r^3N\boxed{\overset{?}{\underset{(2)}{\ge}}}0$$

Again $\because P=-(24R+6r)T\ge0\therefore$ to prove (2), it suffices to prove : LHS of (2) $\overset{?}{\ge}$ P
$$\Leftrightarrow(32R^3-436R^2r-96Rr^2-r^3)s^4+$$
$$\overbrace{r(1792R^4+1792R^3r+588R^2r^2+52Rr^3-2r^4)s^2}^{U}-r^3N\boxed{\overset{?}{\underset{(3)}{\ge}}}0\text{ and }\because U\overset{Gerretsen}{\ge}$$

$U(16Rr-5r^2)>r^3N\left(\because R\overset{Euler}{\ge}2r\right)\therefore$ if $32R^3-436R^2r-96Rr^2-r^3\ge0$,

then (3) is definitely true and if : $32R^3-436R^2r-96Rr^2-r^3<0$, then, since :

$Q=(32R^3-436R^2r-96Rr^2-r^3)T\overset{via\ (ii)}{\ge}0\therefore$ to prove (3), it suffices to prove :

$$\text{LHS of (3)}\overset{?}{\ge}Q\Leftrightarrow(8R^4+43R^3r-461R^2r^2-29Rr^3+14r^4)s^2\boxed{\overset{?}{\underset{(4)}{\ge}}}$$
$$r(128R^5-1648R^4r-1664R^3r^2-611R^2r^3-99Rr^4-6r^5)$$

$\boxed{\textit{Case 1}}$ $X=8R^4+43R^3r-461R^2r^2-29Rr^3+14r^4\ge0$ and then :

$$\text{LHS of (4)}\overset{Gerretsen}{\ge}X(16Rr-5r^2)\overset{?}{\ge}\text{RHS of (4)}$$
$$\Leftrightarrow32t^6+76t^5-167t^3-832t^2+68t+48\overset{?}{\ge}0\left(t=\frac{R}{r}\right)\Leftrightarrow$$
$$(t-2)(32t^5+140t^4+280t^3+393t^2-46t-24)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2$$

$\boxed{\textit{Case 2}}$ $X<0$ and then : LHS of (4) $\overset{Gerretsen}{\ge}X(4R^2+4Rr+3r^2)\overset{?}{\ge}$ RHS of (4)

$$\Leftrightarrow2296t^4-5927t^3+2452t^2+468t-64\overset{?}{\ge}0$$
$$\Leftrightarrow(t-2)(2296t^3-1335t^2-218t+32)\overset{?}{\ge}0\to\text{true}\because t\ge2\Rightarrow\text{(4) is true; again,}$$

$$\sum_{cyc}\frac{r_a}{w_a}\ge\frac{29R^2-2s^2+10Rr+14r^2}{16Rr}\left(\begin{matrix}\text{Inequality in Triangle} - 228 \text{ by Dang}\\ \text{Ngoc Minh; published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\overset{?}{\ge}\frac{R(s^2+8Rr+5r^2)}{r(s^2+2Rr+r^2)}\Leftrightarrow -2s^4+(13R^2+6Rr+12r^2)s^2-$$

$$r(70R^3+31R^2r-38Rr^2-14r^3)\boxed{\overset{?}{\underset{⑤}{\ge}}}0 \text{ and it suffices to prove : LHS of ⑤} \overset{?}{\ge}-2T$$

$$\Leftrightarrow(5R^2-34Rr+16r^2)s^2+r(58R^3+65R^2r+62Rr^2+16r^3)\boxed{\overset{?}{\underset{⑥}{\ge}}}0 \text{ and it's}$$

trivially true when : $5R^2-34Rr+16r^2\ge 0$ and when : $5R^2-34Rr+16r^2<0$,

$$\text{then : LHS of ⑥}\overset{\text{Gerretsen}}{\ge}(5R^2-34Rr+16r^2)(4R^2+4Rr+3r^2)+$$

$$r(58R^3+65R^2r+62Rr^2+16r^3)\overset{?}{\ge}0\Leftrightarrow 10t^4-29t^3+4t^2+12t+32\overset{?}{\ge}0$$

$$\Leftrightarrow(t-2)^2(10t^2+11t+8)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow ⑥\Rightarrow ⑤ \text{ is true; so}$$

$$\sum_{cyc}\frac{r_a}{w_a}\ge\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}\ge\sum_{cyc}\frac{m_a}{h_a}\ \forall\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4260. In ΔABC the following relationship holds:

$$r_b+r_c\ge max\left\{\frac{m_b}{m_c}r_b+\frac{m_c}{m_b}r_c;\frac{w_b}{w_c}r_b+\frac{w_c}{w_b}r_c;\frac{h_b}{h_c}r_b+\frac{h_c}{h_b}r_c\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Bardiya-Nepal

Let $J\in\{m,w,h\}$ represent the median, angle bisector or the altitude.
It suffices to prove that:

$$r_b+r_c\ge\frac{J_b}{J_c}r_b+\frac{J_c}{J_b}r_c$$

WLOG, assume $b\ge c$.

Also let $(a,b,c)=(y+z,z+x,x+y)$ so that $z+x\ge x+y\ \therefore z\ge y$

$$We\ know,\ b\ge c\Longrightarrow J_b\le J_c\Leftrightarrow\frac{1}{J_b}\ge\frac{1}{J_c}.$$

Lemma: $r_bJ_b\ge r_cJ_c$

Proof of the lemma:

1.) *For h, we have to prove that*: $r_bh_b\ge r_ch_c\Leftrightarrow$

$$\frac{\Delta}{s-b}\cdot\frac{2\Delta}{b}\ge\frac{\Delta}{s-c}\cdot\frac{2\Delta}{c}\Leftrightarrow\frac{1}{y}\cdot\frac{1}{z+x}\ge\frac{1}{z}\cdot\frac{1}{x+y}\Leftrightarrow$$

$$\Leftrightarrow zx+yz\ge yz+xy\Leftrightarrow zx\ge xy\ \ which\ is\ true\ since\ z\ge y.$$

2.) *For w, we have to prove that*: $r_bw_b\ge r_cw_c\Leftrightarrow$

$$\frac{\Delta}{s-b}\cdot\frac{2}{a+c}\sqrt{acs(s-b)}\ge\frac{\Delta}{s-c}\cdot\frac{2}{a+b}\sqrt{abs(s-c)}\overset{Squaring\ and\ simplifying}{\Longleftrightarrow}$$

$$\frac{c}{s-b}\cdot\frac{1}{(a+c)^2}\ge\frac{b}{s-c}\cdot\frac{1}{(a+b)^2}$$

$$\Leftrightarrow\frac{c(s-c)}{(a+c)^2}\ge\frac{b(s-b)}{(a+b)^2}\Leftrightarrow\frac{xz+yz}{(x+2y+z)^2}\ge\frac{xy+yz}{(x+y+2z)^2}\ \text{which is true since } z\ge y.$$

3.) For m, we have to prove that: $r_bm_b\ge r_cm_c\Leftrightarrow$

$$\frac{\Delta^2}{(s-b)^2}\left(\frac{2a^2+2c^2-b^2}{4}\right)\ge\frac{\Delta^2}{(s-c)^2}\left(\frac{2a^2+2b^2-c^2}{4}\right)$$

$$\Leftrightarrow\frac{2(y+z)^2+2(x+y)^2-(z+x)^2}{y^2}\ge\frac{2(y+z)^2+2(z+x)^2-(x+y)^2}{z^2}$$

which is equivalent to:

$$\frac{(z-y)}{y^2z^2}\left[(y+z)\left(x-\frac{y^2-yz+z^2}{y+z}\right)^2+\frac{yz(8y^2+7yz+8z^2)}{y+z}\right]\ge0$$

which is also true since $z\ge y$.

Thus, the lemma is now proven!

We can notice that $\left(\frac{1}{J_b},\frac{1}{J_c}\right)$ *and* (r_bJ_b,r_cJ_c) *are similarly sorted.*

Hence, by Rearrangement Inequality we have:

$$\frac{1}{J_b}(r_bJ_b)+\frac{1}{J_c}(r_cJ_c)\ge\frac{1}{J_c}(r_bJ_b)+\frac{1}{J_b}(r_cJ_c)$$

$$\Leftrightarrow r_b+r_c\ge\frac{J_b}{J_c}r_b+\frac{J_c}{J_b}r_c$$

Therefore, $\quad r_b+r_c\ge max\left\{\frac{m_b}{m_c}r_b+\frac{m_c}{m_b}r_c;\frac{w_b}{w_c}r_b+\frac{w_c}{w_b}r_c;\frac{h_b}{h_c}r_b+\frac{h_c}{h_b}r_c\right\}$

Equality holds if the triangle is isosceles $(b=c)$

4261. In ΔABC the following relationship holds:

$$\frac{1}{r_a}+\frac{1}{m_a}\le\frac{4}{h_b+h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

Let $x=s-a,y=s-b,z=s-c$ *then:*

$$a=y+z,b=z+x,c=x+y\ \&\ s=x+y+z$$

$$F^2=s(s-a)(s-b)(s-c)=sxyz,\quad\frac{1}{r_a}=\frac{s-a}{F}=\frac{x}{F}\ (1)$$

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}$$

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}=\frac{4x^2+4xy+4xz+(y-z)^2}{4}=\frac{4xs+(y-z)^2}{4}\ \&$$

$$\frac{4}{h_b+h_c}=\frac{2bc}{F(b+c)}$$

$$\textit{We need to show:}$$

$$\frac{1}{r_a}+\frac{1}{m_a}\le\frac{4}{h_b+h_c}\quad or$$

$$\frac{F}{r_a}+\frac{F}{m_a}\le\frac{4F}{h_b+h_c}$$

$$x+\frac{F}{m_a}\overset{(1)}{\le}\frac{2(x+y)(x+z)}{2x+y+z}or,\frac{F}{m_a}\le\frac{2(x+y)(x+z)}{2x+y+z}-x$$

$$\frac{F}{m_a}\le\frac{x(y+z)+2yz}{2x+y+z}\ \ or\ \ \frac{F^2}{m_a^2}\le\left(\frac{x(y+z)+2yz}{2x+y+z}\right)^2$$

$$\frac{4sxyz}{4xs+(y-z)^2}\le\left(\frac{x(y+z)+2yz}{2x+y+z}\right)^2$$

$$(x(y+z)+2yz)^2(4xs+(y-z)^2)-4sxyz(2x+y+z)^2\ge 0$$

$$(xy+xz+2yz)^2(4x^2+4xy+4xz+y^2\text{-}2yz+z^2)-4xyz(x+y+z)(2x+y+z)^2\ge 0$$

$$(\textit{putting the value of } s=x+y+z)$$

$$4x^4y^2+4x^4z^2+8x^4yz+4x^3y^3+4x^3z^3-4x^4yz-4x^3y^2z-4x^3yz^2+x^2y^4+$$
$$+x^2z^4+8x^2y^3z+8x^2yz^3+10x^2y^2z^2+4y^4z^2-4y^3z^3+4y^2z^4\ge 0$$

$$or,(y-z)^2(4x^4+4x^3(y+z)+x^2(y-z)^2+4y^2z^2)\ge 0\ true$$

$$\textit{Equality holds for } y=z \textit{ or } b=c$$

4262. In any Δ ABC the following relationship holds :

$$\sin\frac{A}{2}.m_a+\sin\frac{B}{2}.m_b+\sin\frac{B}{2}.m_b\le\frac{\sqrt{3}}{2}\cdot s$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\sin\frac{A}{2}.m_a\right)=\sum_{cyc}\frac{\sqrt{a(s-b)(s-c)}.m_a}{\sqrt{4Rrs}}\overset{CBS}{\le}$$

$$\le\frac{1}{\sqrt{4Rrs}}.\sqrt{\sum_{cyc}((s-b)(s-c))}.\sqrt{\frac{1}{4}\sum_{cyc}\left(a\left(2\sum_{cyc}a^2-3a^2\right)\right)}=$$

$$=\frac{\sqrt{4Rr+r^2}}{\sqrt{4Rrs}}\cdot\sqrt{2s(s^2-4Rr-r^2)-\frac{3}{4}\cdot 2s(s^2-6Rr-3r^2)}\overset{?}{\leq}\frac{\sqrt{3}}{2}\cdot s$$

$$\overset{\text{squaring}}{\Leftrightarrow}\frac{(s^2+2Rr+5r^2)(4R+r)}{8R}\overset{?}{\leq}\frac{3s^2}{4}\Leftrightarrow(2R-r)s^2\underbrace{\overset{?}{\geq}}_{(*)}r(4R+r)(2R+5r)$$

$$\text{Now, }(2R-r)s^2\overset{\text{Gerretsen}}{\geq}(2R-r)(16Rr-5r^2)\overset{?}{\geq}r(4R+r)(2R+5r)$$

$$\Leftrightarrow 24Rr(R-2r)\overset{?}{\geq}0\to\text{true via Euler}\Rightarrow(*)\text{ is true}$$

$$\therefore\sin\frac{A}{2}\cdot m_a+\sin\frac{B}{2}\cdot m_b+\sin\frac{B}{2}\cdot m_b\leq\frac{\sqrt{3}}{2}\cdot s\ \forall\ ABC,$$
$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4263. In ΔABC the following relationship holds:

$$\frac{1}{12}\sum_{cyc}m_a(h_a+h_b)(h_a+h_c)\leq\frac{1}{8}\prod_{cyc}(r_a+h_a)\leq m_am_bm_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$h_a+h_b=2F\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{2F(a+b)}{ab}\ \&\ h_a+h_c=\frac{2F(a+c)}{ac}$$

$$\frac{1}{12}\sum_{cyc}m_a(h_a+h_b)(h_a+h_c)=\frac{F^2}{3}\sum\frac{m_a(a+b)(a+c)}{a^2bc}$$

$$r_a+h_a=\frac{F}{s-a}+\frac{2F}{a}=\frac{F(2s-a)}{a(s-a)}=\frac{F(b+c)}{a(s-a)}\ \&\ F^2=s(s-a)(s-b)(s-c)$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a)=\frac{F^3}{8}\frac{(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)}=\frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$\text{we need to show: }\frac{1}{12}\sum_{cyc}m_a(h_a+h_b)(h_a+h_c)\leq\frac{1}{8}\prod_{cyc}(r_a+h_a)$$

$$\text{or, }\frac{F^2}{3}\sum\frac{m_a(a+b)(a+c)}{a^2bc}\leq\frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$\text{or, }8F\sum\frac{m_a(a+b)(a+c)}{a}\leq 3s(a+b)(b+c)(c+a)$$

$$8F\sum\frac{m_a(a+b)(b+c)(a+c)}{a(b+c)}\leq 3s(a+b)(b+c)(c+a)$$

$$\sum\frac{m_a}{a(b+c)}\leq\frac{3s}{8F}=\frac{3}{8r}$$

$$\left(\sum\frac{m_a}{a(b+c)}\right)^2\overset{C-S}{\leq}\left(\sum m_a^2\right)\left(\sum\frac{1}{a^2(b+c)^2}\right)\overset{AM-GM}{\leq}\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{a^24bc}\right)\leq$$

$$\overset{Leibniz}{\leq}\left(\frac{3}{4}\cdot 9R^2\right)\left(\frac{1}{4}\sum\frac{bc}{a^2b^2c^2}\right)\leq\frac{27}{16a^2b^2c^2}\frac{(a+b+c)^2}{3}=\frac{27\times 4s^2}{48\times 16R^2r^2s^2}=\frac{9}{64r^2}$$

$$then \sum \frac{m_a}{a(b+c)} \le \frac{3}{8r} \;\&\; \frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a)\ \ (A)$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a) = \frac{F^3}{8}\frac{(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)} = \frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$= \frac{Fs(a+b)(b+c)(c+a)}{8\times 4RF} = \frac{s(a+b)(b+c)(c+a)}{32R}$$

$$m_am_bm_c \ge \frac{b+c}{2}\cos\frac{A}{2}\times\frac{a+c}{2}\cos\frac{B}{2}\times\frac{b+a}{2}\cos\frac{C}{2}$$

$$= \frac{(a+b)(b+c)(c+a)}{8}\prod\cos\left(\frac{A}{2}\right) =$$

$$= \frac{(a+b)(b+c)(c+a)}{8}\frac{s}{4R} = \frac{s(a+b)(b+c)(c+a)}{32R}$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a) \le m_am_bm_c\ \ (B)$$

From $(A)\ \&\ (B)$ *we get*:

$$\frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a) \le m_am_bm_c$$

Equality holds for an equilateral triangle.

4264. ***In any ΔABC the following relationship holds:***

$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c} \ge \frac{20R^2-Rr+3r^2}{18r(R+r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c} \le \frac{r_a}{4}\left(\frac{1}{h_a}+\frac{1}{h_b}\right)+\frac{r_b}{4}\left(\frac{1}{h_c}+\frac{1}{h_b}\right)+\frac{r_c}{4}\left(\frac{1}{h_a}+\frac{1}{h_c}\right) =$$

$$= \frac{1}{4}\left(r_a\left(\frac{1}{r}-\frac{1}{h_c}\right)+r_b\left(\frac{1}{r}-\frac{1}{h_a}\right)+r_c\left(\frac{1}{r}-\frac{1}{h_b}\right)\right) = \frac{1}{4}\left(\frac{(r_a+r_b+r_c)}{r}-\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right)\right)$$
$$=$$

$$= \frac{1}{4r}(r_a+r_b+r_c)-\frac{1}{4}\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right) = \frac{R}{r}+\frac{1}{4}-\frac{1}{4}\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right)$$

$$\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}} = 3\sqrt[3]{\frac{R}{2r}}$$

$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c} \le \frac{R}{r}+\frac{1}{4}-\frac{3}{4}\sqrt[3]{\frac{R}{2r}}$$

Let's prove that :

$$\frac{R}{r}+\frac{1}{4}-\frac{3}{4}\sqrt[3]{\frac{R}{2r}}\le\frac{20R^2-Rr+3r^2}{18r(R+r)}$$

$$Let:\ \sqrt[3]{\frac{R}{2r}}=x\ge 1\ \ (R\ge 2r)$$

$$Then:\ \ 2x^3-\frac{1}{4}-\frac{3}{4}x\le\frac{80x^6-2x^3+3}{18(2x^3+1)}\Leftrightarrow\frac{8x^3-3x+1}{4}\le\frac{80x^6-2x^3+3}{18(2x^3+1)}$$

$$(8x^3-3x+1)(18x^3+9)\le 160x^6-4x^3+6$$

$$16x^6+54x^4-94x^3+27x-3\ge 0$$

$$(x-1)\left(16x^5+16x^4+70x^3-24x^2-24x+3\right)\ge 0\ \ -to\ prove$$

$$(x-1)\ge 0\ \ \to\ \ x\ge 1\to\sqrt[3]{\frac{R}{2r}}\ge 1\to\ \ R\ge 2r\ \ (True)$$

Remains to prove: $16x^5+16x^4+70x^3-24x^2-24x+3>0$

$$16x^5+16x^4+24x^2(x-1)+24x(x^2-1)+22x^3+3>0\ \ (True)$$

$$Equality\ \ holds\ \ for:\ \ \ a=b=c$$

4265.

In any Δ ABC the following relationship holds :

$$\frac{2m_a}{r_b+r_c}+\frac{r_b+r_c}{2m_a}\le\sqrt{\frac{R+6r}{2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2m_a}{r_b+r_c}=x\ (say)\ and\ we\ are\ to\ prove:x+\frac{1}{x}\overset{?}{\le}\sqrt{\frac{R+6r}{2r}}=t\ (say)$$

$$\Leftrightarrow x^2-xt+1\overset{?}{\le}0\Leftrightarrow 2x\boxed{\underset{(1)}{\overset{?}{\le}}}\,t+\sqrt{t^2-4}\ \text{AND}\ 2x\boxed{\underset{(2)}{\overset{?}{\ge}}}\,t-\sqrt{t^2-4}$$

$$\left(\because t^2-4=\frac{R-2r}{2r}\overset{\text{Euler}}{\ge}0\right)$$

$$\text{Now},(b+c)^2\overset{?}{\ge}32Rr.\cos^2\frac{A}{2}=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0 \Leftrightarrow (b+c-2a)^2\overset{?}{\geq}0 \to \text{true}$$

$$\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2} \Rightarrow m_a \overset{\text{Lascu}}{\geq} \frac{b+c}{2}.\cos\frac{A}{2}\geq\sqrt{8Rr}.\cos^2\frac{A}{2}$$

$$\Rightarrow \frac{2m_a}{r_b+r_c}\geq\frac{4.\sqrt{2Rr}.\cos^2\frac{A}{2}}{4R.\cos^2\frac{A}{2}}=\sqrt{\frac{2r}{R}} \Rightarrow 2x\geq 2.\sqrt{\frac{2r}{R}}\overset{?}{\geq} t-\sqrt{t^2-4}$$

$$=\sqrt{\frac{R+6r}{2r}}-\sqrt{\frac{R-2r}{2r}} \Leftrightarrow \frac{8r}{R}\overset{?}{\geq}\frac{R+6r}{2r}+\frac{R-2r}{2r}-2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}}$$

$$\Leftrightarrow 2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}}\overset{?}{\geq}\frac{R^2+2Rr-8r^2}{Rr}$$

$$\Leftrightarrow \frac{(R+6r)(R-2r)}{r^2}\overset{?}{\geq}\frac{(R^2+2Rr-8r^2)^2}{R^2r^2}$$

$$\left(\because R^2+2Rr-8r^2=R^2-4r^2+2r(R-2r)\overset{\text{Euler}}{\geq}0\right) \Leftrightarrow 32r^3(R-2r)\overset{?}{\geq}0$$

$$\to \text{true via Euler} \Rightarrow \boxed{② \text{ is true}}$$

$$\text{Again}, 2x \overset{\text{Dang Ngoc Minh}}{\leq} \sqrt{\frac{R+2r}{r}}$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 352; published at www.ssmrmh.ro)

$$\overset{?}{\leq} t+\sqrt{t^2-4}=\sqrt{\frac{R+6r}{2r}}+\sqrt{\frac{R-2r}{2r}}$$

$$\Leftrightarrow \frac{R+2r}{r}\overset{?}{\leq}\frac{R+6r}{2r}+\frac{R-2r}{2r}+2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}} \Leftrightarrow 2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}}\overset{?}{\geq}0$$

$$\to \text{true via Euler} \Rightarrow \boxed{① \text{ is true}} \therefore \text{both ① and ② are true and so,}$$

$$\frac{2m_a}{r_b+r_c}+\frac{r_b+r_c}{2m_a}\leq\sqrt{\frac{R+6r}{2r}}\ \forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4266. In any $\Delta\,ABC$ with $a\geq b\geq c$ and $d,e,f\geq 0, e=\text{mid}\{d,e,f\}$, the following relationship holds :

$$d.\left(r_a h_a - h_b h_c\right) + e.\left(r_b h_b - h_c h_a\right) + f.\left(r_c h_c - h_a h_b\right) \ge 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a h_a - h_b h_c = \frac{2r^2s^2}{a(s-a)} - \frac{4r^2s^2}{bc} = 2r^2s^2.\frac{\left(bc - 2a(s-a)\right)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}$$

$$= 2r^2s^2.\frac{(a^2 + bc - ab - ac)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)} = 2r^2s^2.\frac{(a-b)(a-c)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}$$

$$\therefore r_a h_a - h_b h_c = 2r^2s^2.\frac{yz(x-y)(x-z)}{abc(s-a)(s-b)(s-c)} \text{ and analogs}$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \text{ and then : } a = y+z, b = z+x, c = x+y \\ \text{and } s = x+y+z \end{matrix}\right) \Rightarrow LHS$$

$$\ge \frac{2r^2s^2}{abcr^2s}\left(\overbrace{d.yz(x-y)(x-z) + e.zx(y-z)(y-x) + f.xy(z-x)(z-y)}^{M}\right) \to (1)$$

$\boxed{\text{Case 1}}$ $d \ge e \ge f$ and then, we can assign : $e = f + m$ and $d = f + m + n$ (where $m, n \ge 0$) and then, following simplification :

$$\therefore (1), (2) \Rightarrow d.\left(r_a h_a - h_b h_c\right) + e.\left(r_b h_b - h_c h_a\right) + f.\left(r_c h_c - h_a h_b\right) \ge 0$$

$\boxed{\text{Case 2}}$ $f \ge e \ge d$ and then, we can assign : $e = d + m'$ and $f = d + m' + n'$ (where $m', n' \ge 0$) and then, following simplification :

$$M = n'yz(x-y)(x-z) + m'z^2\,(x-y)^2 + f\left(\sum_{cyc} x^2y^2 - xyz\left(\sum_{cyc} x\right)\right) \ge 0$$

$$\left(\begin{matrix} \because m', n', f \ge 0 \text{ and } \because a \ge b \ge c \Rightarrow z \ge y \ge x \Rightarrow \\ (x-y)(x-z) \ge 0 \end{matrix}\right) \therefore M \ge 0 \to (3)$$

$\therefore (1), (3) \Rightarrow d.\left(r_a h_a - h_b h_c\right) + e.\left(r_b h_b - h_c h_a\right) + f.\left(r_c h_c - h_a h_b\right) \ge 0$ and combining both cases, $d.\left(r_a h_a - h_b h_c\right) + e.\left(r_b h_b - h_c h_a\right) + f.\left(r_c h_c - h_a h_b\right) \ge 0$ $\forall\, \Delta\, ABC$ with $a \ge b \ge c$ and $d, e, f \ge 0 \,|\, e = \text{mid}\{d, e, f\}$, $'' = ''$ iff $\Delta\, ABC$ is equilateral (QED)

4267. In any $\Delta\, ABC$ with $a \ge b \ge c$ and $d, e, f \ge 0$, $e = \text{mid}\{d, e, f\}$, the following relationship holds :

$$d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right) \ge 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a - \frac{w_b w_c}{w_a} = \frac{s\tan\frac{A}{2}\cdot\frac{2bc}{b+c}\cdot\cos\frac{A}{2} - \frac{2ca}{c+a}\cos\frac{B}{2}\cdot\frac{2ab}{a+b}\cdot\cos\frac{C}{2}}{w_a}$$

$$= \frac{2bc}{w_a}\left(\frac{s\sin\frac{A}{2}}{b+c} - \frac{2a^2.s}{4R(c+a)(a+b)\cos\frac{A}{2}}\right)$$

$$= \frac{2abcs}{w_a}\left(\frac{(c+a)(a+b)-2a(b+c)}{(a+b)(b+c)(c+a).4R\cos\frac{A}{2}}\right)$$

$$= \frac{2abcs(a-b)(a-c)}{(a+b)(b+c)(c+a).4R.\frac{2bc}{b+c}.\frac{s(s-a)}{bc}}$$

$$= \frac{2abcs(s+s-a)(s-b)(s-c)(x-y)(x-z)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R.2s}$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \textit{ and then}: a = y+z, b = z+x, c = x+y \\ \textit{and } s = x+y+z \end{matrix}\right)$$

$$\therefore r_a - \frac{w_b w_c}{w_a} = \frac{abc(s+x)yz(x-y)(x-z)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}.\textit{ and analogs}$$

$$\Rightarrow d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right)$$

$$= \frac{abc\big(d.(s+x)yz(x-y)(x-z) + e.(s+y)zx(y-z)(y-x) + f.(s+z)xy(z-x)(z-y)\big)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}$$

$$= \frac{abc\left(\begin{matrix} s\left(\overbrace{d.yz(x-y)(x-z) + e.zx(y-z)(y-x) + f.xy(z-x)(z-y)}^{M}\right) + \\ xyz\left(\overbrace{d.(x-y)(x-z) + e.(y-z)(y-x) + f.(z-x)(z-y)}^{N}\right)\end{matrix}\right)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}$$

$= \text{LHS} \rightarrow (1)$

$\boxed{\textit{Case 1}}$ $d \geq e \geq f$ *and then, we can assign* : $e = f + m$ *and* $d = f + m + n$ (where $m, n \geq 0$) *and then, following simplification* :

$$M = nyz(x-y)(x-z) + mz^2(x-y)^2 + f\left(\sum_{cyc} x^2y^2 - xyz\left(\sum_{cyc} x\right)\right) \geq 0$$

$(\because m, n, f \geq 0$ *and* $\because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow (x-y)(x-z) \geq 0) \therefore M \geq 0 \rightarrow (2)$

$$\textit{Also}, N = n(x-y)(x-z) + m(x-y)^2 + f\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) \geq 0$$

$(\because m, n, f \geq 0$ *and* $\because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow (x-y)(x-z) \geq 0) \therefore N \geq 0 \rightarrow (3)$

$\therefore ①,②,③ \Rightarrow d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right) \geq 0$

$\boxed{\text{Case 2}}$ $f \geq e \geq d$ *and then, we can assign* : $e = d + m'$ *and* $f = d + m' + n'$ (where $m', n' \geq 0$) *and then, following simplification* :

$$M = n'yz(x-y)(x-z) + m'z^2(x-y)^2 + f\left(\sum_{cyc} x^2y^2 - xyz\left(\sum_{cyc} x\right)\right) \geq 0$$

$\left(\begin{matrix}\because m', n', f \geq 0 \text{ and } \because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow \\ (x-y)(x-z) \geq 0\end{matrix}\right) \therefore M \geq 0 \to ④$

$$Also, N = n'(x-y)(x-z) + m'(x-y)^2 + f\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) \geq 0$$

$\left(\begin{matrix}\because m', n', f \geq 0 \text{ and } \because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow \\ (x-y)(x-z) \geq 0\end{matrix}\right) \therefore N \geq 0 \to ⑤ \therefore ①,④,⑤ \Rightarrow$

$d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right) \geq 0$ *and combining* both cases, $d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right) \geq 0$

$\forall \Delta ABC$ with $a \geq b \geq c$ *and* $d, e, f \geq 0 \,|\, e = mid\{d, e, f\}$,
$'' = ''$ iff ΔABC is equilateral (QED)

4268. **In any ΔABC the following relationship holds :**

$$r_a w_b w_c + r_b w_c w_a + r_c w_a w_b \geq \frac{9\sqrt{3}}{8}.abc$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\sum_{cyc}\frac{r_a}{w_a} \geq \frac{16Rr^2s^2}{s^2+2Rr+r^2}.\frac{29R^2-2s^2+10Rr+14r^2}{16Rr}$$

$\left(\begin{matrix}\text{Reference : Inequality in Triangle} - 228 \text{ by Dang Ngoc Minh;} \\ \text{publshed at www.ssmrmh.ro}\end{matrix}\right) \overset{?}{\geq} \frac{9\sqrt{3}}{8}.abc$

$= \frac{9\sqrt{3}.Rrs}{2} \overset{\text{squaring}}{\Leftrightarrow} 16s^6 - (707R^2 + 160Rr + 224r^2)s^4 +$
$(3364R^4 + 1348R^3r + 3162R^2r^2 + 1120Rr^3 + 784r^4)s^2 -$
$243R^2r^2(2R+r)^2 \boxed{\underset{①}{\overset{?}{\geq}}} 0$ and $\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ to prove ①,

$$\text{it suffices to prove : LHS of } ① \overset{?}{\geq} 16(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow -(707R^2-608Rr+464r^2)s^4$$
$$+(3364R^4+1348R^3r-9126R^2r^2+8800Rr^3-416r^4)s^2-$$
$$r^2(972R^4-64564R^3r+61683R^2r^2-19200Rr^3+2000r^4) \boxed{\underset{②}{\overset{?}{\geq}}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0, \text{where } m=$$
$$2R^2+10Rr-r^2 \;\&\; n=2(R-2r)\sqrt{R^2-2Rr} \therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$
$$\therefore \text{in order to prove } ②, \text{it suffices to prove : LHS of } ② \overset{?}{\geq}$$
$$-(707R^2-608Rr+464r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$\Leftrightarrow (134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4)s^2+$$
$$r\left(11312R^5-1487R^4r+18390R^3r^2-11500R^2r^3+6040Rr^4-384r^5\right) \boxed{\underset{③}{\overset{?}{\geq}}} 0$$

$$\text{and} \because R \overset{\text{Euler}}{\geq} 2r \therefore ③ \text{ is trivially true if :}$$
$$134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4\geq 0 \text{ and if :}$$
$$134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4<0, \text{then LHS of } ③ \overset{\text{Gerretsen}}{\geq}$$
$$(134R^4-2590R^3r+648R^2r^2-424Rr^3+128r^4)(4R^2+4Rr+3r^2)+$$
$$r\left(11312R^5-1487R^4r+18390R^3r^2-11500R^2r^3+6040Rr^4-384r^5\right)\overset{?}{\geq} 0$$
$$\Leftrightarrow 536t^5+1488t^4-8853t^3+11516t^2-10740t+5280\overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(536t^4+2560t^3-3733t^2+4050t-2640)\overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \forall \Delta ABC \;\&\; \text{so}, r_a w_b w_c + r_b w_c w_a + r_c w_a w_b \geq \frac{9\sqrt{3}}{8}.abc$$
$$\forall ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4269. In any ΔABC the following relationship holds :

$$\frac{1}{h_a+r_b}+\frac{1}{h_a+r_c}\geq\frac{1}{2h_a}+\frac{1}{r_b+r_c}\geq\frac{1}{\sqrt{r_br_c}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{h_a+r_b}+\frac{1}{h_a+r_c}\overset{?}{\geq}\frac{1}{2h_a}+\frac{1}{r_b+r_c}$$
$$\Leftrightarrow \frac{a(s-b)}{rs(a+2(s-b))}+\frac{a(s-c)}{rs(a+2(s-c))}\overset{?}{\geq}\frac{a}{4rs}+\frac{(s-b)(s-c)}{rsa}$$

$$\Leftrightarrow \frac{a^2\big((s-b)+(s-c)\big)+4a(s-b)(s-c)}{\big(a+2(s-b)\big)\big(a+2(s-c)\big)} \overset{?}{\geq} \frac{a^2+4(s-b)(s-c)}{4a}$$

$$\Leftrightarrow \frac{a^3+4a(s-b)(s-c)}{\big(a+2(s-b)\big)\big(a+2(s-c)\big)} \overset{?}{\geq} \frac{a^2+4(s-b)(s-c)}{4a}$$

$$\Leftrightarrow 4a^2 \overset{?}{\geq} a^2+4(s-b)(s-c)+2a\big((s-b)+(s-c)\big)$$

$$\Leftrightarrow 4a^2 \overset{?}{\geq} a^2+4(s-b)(s-c)+2a^2 \Leftrightarrow a^2 \overset{?}{\geq} a^2-(b-c)^2 \Leftrightarrow (b-c)^2 \overset{?}{\geq} 0$$

$$\to \text{true and again, } \frac{1}{2h_a}+\frac{1}{r_b+r_c} \overset{?}{\geq} \frac{1}{\sqrt{r_b r_c}}$$

$$\Leftrightarrow \frac{a^2+4(s-b)(s-c)}{4a.rs} \overset{?}{\geq} \frac{\sqrt{(s-b)(s-c)}}{rs}$$

$$\Leftrightarrow a^2-4a.\sqrt{(s-b)(s-c)}+4(s-b)(s-c) \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(a-2\sqrt{(s-b)(s-c)}\right)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore \frac{1}{h_a+r_b}+\frac{1}{h_a+r_c} \geq \frac{1}{2h_a}+\frac{1}{r_b+r_c} \geq \frac{1}{\sqrt{r_b r_c}}\ \forall\, ABC, '' = ''\text{ iff } b=c\ (QED)$$

4270. In any Δ ABC the following relationship holds :

$$\frac{21}{16}\cdot\sum_{cyc} b^2c^2+8.\frac{m_a^2m_b^2m_c^2}{\sum_{cyc}a^2} \geq (m_am_b+m_bm_c+m_cm_a)^2 \geq \frac{9}{32}\cdot\sum_{cyc}a^4+30.\frac{m_a^2m_b^2m_c^2}{\sum_{cyc}a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, we seek to prove : } 2\sum_{cyc}a^2b^2+\frac{9a^2b^2c^2}{\sum_{cyc}a^2} \overset{?}{\geq} 3abc\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 2\big((s^2+4Rr+r^2)^2-16Rrs^2\big)+\frac{72R^2r^2s^2}{s^2-4Rr-r^2} \overset{?}{\geq} 24Rrs^2$$

$$\Leftrightarrow s^6-(24Rr-r^2)s^4+r^2(132R^2+20Rr-r^2)s^2-r^3(4R+r)^3 \boxed{\overset{?}{\underset{(1)}{\geq}}} 0 \text{ and } \because$$

$$(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} (s^2-16Rr+5r^2)^3 \Leftrightarrow (12R-7r)s^4-$$

$$\mathrm{r}(318\mathrm{R}^2-250\mathrm{Rr}+38\mathrm{r}^2)\mathrm{s}^2+\mathrm{r}^2(2016\mathrm{R}^3-1944\mathrm{R}^2\mathrm{r}+594\mathrm{Rr}^2-63\mathrm{r}^3)\overset{?}{\underset{②}{\geq}}0$$

$$and \because \mathrm{P}=(12\mathrm{R}-7\mathrm{r})\left((\mathrm{s}^2-16\mathrm{Rr}+5\mathrm{r}^2)(\mathrm{R}-\mathrm{r})-\mathrm{r}^2(\mathrm{R}-2\mathrm{r})\right)^2\overset{\text{Rouche + Euler}}{\geq}0$$

$$\therefore \text{in order to prove ②, it suffices to prove : } (\mathrm{R}-\mathrm{r})^2.\text{LHS of ②}\overset{?}{\geq}\mathrm{P}$$

$$\Leftrightarrow (66\mathrm{R}^4-202\mathrm{R}^3\mathrm{r}+200\mathrm{R}^2\mathrm{r}^2-68\mathrm{Rr}^3+4\mathrm{r}^4)\mathrm{s}^2\overset{?}{\underset{③}{\geq}}$$

$$\mathrm{r}\left(1056\mathrm{R}^5-3496\mathrm{R}^4\mathrm{r}+3934\mathrm{R}^3\mathrm{r}^2-1717\mathrm{R}^2\mathrm{r}^3+228\mathrm{Rr}^4\right)$$

$$\text{Now} \because 66\mathrm{R}^4-202\mathrm{R}^3\mathrm{r}+200\mathrm{R}^2\mathrm{r}^2-68\mathrm{Rr}^3+4\mathrm{r}^4=$$

$$(\mathrm{R}-2\mathrm{r})(66\mathrm{R}^3-70\mathrm{R}^2\mathrm{r}+60\mathrm{Rr}^2+52\mathrm{r}^3)+108\mathrm{r}^3\overset{\text{Euler}}{\geq}108\mathrm{r}^3>0\ \therefore \text{LHS of ③}$$

$$\overset{\text{Rouche + Euler}}{\geq}(66\mathrm{R}^4-202\mathrm{R}^3\mathrm{r}+200\mathrm{R}^2\mathrm{r}^2-68\mathrm{Rr}^3+4\mathrm{r}^4)\left(\frac{16\mathrm{Rr}-5\mathrm{r}^2+}{\frac{\mathrm{r}^2(\mathrm{R}-2\mathrm{r})}{\mathrm{R}-\mathrm{r}}}\right)$$

$$\overset{?}{\geq}\text{RHS of ③}\Leftrightarrow 8\mathrm{t}^4-43\mathrm{t}^3+79\mathrm{t}^2-56\mathrm{t}+12\overset{?}{\geq}0\ \left(\mathrm{t}=\frac{\mathrm{R}}{\mathrm{r}}\right)$$

$$\Leftrightarrow (\mathrm{t}-2)^2(8\mathrm{t}^2-11\mathrm{t}+3)\overset{?}{\geq}0\to \text{true}\because \mathrm{t}\overset{\text{Euler}}{\geq}2\Rightarrow ③\Rightarrow ②\Rightarrow ①\text{ is true}$$

$$\therefore 2\sum_{\text{cyc}}a^2\mathrm{b}^2+\frac{9a^2\mathrm{b}^2\mathrm{c}^2}{\sum_{\text{cyc}}a^2}\geq 3a\mathrm{bc}\left(\sum_{\text{cyc}}a\right);\text{implementing it on } a\ \Delta\,\text{XYZ with sides}$$

$\frac{2\mathrm{m}_a}{3},\frac{2\mathrm{m}_\mathrm{b}}{3},\frac{2\mathrm{m}_\mathrm{c}}{3}$, whose medians & area are $\frac{a}{2},\frac{\mathrm{b}}{2},\frac{\mathrm{c}}{2}$ *and* $\frac{\mathrm{F}}{3}$ respectively, we get :

$$2.\frac{16}{81}.\sum_{\text{cyc}}\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2+9.\frac{\frac{64}{729}.\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2\mathrm{m}_\mathrm{c}^2}{\frac{4}{9}.\sum_{\text{cyc}}\mathrm{m}_a^2}\geq 3.\frac{8}{27}\mathrm{m}_a\mathrm{m}_\mathrm{b}\mathrm{m}_\mathrm{c}.\frac{2}{3}.\left(\sum_{\text{cyc}}\mathrm{m}_a\right)$$

$$\Rightarrow 2.\frac{16}{81}.\frac{9}{16}.\sum_{\text{cyc}}\mathrm{b}^2\mathrm{c}^2+9.\frac{\frac{64}{729}.\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2\mathrm{m}_\mathrm{c}^2}{\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}}a^2}\geq 3.\frac{8}{27}\mathrm{m}_a\mathrm{m}_\mathrm{b}\mathrm{m}_\mathrm{c}.\frac{2}{3}.\left(\sum_{\text{cyc}}\mathrm{m}_a\right)$$

$$\Rightarrow \frac{2}{9}.\sum_{\text{cyc}}\mathrm{b}^2\mathrm{c}^2+\frac{8}{27}.\frac{9}{16}.\sum_{\text{cyc}}\mathrm{b}^2\mathrm{c}^2+\frac{64}{27}.\frac{\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2\mathrm{m}_\mathrm{c}^2}{\sum_{\text{cyc}}a^2}\geq$$

$$\frac{8}{27}\left(2\mathrm{m}_a\mathrm{m}_\mathrm{b}\mathrm{m}_\mathrm{c}\left(\sum_{\text{cyc}}\mathrm{m}_a\right)+\sum_{\text{cyc}}\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2\right)$$

$$\Rightarrow \boxed{\frac{21}{16}.\sum_{\text{cyc}}\mathrm{b}^2\mathrm{c}^2+8.\frac{\mathrm{m}_a^2\mathrm{m}_\mathrm{b}^2\mathrm{m}_\mathrm{c}^2}{\sum_{\text{cyc}}a^2}\geq(\mathrm{m}_a\mathrm{m}_\mathrm{b}+\mathrm{m}_\mathrm{b}\mathrm{m}_\mathrm{c}+\mathrm{m}_\mathrm{c}\mathrm{m}_a)^2}$$

$$\text{We now seek to prove : }\left(\sum_{\text{cyc}}ab\right)^2\overset{?}{\geq}\sum_{\text{cyc}}a^2\mathrm{b}^2+\frac{45a^2\mathrm{b}^2\mathrm{c}^2}{32\sum_{\text{cyc}}a^2}-\frac{\mathrm{F}^2}{2}\overset{?}{\geq}0\Leftrightarrow$$

$$(4R+2r)\boxed{\overset{?}{\underset{④}{\geq}}}r(61R^2+12Rr+2r^2);\ \&\ LHS_{④}\overset{Gerretsen}{\geq}\geq(4R+2r)(16Rr-5r^2)$$

$$\overset{?}{\geq}\text{ RHS of } ④\Leftrightarrow R^2-4r^2\overset{?}{\geq}0\to\text{true via Euler}\ \therefore\ \frac{1}{16}\cdot\left(\sum_{cyc}ab\right)^2\geq$$

$$\frac{1}{16}\cdot\sum_{cyc}a^2b^2+\frac{45a^2b^2c^2}{32\sum_{cyc}a^2}-\frac{F^2}{2}\text{ and implementing it on a triangle XYZ, we get :}$$

$$\frac{1}{16}\cdot\frac{16}{81}\left(\sum_{cyc}m_am_b\right)^2\geq\frac{1}{16}\cdot\frac{16}{81}\cdot\frac{9}{32}\left(\sum_{cyc}a^4+16F^2\right)+\frac{45.\frac{64}{729}\cdot m_a^2m_b^2m_c^2}{32.\frac{4}{9}\cdot\frac{3}{4}\cdot\sum_{cyc}a^2}-\frac{F^2}{18}\Rightarrow$$

$$\boxed{(m_am_b+m_bm_c+m_cm_a)^2\geq\frac{9}{32}\cdot\sum_{cyc}a^4+30.\frac{m_a^2m_b^2m_c^2}{\sum_{cyc}a^2}}\ ''=''\text{ iff } a=b=c\ (QED)$$

4271. In any Δ ABC the following relationship holds :

$$max\left\{\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\right\}\leq\frac{2}{h_b+h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=s-a, y=s-b, z=s-c \text{ and then : } a=y+z, b=z+x,$$

$$c=x+y\text{ and } s=x+y+z\text{ and now : }\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c}\overset{?}{\leq}\frac{2}{h_b+h_c}$$

$$\Leftrightarrow\frac{(s-a)(s-b)}{rsc}+\frac{(s-a)(s-c)}{rsb}\overset{?}{\leq}\frac{2bc}{2rs(b+c)}$$

$$\Leftrightarrow(z+x)^2(x+y)^2\overset{?}{\geq}\big(xy(z+x)+zx(x+y)\big)(2x+y+z)$$

$$\overset{\text{expanding and simplifying}}{\Leftrightarrow}x^4-2x^2yz+y^2z^2\overset{?}{\geq}0\Leftrightarrow(x^2-yz)^2\overset{?}{\geq}0\to\text{true}$$

$$\text{and again, }\frac{1}{2r_a}+\frac{1}{r_b+r_c}\overset{?}{\leq}\frac{2}{h_b+h_c}\Leftrightarrow\frac{s-a}{2rs}+\frac{(s-b)(s-c)}{rsa}\overset{?}{\leq}\frac{2bc}{2rs(b+c)}$$

$$\Leftrightarrow(2x+y+z)(x(y+z)+2yz)\overset{?}{\geq}2(y+z)(z+x)(x+y)$$

$$\overset{\text{expanding and simplifying}}{\Leftrightarrow}x(y^2-2yz+z^2)\overset{?}{\geq}0\Leftrightarrow x(y-z)^2\overset{?}{\geq}0\to\text{true}$$

$$\therefore\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\leq\frac{2}{h_b+h_c}$$

$$\Rightarrow max\left\{\frac{1}{r_a+r_b}+\frac{1}{r_a+r_c},\frac{1}{2r_a}+\frac{1}{r_b+r_c}\right\}\leq\frac{2}{h_b+h_c}\ \forall\ ABC,$$

$$'' = ''\text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4272. In any Δ ABC the following relationship holds :

$$(m_a+m_b+m_c)^4 \le 16\sum_{cyc} b^2c^2 - \frac{117a^2b^2c^2}{16\sum_{cyc} a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We seek to prove : } \frac{1}{16}\left(\sum_{cyc} a\right)^4 \overset{?}{\le} \frac{16}{9}\sum_{cyc} a^2b^2 - \frac{52}{27}\cdot\frac{m_a^2m_b^2m_c^2}{\sum_{cyc} a^2} \text{ and since :}$$

$$m_a^2m_b^2m_c^2 = \frac{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3}{16}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle} - 124 \text{ by Dang Ngoc Minh;}\\ \text{publshed at www.ssmrmh.ro}\end{matrix}\right)$$

$$\therefore \text{ the inequality becomes : } s^4 \overset{?}{\le} \frac{16}{9}\left((s^2+4Rr+r^2)^2 - 16Rrs^2\right) -$$

$$\frac{52}{27}\cdot\frac{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3}{32(s^2-4Rr-r^2)}$$

$$\Leftrightarrow 155s^6 - (3588Rr-171r^2)s^4 + r^2(19212R^2+4632Rr+45r^2)s^2 -$$

$$r^3(23744R^3+17808R^2r+4452Rr^2+371r^3) \boxed{\overset{?}{\underset{(1)}{\ge}}} 0 \text{ and } \because (s^2-16Rr+5r^2)^3$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ to prove (1), it suffices to prove : LHS of (1)} \overset{?}{\ge} 155(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (642R-359r)s^4 - r(16638R^2-13172Rr+1930r^2)s^2 +$$

$$r^2(32(3183R^3-3193R^2r+945Rr^2-103r^3)+8R^2r+18Rr^2+5r^3) \boxed{\overset{?}{\underset{(2)}{\ge}}} 0 \text{ and}$$

$$\because (642R-359r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ to prove (2), suffices to prove :}$$

$$\text{LHS of (2)} \overset{?}{\ge} (642R-359r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (1953R^2-2368Rr+830r^2)s^2 \boxed{\overset{?}{\underset{(3)}{\ge}}}$$

$$r(31248R^3-46228R^2r+21616Rr^2-2842r^3)$$

$$\text{Finally, } (1953R^2-2368Rr+830r^2)s^2 \overset{\text{Rouche + Euler}}{\ge}$$

$$(1953R^2-2368Rr+830r^2)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\ge} \text{RHS of (3)}$$

$$\Leftrightarrow 528t^3 - 1345t^2 + 754t - 352 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(528t^2 - 289t + 176) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true}$$

$\therefore \frac{1}{16}\left(\sum_{\text{cyc}} a\right)^4 \leq \frac{16}{9}.\sum_{\text{cyc}} a^2b^2 - \frac{52}{27}.\frac{m_a^2m_b^2m_c^2}{\sum_{\text{cyc}} a^2}$ and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, whose medians as a consequence of trivial calculations are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ respectively, we arrive at :

$$\frac{1}{16}.\frac{16}{81}\left(\sum_{\text{cyc}} m_a\right)^4 \leq \frac{16}{9}.\frac{16}{81}.\sum_{\text{cyc}} m_a^2m_b^2 - \frac{52}{27}.\frac{1}{64}.\frac{a^2b^2c^2}{\frac{4}{9}.\sum_{\text{cyc}} m_a^2}$$

$$= \frac{16}{9}.\frac{16}{81}.\frac{9}{16}.\sum_{\text{cyc}} a^2b^2 - \frac{52}{27}.\frac{1}{64}.\frac{a^2b^2c^2}{\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}} a^2}$$

$$\Rightarrow (m_a + m_b + m_c)^4 \leq 16\sum_{\text{cyc}} b^2c^2 - \frac{117a^2b^2c^2}{16\sum_{\text{cyc}} a^2} \ \forall \text{ ABC},$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4273. **In ΔABC the following relationship holds:**

$$m_ah_bh_c + m_bh_ch_a + m_ch_ah_b \leq 3r_ar_br_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$4m_a\sqrt{a^2+b^2+c^2} = 2\sqrt{2b^2+2c^2-a^2}\sqrt{a^2+b^2+c^2} \overset{AM-GM}{\leq} 3(b^2+c^2)$$

$$m_a \leq \frac{3(b^2+c^2)}{4\sqrt{a^2+b^2+c^2}} \ (1)$$

$$m_ah_bh_c + m_bh_ch_a + m_ch_ah_b = 4F^2\left(\frac{m_a}{bc} + \frac{m_b}{ca} + \frac{m_c}{ab}\right) = \frac{4F^2}{abc}(am_a + bm_b + cm_c) =$$

$$= \frac{F}{R}(am_a + bm_b + cm_c) \overset{(1)}{\leq} \frac{F}{R}\sum \frac{3a(b^2+c^2)}{4\sqrt{a^2+b^2+c^2}} = \frac{F}{4R}\frac{3}{\sqrt{a^2+b^2+c^2}}\sum a(b^2+c^2)$$

$$\leq$$

$$\stackrel{CBS}{\leq}\frac{F}{R}\frac{3}{4\sqrt{a^2+b^2+c^2}}\cdot\frac{1}{3}\sum a\cdot\left(\sum b^2+c^2\right)=\frac{F}{R}\frac{3}{4\sqrt{a^2+b^2+c^2}}\cdot\frac{1}{3}\sum a\cdot\left(2\sum b^2\right)\stackrel{}{=}$$

$$=\frac{F}{4R}2s\cdot 2\sqrt{a^2+b^2+c^2}\stackrel{Leibniz}{\leq}\frac{F}{4R}4s\cdot\sqrt{9R^2}=3Rs\cdot\frac{F}{R}=3Fs=3r_ar_br_c.$$

Equality holds for an equilateral triangle.

4274. In any Δ ABC the following relationship holds :

$$8m_bm_c\geq max\left\{3a^2-12Rr+60r^2,4a^2+b^2+c^2-\frac{9b^2c^2(b-c)^2}{16F^2}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata India

$$(m_b-m_c)^2\leq\frac{s^2+2Rr-31r^2}{2}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle} - 345 \text{ by Dang Ngoc Minh;}\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$\Rightarrow 4m_b^2+4m_c^2-8m_bm_c\leq 2s^2+4Rr-62r^2$$

$$\Rightarrow 8m_bm_c\geq 2c^2+2a^2-b^2+2a^2+2b^2-c^2-2s^2-4Rr+62r^2$$

$$=4a^2+b^2+c^2-2s^2-4Rr+62r^2=3a^2+\sum_{cyc}a^2-2s^2-4Rr+62r^2$$

$$=3a^2+2(s^2-4Rr-r^2)-2s^2-4Rr+62r^2=3a^2-12Rr+60r^2$$

$$\therefore\boxed{8m_bm_c\geq 3a^2-12Rr+60r^2,''='' \text{ iff } a=b=c}\text{ and again,}$$

$$8m_bm_c\stackrel{?}{\geq}4a^2+b^2+c^2-\frac{9b^2c^2(b-c)^2}{16F^2}$$

$$\Leftrightarrow 8m_bm_c\stackrel{?}{\geq}4m_b^2+4m_c^2-\frac{9b^2c^2(b-c)^2}{16F^2}\Leftrightarrow\frac{9b^2c^2(b-c)^2}{16F^2}\stackrel{?}{\geq}4(m_b-m_c)^2$$

$$=\frac{4\left(m_b^2-m_c^2\right)^2}{(m_b+m_c)^2}=\frac{\left((2c^2+2a^2-b^2)-(2a^2+2b^2-c^2)\right)^2}{4(m_b+m_c)^2}=\frac{9(b-c)^2(b+c)^2}{4(m_b+m_c)^2}$$

$$\Leftrightarrow\frac{bc}{2F}\cdot(m_b+m_c)\stackrel{?}{\geq}b+c\left(\because (b-c)^2\stackrel{?}{\geq}0\right)\to\text{true}$$

$$\because\frac{bc}{2F}\cdot(m_b+m_c)\geq\frac{bc}{2F}\cdot(h_b+h_c)=\frac{abc(b+c)}{2F.2R}=\frac{4RF(b+c)}{4RF}=b+c$$

$$\therefore\boxed{8m_bm_c\geq 4a^2+b^2+c^2-\frac{9b^2c^2(b-c)^2}{16F^2},''='' \text{ iff } a=b=c}\text{ and so,}$$

$$8m_bm_c\geq max\left\{3a^2-12Rr+60r^2,4a^2+b^2+c^2-\frac{9b^2c^2(b-c)^2}{16F^2}\right\}\forall\text{ ABC,}$$

$'' = ''$ iff Δ ABC is equilateral (QED)

4275. *In any acute ΔABC following relationship holds:*

$$\frac{r_b+r_c}{\sqrt{2(b^2+c^2)}} \geq \cos\frac{A}{2} \geq \frac{2h_a+r_b+r_c}{2(b+c)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$(1st\ Part)$

$$Let\ x = s-a, y = s-b, z = s-c\ then$$

$$a = y+z, b = z+x, c = x+y\ \&\ s = x+y+z$$

$$r_b = \frac{F}{s-b} = \frac{F}{y}, r_c = \frac{F}{s-c} = \frac{F}{z}\ \&\ r_b+r_c = \frac{F(y+z)}{yz}$$

$$F^2 = s(s-a)(s-b)(s-c) = sxyz$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{x(x+y+z)}{(x+z)(x+y)}}$$

$$We\ need\ to\ show: \frac{r_b+r_c}{\sqrt{2(b^2+c^2)}} \geq \cos\frac{A}{2}\ or\ \frac{(r_b+r_c)^2}{2(b^2+c^2)} \geq \cos^2\frac{A}{2}$$

$$\frac{\left(\frac{F(y+z)}{yz}\right)^2}{2((x+z)^2+(x+y)^2)} \geq \frac{x(x+y+z)}{(x+z)(x+y)}$$

$$\frac{\frac{sxyz(y+z)^2}{(yz)^2}}{2((x+z)^2+(x+y)^2)} \geq \frac{x(x+y+z)}{(x+z)(x+y)}$$

$$(y+z)^2(x+z)(x+y) \geq 2yz((x+z)^2+(x+y)^2)$$

$$x^2y^2+x^2z^2-2x^2yz+xy^3-xy^2z-xyz^2+xz^3-y^3z+3y^2z^2-yz^3 \geq 0$$

$$x^2(y-z)^2+x(\acute{y}-z)^2(y+z)-yz(y-z)^2 \geq 0$$

$$(y-z)^2(x^2+x(y+z)-yz) \geq 0\ true$$

$$\left(\begin{matrix} as\ triangle\ is\ acute\ then\ a^2 < b^2+c^2 or \\ (y+z)^2 < (x+z)^2+(x+y)^2 or, x^2+x(y+z)-yz > 0 \end{matrix}\right)$$

$(2nd\ Part)$

$$acute\ triangle, a^2+c^2 > b^2\ or$$

$$(y+x)^2+(z+y)^2 > (x+z)^2\ or, xy+y^2+yz-xz > 0$$

$$a^2+b^2 > c^2\ or, (y+z)^2+(z+x)^2 > (x+y)^2 or, xz+yz+z^2-xy > 0$$

$$using\ above\ we\ get\ (x(y-z)+y^2+3yz) = (xy+yz+y^2-xz)+2yz > 0\ (A)$$

$$and$$

$$(x(y-z)-3yz-z^2))=xy-yz-3yz-z^2=-(xz+z^2+yz-xy)-2yz<0(B)$$

$$\frac{2h_a+r_b+r_c}{2(b+c)}=\frac{\frac{4F}{a}+\frac{F}{s-b}+\frac{F}{s-c}}{2(b+c)}=\frac{F\left(\frac{4}{y+z}+\frac{1}{y}+\frac{1}{z}\right)}{2(2x+y+z)}=\frac{F(y^2+6yz+z^2)}{2yz(y+z)(2x+y+z)}$$

We need to show:

$$\frac{2h_a+r_b+r_c}{2(b+c)}\le\cos\frac{A}{2}$$

$$\frac{F(y^2+6yz+z^2)}{2yz(y+z)(2x+y+z)}\le\sqrt{\frac{x(x+y+z)}{(x+z)(x+y)}}$$

$$\frac{sxyz(y^2+6yz+z^2)^2}{4y^2z^2(y+z)^2(2x+y+z)^2}\le\frac{x(x+y+z)}{(x+z)(x+y)}$$

$$4yz(y+z)^2(2x+y+z)^2-(x+y)(x+z)(y^2+6yz+z^2)\le 0$$

$$-(y-z)^2(x(y-z)+y^2+3yz)(x(y-z)-3yz-z^2))\ \ge 0\ \ \textit{true by}\ (A)\ \&\ (B)$$

Equality holds for an equilateral triangle.

4276. ***In any ΔABC following relationship holds:***

$$2m_a+r_a\le\left(2\sqrt{2}+2\right)R-\left(4\sqrt{2}-5\right)r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\textit{Let } x=\sin\frac{A}{2}\ \&\ t=\cos\frac{B-C}{2}\ \&\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}=2x\sqrt{1-x^2}$$

$$\textit{now } 0<x<1\ \textit{ and }\left|\frac{B-C}{2}\right|<\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}\ \&$$

$$t=\cos\frac{B-C}{2}>\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)=\sin\frac{A}{2}=x$$

$$\textit{so } x<t<1$$

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ \&\ A+B+C=\pi$$

$$2\sin\frac{B}{2}\sin\frac{C}{2}=\cos\frac{B-C}{2}-\cos\frac{B+C}{2}=\cos\frac{B-C}{2}-\sin\frac{A}{2}=t-x$$

$$\sin\frac{B}{2}\sin\frac{C}{2}=\frac{t-x}{2}$$

$$r_a=\ r=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ \&\ 2\cos\frac{B}{2}\cos\frac{C}{2}=\cos\frac{B+C}{2}+\cos\frac{B-C}{2}=$$

$$=\sin\frac{A}{2}+\cos\frac{B-C}{2}=t+x\ \ so,$$

$$\cos\frac{B}{2}\cos\frac{C}{2}=\frac{(t+x)}{2}$$

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}=\frac{4R^2(2\sin^2 B+2\sin^2 C-\sin^2 A)}{4}\ \ then$$

$$\frac{m_a^2}{R^2}=2\sin^2 C+2\sin^2 B-\sin^2 A=2(1-\cos(B+C)\cos(B-C))\ -\sin^2 A$$

$$=2(1+\cos A\cos(B-C))-\sin^2 A=$$

$$=2\left(1+\left(1-2\sin^2\frac{A}{2}\right)\left(2\cos^2\frac{B-C}{2}-1\right)\right)-\sin^2 A=$$

$$=2\left(1-(2x^2-1)(2t^2-1)\right)-4x^2(1-x^2)=4t^2-8x^2t^2+4x^4$$

$$Then\ \frac{2m_a}{R}=4\sqrt{t^2-2x^2t^2+x^4},\frac{r_a}{R}=2x(t+x),\frac{r}{R}=2x(t-x)$$

Using above result we need to show:

$$2m_a+r_a\le(2\sqrt{2}+2)R-(4\sqrt{2}-5)r$$

$$\frac{2m_a}{R}+\frac{r_a}{R}+(4\sqrt{2}-5)\frac{r}{R}\le(2\sqrt{2}+2)$$

$$4\sqrt{t^2-2x^2t^2+x^4}+2x(t+x)+2x(t-x)(4\sqrt{2}-5)\le(2\sqrt{2}+2)$$

$$4\sqrt{(t-x^2)^2}+2x(t+x)+2x(t-x)(4\sqrt{2}-5)\dot{\le}(2\sqrt{2}+2)$$

$$4\sqrt{(1-x^2)^2}+2x(1+x)+2x(1-x)(4\sqrt{2}-5)\overset{x<t<1}{\dot{\le}}(2\sqrt{2}+2)$$

$$4(1-x^2)+2x(1+x)+2x(1-x)(4\sqrt{2}-5)\overset{x<t<1}{\dot{\le}}(2\sqrt{2}+2)$$

$$4+8(\sqrt{2}-1)x(1-x)-(2\sqrt{2}+2)\le 0\ or,(\sqrt{2}-1)(2x-1)^2\ge 0$$

Equality holds for an equilateral triangle.

4277. In ΔABC the following relationship holds:

$$m_a w_a < bc$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$m_a^2 - \frac{(b+c)^2}{4} = \frac{2b^2+2c^2-a^2-b^2-c^2-2bc}{4} = \frac{(b-c)^2-a^2}{4}$$
$$= \frac{(b-c+a)(b-c-a)}{4} < 0 \text{ so } m_a < \frac{b+c}{2}$$
$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} \le \frac{2bc}{b+c} \text{ as } \cos\frac{A}{2} \le 1$$
$$\text{now } m_a w_a < \frac{2bc}{b+c}\cdot\frac{b+c}{2} = bc$$

4278. In any Δ ABC with $a \ge b \ge c$ and

$$0 \le d \le e \le f \text{ the following relationship holds:}$$

$$d(3bc - 4w_a r_a) + e(3ca - 4w_b r_b) + f(3ab - 4w_c r_c) \ge 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } a \ge b \ge c \Rightarrow 3bc \le 3ca \therefore \text{ in order to prove:}$$
$$3bc - 4w_a r_a \overset{?}{\le} 3ca - 4w_b r_b, \text{ it suffices to prove:}$$
$$\frac{2ca}{c+a}.\cos\frac{B}{2}.s\tan\frac{B}{2} \overset{?}{\le} \frac{2bc}{b+c}.\cos\frac{A}{2}.s\tan\frac{A}{2}$$
$$\Leftrightarrow \frac{a^2}{(c+a)^2}.\frac{(s-c)(s-a)}{ca} \overset{?}{\le} \frac{b^2}{(b+c)^2}.\frac{(s-b)(s-c)}{bc} \text{ and } \because \frac{1}{c+a} \le \frac{1}{b+c}$$
$$\therefore \text{ it suffices to prove: } a(s-a) \overset{?}{\le} b(s-b) \Leftrightarrow (a+b)(a-b) \overset{?}{\ge} s(a-b)$$
$$\Leftrightarrow (a-b)(2s-c-s) \overset{?}{\ge} 0 \Leftrightarrow (a-b)(s-c) \overset{?}{\ge} 0 \to \text{true} \because a \ge b$$
$$\therefore 3bc - 4w_a r_a \boxed{\overset{①}{\le}} 3ca - 4w_b r_b \text{ and again, since } a \ge b \ge c \Rightarrow 3ca \le 3ab$$
$$\therefore \text{ in order to prove: } 3ca - 4w_b r_b \overset{?}{\le} 3ab - 4w_c r_c, \text{ it suffices to prove:}$$

$$\frac{2ab}{a+b}\cdot\cos\frac{C}{2}\cdot s\tan\frac{C}{2}\overset{?}{\leq}\frac{2ca}{c+a}\cdot\cos\frac{B}{2}\cdot s\tan\frac{B}{2}$$

$$\Leftrightarrow\frac{b^2}{(a+b)^2}\cdot\frac{(s-a)(s-b)}{ab}\overset{?}{\leq}\frac{c^2}{(c+a)^2}\cdot\frac{(s-c)(s-a)}{ca}\text{ and }\because\frac{1}{a+b}\leq\frac{1}{c+a}$$

$$\therefore\text{ it suffices to prove : }b(s-b)\overset{?}{\leq}c(s-c)\Leftrightarrow(b+c)(b-c)\overset{?}{\geq}s(b-c)$$

$$\Leftrightarrow(b-c)(2s-a-s)\overset{?}{\geq}0\Leftrightarrow(b-c)(s-a)\overset{?}{\geq}0\to\text{true}\because b\geq c$$

$$\therefore 3ca-4w_br_b\boxed{\overset{②}{\leq}}3ab-4w_cr_c\therefore ①\text{ and }②\Rightarrow$$

$$d\leq e\leq f\text{ and }3bc-4w_ar_a\leq 3ca-4w_br_b\leq 3ab-4w_cr_c$$

$$\therefore d.(3bc-4w_ar_a)+e.(3ca-4w_br_b)+f.(3ab-4w_cr_c)\overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3}(d+e+f).\sum_{cyc}(3bc-4w_ar_a)\to(*)$$

$$\text{Now, }4\sum_{cyc}w_ar_a\overset{\text{AM-GM}}{\leq}4.\sum_{cyc}\left(\sqrt{s(s-a)}.\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}\right)\overset{\text{CBS}}{\leq}$$

$$4s.\sqrt{3\sum_{cyc}((s-b)(s-c))}=4s.\sqrt{3(4Rr+r^2)}\overset{?}{\leq}3\sum_{cyc}bc=3(s^2+4Rr+r^2)$$

$$\overset{\text{squaring}}{\Leftrightarrow}3s^4-(40Rr+10r^2)s^2+3r^2(4R+r)^2\boxed{\overset{?}{\underset{(\bullet)}{\geq}}}0\text{ and }\because$$

$$3(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(\bullet),\text{ it suffices to prove :}$$

$$\text{LHS of }(\bullet)\overset{?}{\geq}3(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(7R-5r)s^2\boxed{\overset{?}{\underset{(\bullet\bullet)}{\geq}}}r(90R^2-63Rr+9r^2);\text{ and finally, }(7R-5r)s^2\overset{\text{Gerretsen}}{\geq}$$

$$(7R-5r)(16Rr-5r^2)\overset{?}{\geq}r(90R^2-63Rr+9r^2)\Leftrightarrow 2r(11R-4r)(R-2r)\overset{?}{\geq}0$$

$$\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}\therefore\sum_{cyc}(3bc-4w_ar_a)\geq 0\to(**)\text{ and }\because$$

$$d,e,f\geq 0\Rightarrow d+e+f\geq 0\therefore(*)\text{ and }(**)\Rightarrow$$

$$d.(3bc-4w_ar_a)+e.(3ca-4w_br_b)+f.(3ab-4w_cr_c)\geq 0\ \forall\ \Delta\ ABC$$

with $a\geq b\geq c$ and $0\leq d\leq e\leq f$, " = " iff Δ ABC is equilateral (QED)

4279. In any Δ ABC with $a\geq b\geq c$ and

$$0\leq d\leq e\leq f\text{ the following relationship holds :}$$

$$d(4r_br_c-3bc)+e(4r_cr_a-3ca)+f(4r_ar_b-3ab)\geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Charaborty-Kolkata-India

$$4r_br_c-3bc\overset{?}{\le}4r_cr_a-3ca\Leftrightarrow 3c(a-b)\overset{?}{\le}4s\big((s-b)-(s-a)\big)$$

$$\Leftrightarrow(4(s-c)+c)(a-b)\overset{?}{\ge}0\to\text{true}\because a\ge b\therefore 4r_br_c-3bc\boxed{\overset{①}{\le}}4r_cr_a-3ca$$

$$\text{Also},4r_cr_a-3ca\overset{?}{\le}4r_ar_b-3ab\Leftrightarrow 3a(b-c)\overset{?}{\le}4s\big((s-c)-(s-b)\big)$$

$$\Leftrightarrow(4(s-a)+a)(b-c)\overset{?}{\ge}0\to\text{true}\because b\ge c\therefore 4r_cr_a-3ca\boxed{\overset{①}{\le}}4r_ar_b-3ab$$

$$\therefore ①\text{ and }②\Rightarrow d\le e\le f\text{ and }4r_br_c-3bc\le 4r_cr_a-3ca\le 4r_ar_b-3ab$$

$$\therefore d.(4r_br_c-3bc)+e.(4r_cr_a-3ca)+f.(4r_ar_b-3ab)\overset{\text{Chebyshev}}{\ge}$$

$$\frac{1}{3}(d+e+f).\sum_{\text{cyc}}(4s(s-a)-3bc)=\frac{1}{3}(d+e+f).\big(4s^2-3(s^2+4Rr+r^2)\big)$$

$$=\frac{1}{3}(d+e+f).\big(s^2-16Rr+5r^2+4r(R-2r)\big)\overset{\substack{\text{Gerretsen}\\\text{and}\\\text{Euler}}}{\ge}0$$

$(\because d,e,f\ge 0\Rightarrow d+e+f\ge 0)$ *and so*,

$d.(4r_br_c-3bc)+e.(4r_cr_a-3ca)+f.(4r_ar_b-3ab)\ge 0\ \forall\,\Delta\,ABC$
with $a\ge b\ge c$ and $0\le d\le e\le f$, " = " iff Δ ABC is equilateral (QED)

4280. In ΔABC the following relationship holds:

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{h_c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{h_a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{h_b}}\ge 2\sqrt[4]{108}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$s^2\overset{Mitrinovic}{\le}\frac{27}{4}R^2\ or,\frac{R}{s}\ge\frac{2}{3\sqrt{3}}\ (1)$$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{h_c}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{h_a}}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{h_b}}=\sum\frac{\sqrt{a}+\sqrt{b}}{\sqrt{h_c}}=\sum\frac{\sqrt{a}+\sqrt{b}}{\sqrt{\frac{ab}{2R}}}=\sqrt{2R}\sum\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)=$$

$$=2\sqrt{2R}\sum\frac{1}{\sqrt{a}}=2\sqrt{2R}\sum\frac{1^{\frac{3}{2}}}{\sqrt{a}}\overset{Radon}{\ge}2\sqrt{2R}\frac{(1+1+1)^{\frac{3}{2}}}{\sqrt{a+b+c}}=2\sqrt{2R}\frac{(1+1+1)^{\frac{3}{2}}}{\sqrt{2s}}=$$

$$=2\sqrt{\frac{R}{s}3\sqrt{3}}\ge 2\sqrt{\frac{2}{3\sqrt{3}}\times 3\sqrt{3}}=2\sqrt{6\sqrt{3}}=2\sqrt[4]{36\times 3}=2\sqrt[4]{108}$$

Equality holds for an equilateral triangle.

4281. **In any Δ ABC such that $a \neq b \neq$ c, the following relationship holds :**

$$\frac{1}{\cot\frac{A}{2}\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}+\frac{1}{\cot\frac{B}{2}\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\left(\cot\frac{B}{2}-\cot\frac{A}{2}\right)}+$$
$$+\frac{1}{\cot\frac{C}{2}\left(\cot\frac{C}{2}-\cot\frac{A}{2}\right)\left(\cot\frac{C}{2}-\cot\frac{B}{2}\right)}<\frac{1}{3\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x \neq y \neq z, \sum_{cyc}\frac{1}{x(x-y)(x-z)}$$
$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(-yz(y-z)-zx(z-x)-xy(x-y)\right)$$
$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(yz(y-z)-x(y-z)(y+z)+x^2(y-z)\right)$$
$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)\left(y(z-x)-x(z-x)\right)$$
$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)(z-x)(x-y)$$
$$\therefore \sum_{cyc}\frac{1}{x(x-y)(x-z)}=\frac{1}{xyz} \text{ and via } x \equiv \cot\frac{A}{2}, y \equiv \cot\frac{B}{2}, z \equiv \cot\frac{C}{2},$$
$$\frac{1}{\cot\frac{A}{2}\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}+\frac{1}{\cot\frac{B}{2}\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\left(\cot\frac{B}{2}-\cot\frac{A}{2}\right)}+$$
$$\frac{1}{\cot\frac{C}{2}\left(\cot\frac{C}{2}-\cot\frac{A}{2}\right)\left(\cot\frac{C}{2}-\cot\frac{B}{2}\right)}=\frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=\frac{r_a r_b r_b}{s^3}=\frac{s^2 r}{s^3}=\frac{r}{s}$$
$$\overset{\text{Mitrinovic}}{<}\frac{1}{3\sqrt{3}} \text{ (equality doesn't occur because } a \neq b \neq c)$$
$$\therefore \frac{1}{\cot\frac{A}{2}\left(\cot\frac{A}{2}-\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}+\frac{1}{\cot\frac{B}{2}\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\left(\cot\frac{B}{2}-\cot\frac{A}{2}\right)}+$$
$$\frac{1}{\cot\frac{C}{2}\left(\cot\frac{C}{2}-\cot\frac{A}{2}\right)\left(\cot\frac{C}{2}-\cot\frac{B}{2}\right)}<\frac{1}{3\sqrt{3}}\ \forall\ \Delta\ ABC\ (QED)$$

4282.

In any Δ ABC such that $a \neq b \neq c$, the following relationship holds :

$$\frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+$$
$$\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}>8$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } x\neq y\neq z, \sum_{cyc}\frac{1}{x(x-y)(x-z)}$$
$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(-yz(y-z)-zx(z-x)-xy(x-y)\right)$$
$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.\left(yz(y-z)-x(y-z)(y+z)+x^2(y-z)\right)$$
$$=-\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)\left(y(z-x)-x(z-x)\right)$$
$$=\frac{1}{xyz(x-y)(y-z)(z-x)}.(y-z)(z-x)(x-y)$$
$$\therefore\sum_{cyc}\frac{1}{x(x-y)(x-z)}=\frac{1}{xyz}\text{ and via } x\equiv\sin\frac{A}{2}, y\equiv\sin\frac{B}{2}, z\equiv\sin\frac{C}{2},$$
$$\frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+$$
$$\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}=\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{4R}{r}\overset{\text{Euler}}{>}8$$
$$(\text{equality doesn't occur because } a\neq b\neq c)$$
$$\therefore\frac{1}{\sin\frac{A}{2}\left(\sin\frac{A}{2}-\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}-\sin\frac{C}{2}\right)}+\frac{1}{\sin\frac{B}{2}\left(\sin\frac{B}{2}-\sin\frac{C}{2}\right)\left(\sin\frac{B}{2}-\sin\frac{A}{2}\right)}+$$
$$\frac{1}{\sin\frac{C}{2}\left(\sin\frac{C}{2}-\sin\frac{A}{2}\right)\left(\sin\frac{C}{2}-\sin\frac{B}{2}\right)}>8\ \forall\ \Delta\ ABC\ (QED)$$

4283. In $\Delta\ ABC$ the following relationship holds:

$$\sum_{cyc}\frac{a^4}{a+3c}\geq 18\sqrt{3}r^3$$

Proposed by Adgozel Adgozalov-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\frac{a^4}{a+3c}+\frac{b^4}{b+3a}+\frac{c^4}{c+3b}\overset{HOLDER}{\geq}\frac{(a+b+c)^4}{9(a+3c+b+3a+c+3b)}=$$

$$=\frac{(2s)^4}{9\cdot 8s}=\frac{2s^3}{9}\overset{MITRINOVICI}{\geq}\frac{2\cdot\left(3\sqrt{3}r\right)^3}{9}=\frac{2\cdot 27\cdot 3\sqrt{3}r^3}{9}=18\sqrt{3}r^3$$

Equality holds for $a=b=c$.

4284.

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{2R}{r}}\cdot\left(max(r_a,r_b,r_c)+\frac{h_a+h_b+h_c}{2}\right)\geq 2(n_a+n_b+n_c)-\sum_{cyc}\sqrt{(b-c)^2+r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\leq}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{Bogdan\ Fustei}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\sqrt{\frac{R}{2r}}\overset{squaring}{\Leftrightarrow}$$

$$R^2(4R+r)^2+6r^2(4R+r)^2\boxed{\underset{(*)}{\overset{?}{\geq}}}6r(3R-r)s^2$$

$$\text{Now, }6r(3R-r)s^2\overset{Rouche}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\text{and }\because R-2r\overset{Euler}{\geq}0\therefore\text{ it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\geq}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\geq}0$$

$$\to \text{true (strict inequality)} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true} \therefore \sum_{\text{cyc}} n_a \leq \sqrt{\frac{R}{2r}}.(4R+r) \to ①$$

$$\text{and also}, \sum_{\text{cyc}} n_a \leq \sum_{\text{cyc}} AI + \sum_{\text{cyc}} \sqrt{(b-c)^2+r^2} \to ②$$

(Reference : Inequality in Triangle by Bogdan Fustei – 103; published at www.ssmrmh.ro) $\therefore ① + ② \Rightarrow$

$$2\sum_{\text{cyc}} n_a - \sum_{\text{cyc}} \sqrt{(b-c)^2+r^2} \leq \sqrt{\frac{R}{2r}}.(4R+r) + \sum_{\text{cyc}} AI \overset{?}{\leq}$$

$$\sqrt{\frac{2R}{r}}.\left(2R + \frac{h_a+h_b+h_c}{2} - r\right) = \sqrt{\frac{R}{2r}}.\left(4R - 2r + \frac{s^2+4Rr+r^2}{2R}\right)$$

$$\Leftrightarrow \sum_{\text{cyc}} AI \boxed{\overset{?}{\leq}(\blacksquare)} \sqrt{\frac{R}{2r}}.\left(\frac{s^2-2Rr+r^2}{2R}\right) \text{ and we have :}$$

$$\sum_{\text{cyc}} AI = r.\sum_{\text{cyc}} \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} \overset{\text{CBS}}{\leq} \frac{r}{r.\sqrt{s}}.\sqrt{\sum_{\text{cyc}} ab}.\sqrt{\sum_{\text{cyc}}(s-a)}$$

$$= \sqrt{s^2+4Rr+r^2} \overset{?}{\leq} \sqrt{\frac{R}{2r}}.\left(\frac{s^2-2Rr+r^2}{2R}\right)$$

$$\Leftrightarrow (s^2-2Rr+r^2)^2 \overset{?}{\geq} 8Rr(s^2+4Rr+r^2)$$

$$\Leftrightarrow s^4 - (12Rr-2r^2)s^2 - r^2(28R^2+12Rr-r^2) \boxed{\overset{?}{\geq}(\bullet)} 0 \text{ and via Gerretsen,}$$

$$\text{LHS of } (\bullet) \geq (4Rr-3r^2)s^2 - r^2(28R^2+12Rr-r^2) \overset{\text{Gerretsen}}{\geq}$$

$$(4Rr-3r^2)(16Rr-5r^2) - r^2(28R^2+12Rr-r^2) = 4r^2(R-2r)(9R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\Rightarrow (\bullet) \Rightarrow (\blacksquare) \text{ is true and so,}$$

$$\sqrt{\frac{2R}{r}}.\left(2R + \frac{h_a+h_b+h_c}{2} - r\right) \geq 2(n_a+n_b+n_c) - \sum_{\text{cyc}} \sqrt{(b-c)^2+r^2} \to (m)$$

$$\text{and now}, max(r_a, r_b, r_c) \geq \sqrt{4R^2-7r^2}$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 329; published at www.ssmrmh.ro) $\overset{?}{\geq} 2R - r$

$$\Leftrightarrow 4R^2 - 7r^2 \overset{?}{\geq} (2R-r)^2 = 4R^2 - 4Rr + r^2 \Leftrightarrow 4Rr \overset{?}{\geq} 8r^2 \to \text{true via Euler}$$

$$\therefore max(r_a, r_b, r_c) \geq 2R - r \to (n) \therefore (m) \text{ and } (n) \Rightarrow$$

$$\sqrt{\frac{2R}{r}}.\left(max(r_a, r_b, r_c) + \frac{h_a+h_b+h_c}{2}\right) \geq 2(n_a+n_b+n_c) - \sum_{\text{cyc}} \sqrt{(b-c)^2+r^2}$$

$\forall\, ABC, ''='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

4285. In any $\Delta\, ABC$ the following relationship holds :

$$\sqrt{\frac{2R}{r}}\cdot\left(2R+\frac{h_a+h_b+h_c}{2}-r\right)\geq 2(n_a+n_b+n_c)-\sum_{cyc}\sqrt{(b-c)^2+r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc} n_a}{4R+r}\overset{CBS}{\leq}\frac{\sqrt{3\sum_{cyc} n_a^2}}{4R+r}\overset{\text{Bogdan Fustei}}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\leq}\sqrt{\frac{R}{2r}}\overset{\text{squaring}}{\Leftrightarrow}$$

$$R^2(4R+r)^2+6r^2(4R+r)^2\boxed{\overset{?}{\underset{(*)}{\geq}}}6r(3R-r)s^2$$

$$\text{Now, } 6r(3R-r)s^2\overset{\text{Rouche}}{\leq}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\leq}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\geq}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$\text{and}\because R-2r\overset{\text{Euler}}{\geq}0\therefore \text{it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\geq}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\geq}0$$

$$\rightarrow \text{true (strict inequality)}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}\therefore\sum_{cyc} n_a\leq\sqrt{\frac{R}{2r}}.(4R+r)\rightarrow(1)$$

$$\text{and also}, \sum_{cyc} n_a \leq \sum_{cyc} AI + \sum_{cyc} \sqrt{(b-c)^2 + r^2} \to ②$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Bogdan Fustei} - 103;\\ \text{published at www.ssmrmh.ro}\end{array}\right) \therefore ① + ② \Rightarrow$$

$$2\sum_{cyc} n_a - \sum_{cyc} \sqrt{(b-c)^2 + r^2} \leq \sqrt{\frac{R}{2r}}.(4R+r) + \sum_{cyc} AI \overset{?}{\leq}$$

$$\sqrt{\frac{2R}{r}}.\left(2R + \frac{h_a + h_b + h_c}{2} - r\right) = \sqrt{\frac{R}{2r}}.\left(4R - 2r + \frac{s^2 + 4Rr + r^2}{2R}\right)$$

$$\Leftrightarrow \sum_{cyc} AI \boxed{\underset{(\blacksquare)}{\overset{?}{\leq}}} \sqrt{\frac{R}{2r}}.\left(\frac{s^2 - 2Rr + r^2}{2R}\right) \text{ and we have :}$$

$$\sum_{cyc} AI = r.\sum_{cyc} \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} \overset{CBS}{\leq} \frac{r}{r.\sqrt{s}}.\sqrt{\sum_{cyc} ab}.\sqrt{\sum_{cyc}(s-a)}$$

$$= \sqrt{s^2 + 4Rr + r^2} \overset{?}{\leq} \sqrt{\frac{R}{2r}}.\left(\frac{s^2 - 2Rr + r^2}{2R}\right)$$

$$\Leftrightarrow (s^2 - 2Rr + r^2)^2 \overset{?}{\geq} 8Rr(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow s^4 - (12Rr - 2r^2)s^2 - r^2(28R^2 + 12Rr - r^2) \boxed{\underset{(\bullet)}{\overset{?}{\geq}}} 0 \text{ and via Gerretsen,}$$

$$\text{LHS of } (\bullet) \geq (4Rr - 3r^2)s^2 - r^2(28R^2 + 12Rr - r^2) \overset{Gerretsen}{\geq}$$

$$(4Rr - 3r^2)(16Rr - 5r^2) - r^2(28R^2 + 12Rr - r^2) = 4r^2(R - 2r)(9R - 2r) \overset{Euler}{\geq} 0$$

$$\Rightarrow (\bullet) \Rightarrow (\blacksquare) \text{ is true and so,}$$

$$\sqrt{\frac{2R}{r}}.\left(2R + \frac{h_a + h_b + h_c}{2} - r\right) \geq 2(n_a + n_b + n_c) - \sum_{cyc} \sqrt{(b-c)^2 + r^2} \ \forall\ ABC,$$

$$'' = '' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4286. In any Δ ABC the following relationship holds :

$$\frac{n_a}{h_a} \geq \frac{1}{2}\left(\frac{g_c}{l_b} + \frac{g_b}{l_c}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$
$c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set $''S''$ of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,
$y^3 + z^3 = x^3(m^3 - 3nn)$, and now, $g_b g_c \overset{?}{\leq} r_a w_a \overset{\text{squaring and via Bogdan Fustei}}{\Leftrightarrow}$

$$\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\leq} \frac{s(s-b)(s-c)}{s-a} \cdot \frac{4bc}{(b+c)^2} \cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\geq} (4zx + xy + yz)(4xy + zx + yz)(2x + y + z)^2$$

$$\Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4.yz - 32x^3.yz(y+z) -$$

$$5x^2.yz(y+z)^2 + 3x.yz(y^3+z^3) + 13x.y^2z^2(y+z) + 3y^2z^2(y+z)^2 \boxed{\overset{?}{\geq}} 0 \text{ and via} \quad (1)$$

set of relations $''S''$, to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2 + 4m)}^{\Omega_1} n^2 + \overbrace{(3m^3 - 5m^2 - 32m - 28)}^{\Omega_2} n + \overbrace{(8m^2 + 16m + 4)}^{\Omega_3} + \boxed{\overset{?}{\geq}} 0 \quad (2)$$

and it's trivially true when : $\Omega_2 \geq 0$ and when : $\Omega_2 < 0$, then :

$$3m^3 - 5m^2 - 32m - 28 = (m-5)(3m^2 + 10m + 18) + 62 < 0$$

$$\Rightarrow (m-5)(3m^2 + 10m + 18) < -62 < 0 \Rightarrow m < 5 \rightarrow \text{(i)}$$

Now, LHS of (2) is a quadratic polynomial in $''n''$ with discriminant, $\boxed{\delta} =$
$\sigma_2^2 - 4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196)$ and when :
$9m^3 - 75m^2 + 40m + 196 \leq 0$, then $\delta \leq 0 \Rightarrow$ (2) is trivially true and when :
$9m^3 - 75m^2 + 40m + 196 > 0$, then : $9m^3 - 75m^2 + 40m + 196$

$$= \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m-7) - \frac{14}{3} > 0$$

$$\Rightarrow \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m-7) > \frac{14}{3} > 0 \text{ and } \because m \overset{\text{via (i)}}{\leq} 5$$

$$\Rightarrow 3m(m-5) - 3m - \frac{86}{3} < 0 \therefore m < \frac{7}{3} \rightarrow \text{(ii) and so, when : } \delta > 0,$$

then it suffices to prove : $2\Omega_1.n \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow$ and $\because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$

$$\therefore \text{ suffices to prove : } \Omega_1.\frac{m^2}{2} \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$

$$and \because 3m^2-2m-14 = \left(m+\frac{5}{3}\right)(3m-7) - \frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{ it suffices to prove :}$$

$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0} \therefore g_b g_c \leq r_a w_a$$

$$and\ analogously, \boxed{\boxed{g_c g_a \leq r_b l_b}} \to (*) \text{ and } \boxed{\boxed{g_a g_b \leq r_c l_c}} \to (**)$$

$$\text{Now, } n_a \geq \frac{m_a l_a}{g_a} \Rightarrow \frac{2n_a}{h_a} \overset{\text{Bogdan Fustei}}{\geq} \frac{2m_a l_a}{h_a g_a} \overset{\text{Lascu + AM-GM}}{\geq} 4R.\frac{s(s-a)}{bc}.\frac{1}{g_a}$$

$$= 4R\cos^2\frac{A}{2}.\frac{1}{g_a} = \frac{r_b+r_c}{g_a} \overset{?}{\geq} \frac{g_c}{l_b} + \frac{g_b}{l_c} \Leftrightarrow r_b + r_c \overset{?}{\geq} \frac{g_c g_a}{l_b} + \frac{g_a g_b}{l_c} \to \text{true}$$

$$\because \frac{g_c g_a}{l_b} + \frac{g_a g_b}{l_c} \overset{\text{via (*) and (**)}}{\leq} \frac{r_b l_b}{l_b} + \frac{r_c l_c}{l_c} = r_b + r_c \therefore \frac{n_a}{h_a} \geq \frac{1}{2}\left(\frac{g_c}{l_b} + \frac{g_b}{l_c}\right) \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4287. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\tan\frac{A}{2}.\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan^2\frac{C}{2}\right)}\right) \geq 1$$

Proposed by Neculai Stanciu, Viorel Ovidiu Ignatescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS} \geq \sum_{cyc}\left(\tan\frac{A}{2}.\sqrt{\frac{1}{3}.\frac{3}{4}.\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)^2}\right) =$$

$$= \frac{1}{2}\sum_{cyc}\left(\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)\right) = \sum_{cyc}\left(\tan\frac{A}{2}\tan\frac{B}{2}\right) = 1$$

$$\therefore \sum_{cyc}\left(\tan\frac{A}{2}.\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan^2\frac{C}{2}\right)}\right) \geq 1 \ \forall \text{ ABC},$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4288. *In ΔABC the following relationship holds*:

$$\sqrt{\sum_{cyc}\frac{1}{(r_a+r_b)\sin(A)\sin(B)}} \geq \frac{3\sqrt{R}}{s}$$

Proposed by Neculai Stanciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{(r_a+r_b)\sin(A)\sin(B)}=\frac{4R^2}{(r_a+r_b)ab}=\frac{4R^2.c^2}{(r_a+r_b)abc.c}=$$

$$=\frac{4R^2.c^2}{(r_a+r_b)4RF.c}=\frac{R.c^2}{(r_a+r_b)F.c}$$

$$\sum_{cyc}\frac{R.c^2}{(r_a+r_b)F.c}\overset{Bergstrom}{\geq}\frac{R}{F}\cdot\frac{\left(\sum_{cyc}c\right)^2}{\sum_{cyc}(r_a+r_b).c}=\frac{R}{F}\cdot\frac{4s^2}{4s(R+r)}=$$

$$=\frac{R}{sr}\cdot\frac{4s^2}{4s(R+r)}=\frac{R}{Rr+r^2}\geq\frac{R}{\frac{s^2}{9}}=\frac{9R}{s^2}\ because:\ \therefore Rr+r^2\leq\frac{2s^2}{27}+\frac{s^2}{27}=\frac{s^2}{9}$$

$$\sqrt{\sum_{cyc}\frac{1}{(r_a+r_b)\sin(A)\sin(B)}}\geq\sqrt{\frac{9R}{s^2}}=\frac{3\sqrt{R}}{s}$$

$$Equality\ holds\ for:\ \ a=b=c.$$

4289. In ΔABC the following relationship holds:

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{r_a}}+\frac{\sqrt{c}+\sqrt{b}}{\sqrt{r_b}}+\frac{\sqrt{a}+\sqrt{c}}{\sqrt{r_c}}\geq 2\sqrt[4]{108}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{r_a}}+\frac{\sqrt{c}+\sqrt{b}}{\sqrt{r_b}}+\frac{\sqrt{a}+\sqrt{c}}{\sqrt{r_c}}\overset{AM-GM}{\geq}$$

$$\geq\frac{2\sqrt[4]{ab}}{\sqrt{r_a}}+\frac{2\sqrt[4]{bc}}{\sqrt{r_b}}+\frac{2\sqrt[4]{ac}}{\sqrt{r_c}}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{2\sqrt[4]{ab}.2\sqrt[4]{bc}.2\sqrt[4]{ac}}{\sqrt{r_a.r_b.r_c}}}=$$

$$=6\sqrt[6]{\frac{\text{abc}}{r_a.r_b.r_c}}=6\sqrt[6]{\frac{4R.F}{s^2.r}}=6\sqrt[6]{\frac{4R.sr}{s^2.r}}=6\sqrt[6]{\frac{4R}{s}}\overset{Mitrinovici}{\geq}$$

$$\geq 6\sqrt[6]{\frac{4}{s}\cdot\frac{2s}{3\sqrt{3}}}=6\sqrt[6]{\left(\frac{2^3}{\sqrt{27}}\right)}=6\sqrt{\frac{2}{\sqrt{3}}}=2\sqrt[4]{108}$$

$$Equality\ holds\ for:\ \ a=b=c$$

4290. In ΔABC the following relationship holds:

$$\frac{a\cot\frac{A}{2}}{h_a}+\frac{b\cot\frac{B}{2}}{h_b}+\frac{c\cot\frac{C}{2}}{h_c}\geq 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a\cot\frac{A}{2}}{h_a}+\frac{b\cot\frac{B}{2}}{h_b}+\frac{c\cot\frac{C}{2}}{h_c}=\sum_{cyc}\frac{a\cot\frac{A}{2}}{h_a}=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\tan\frac{A}{2}}=\frac{1}{2F}\sum_{cyc}\frac{a^2}{\tan\frac{A}{2}}\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{2F}\cdot 3\sqrt[3]{\frac{(abc)^2}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}=\frac{3}{2F}\sqrt[3]{\frac{(4Rrs)^2}{\frac{r}{s}}}=\frac{3}{2F}\cdot\sqrt[3]{\frac{16R^2r^2s^2\cdot s}{r}}=\frac{3s}{2F}\cdot 2\sqrt[3]{2R^2r}\geq$$

$$\overset{EULER}{\geq}\frac{6s}{2rs}\sqrt[3]{2\cdot(2r)^2\cdot r}=\frac{3}{r}\cdot 2r=6$$

Equality holds for $a=b=c$.

4291. In ΔABC the following relationship holds:

$$\sin A\sin B\sin C\geq\frac{3\sqrt{3}r^3}{R^3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin A\sin B\sin C=\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}=\frac{abc}{8R^3}\geq\frac{3\sqrt{3}r^3}{R^3}$$

$$\Leftrightarrow abc\geq 24\sqrt{3}r^3\Leftrightarrow 4Rrs\geq 24\sqrt{3}r^3\Leftrightarrow Rs\geq 6\sqrt{3}r^2\quad\text{(to prove)}$$

$$Rs\overset{EULER}{\geq}2r\cdot s\overset{MITRINOVIC}{\geq}2r\cdot 3\sqrt{3}r=6\sqrt{3}r^2$$

4292. In ΔABC the following relationship holds:

$$\cos A\cos B\cos C\leq\frac{r}{4R}$$

Proposed by Nguyen Hung Cuong – Vietanam

Solution by Daniel Sitaru – Romania

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}\le\frac{r}{4R}\Leftrightarrow$$

$$\Leftrightarrow s^2-(4R^2+4Rr+r^2)\le Rr$$

$$s^2\le 4R^2+4Rr+r^2+Rr=4R^2+5Rr+r^2$$

$$s^2\overset{GERRETSEN}{\le}4R^2+4Rr+3r^2\le 4R^2+5Rr+r^2\Leftrightarrow 2r^2\le Rr\Leftrightarrow R\ge 2r\quad\text{(Euler)}$$

Equality holds for $A=B=C$.

4293. In any Δ ABC the following relationship holds :

$$\frac{1}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}+$$
$$\frac{1}{\sin^2\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}+\sin^2\frac{A}{2}}\ge 4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove :

$$\frac{s^2\left((s^2-8Rr-2r^2)^2+(4Rr+r^2)^2+(s^2-8Rr-2r^2)(4Rr+r^2)\right)}{(s^2-8Rr-2r^2)r^2((4R+r)^2-2s^2)+r^2s(s^3-12Rrs)+(4Rr+r^2)^3-12Rr^3s^2+2r^2s^2(4Rr+r^2)-6r^4s^2}\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 9$$

$$\Leftrightarrow s^6-(12Rr-6r^2)s^4-r^2(4R+r)^2s^2+9r^3(4R+r)^3\ \boxed{\overset{?}{\underset{①}{\ge}}}\ 0\ \text{and}\because$$

$$P=(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0\therefore\text{to prove ①, it suffices to prove : }LHS_{①}\overset{?}{\ge}P$$

$$\Leftrightarrow(36R-9r)s^4-r(864R^2-432Rr+81r^2)s^2+$$

$$4r^2(1168R^3-852R^2r+327Rr^2-29r^3)\ \boxed{\overset{?}{\underset{②}{\ge}}}\ 0$$

$$\text{and}\because(36R-9r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\therefore\text{to prove ②, suffices to prove :}$$

$$\text{LHS of } ② \overset{?}{\geq} (36R-9r)(s^2-16Rr+5r^2)^2 \Leftrightarrow (288R^2-216Rr+9r^2)s^2$$

$$\boxed{\overset{?}{\underset{③}{\geq}}} r(4544R^3-4656R^2r+1032Rr^2-109r^3)$$

$$\text{Finally, LHS of } ③ \overset{\text{Gerretsen}}{\geq} (288R^2-216Rr+9r^2)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } ③$$

$$\Leftrightarrow 4t^3-15t^2+12t+4 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)^2(4t+1) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \Rightarrow \boxed{\boxed{(*) \text{ is true}}} \text{ and now, since } 1 \overset{\text{Jensen}}{\geq} \frac{4}{9}\left(\sum_{\text{cyc}} \sin\frac{A}{2}\right)^2$$

$$\therefore \text{ in order to prove : } \sum_{\text{cyc}} \frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}} \overset{?}{\geq} 4, \text{ it suffices to prove :}$$

$$\left(\sum_{\text{cyc}} \sin\frac{A}{2}\right)^2 \cdot \sum_{\text{cyc}} \frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}} \overset{?}{\geq} 9$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} x\right)^2 \cdot \sum_{\text{cyc}} \frac{(z^2+zx+x^2)(x^2+xy+y^2)}{(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)} \overset{?}{\geq} 9 \left(\begin{matrix} x=\sin\frac{A}{2}, y=\sin\frac{B}{2}, \\ z=\sin\frac{C}{2} \end{matrix}\right)$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} x\right)^2 \cdot \frac{\left(\sum_{\text{cyc}} x^2\right)^2+\left(\sum_{\text{cyc}} xy\right)^2+\left(\sum_{\text{cyc}} x^2\right)\left(\sum_{\text{cyc}} xy\right)}{\left(\begin{matrix}\left(\sum_{\text{cyc}} x^2\right)\left(\sum_{\text{cyc}} x^2y^2\right)+xyz\left(\sum_{\text{cyc}} x^3\right)+\sum_{\text{cyc}} x^3y^3+ \\ 2xyz\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right)-6x^2y^2z^2\end{matrix}\right)} \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 9$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x$ >0 and $Z+X-Y=2y>0 \Rightarrow X, Y, Z$ *form sides of a* triangle $\Rightarrow X+Y>Z$, $Y+Z>X, Z+X>Y$ *with semiperimeter, circumradius* & inradius $= s', R, r'$ (*say*)

and then : $\sum_{\text{cyc}} x = s', xyz = r'^2s' \sum_{\text{cyc}} xy = 4R'r'+r'^2, \sum_{\text{cyc}} x^2 = s'^2-8R'r'-2r'^2,$

$$\sum_{\text{cyc}} x^3 = s'^3-12R'r's', \sum_{\text{cyc}} x^2y^2 = r'^2((4R'+r')^2-2s'^2) \text{ and}$$

$$\sum_{\text{cyc}} x^3y^3 = (4R'r'+r'^2)^3-12R'r'^3s'^2 \text{ and so, } (\bullet) \Leftrightarrow$$

$$\frac{s'^2\left((s'^2-8R'r'-2r'^2)^2+(4R'r'+r'^2)^2+(s'^2-8R'r'-2r'^2)(4R'r'+r'^2)\right)}{(s'^2-8R'r'-2r'^2)r'^2((4R'+r')^2-2s'^2)+r'^2s'^{(s'^3-12R'r's')}+(4R'r'+r'^2)^3-12R'r'^3s'^2+2r'^2s'^2(4R'r'+r'^2)-6r'^4s'^2}\ \boxed{\overset{?}{\geq}}\ 9$$

$$\rightarrow \text{true via } (*) \Rightarrow \boxed{\boxed{(\bullet)\text{ is true}}}\ \textit{and so,}$$

$$\frac{1}{\sin^2\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}}+\frac{1}{\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}}+$$
$$\frac{1}{\sin^2\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}+\sin^2\frac{A}{2}}\geq 4\ \forall\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4294. In any Δ ABC the following relationship holds :

$$\frac{r_a^3}{r_b^2-r_br_c+r_c^2}+\frac{r_b^3}{r_c^2-r_cr_a+r_a^2}+\frac{r_c^3}{r_a^2-r_ar_b+r_b^2}\geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a^3}{r_b^2-r_br_c+r_c^2}=\sum_{cyc}\frac{r_a^4}{r_ar_b^2-r_ar_br_c+r_c^2r_a}\overset{\text{Bergstrom}}{\geq}$$
$$\frac{\left(\sum_{cyc}r_a^2\right)^2}{\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-6r_ar_br_c}=\frac{((4R+r)^2-2s^2)^2}{(4R+r)s^2-6rs^2}\overset{?}{\geq}9r$$
$$\Leftrightarrow (4R+r)^4+4s^4-(64R^2+68Rr-41r^2)s^2\underbrace{\overset{?}{\geq}}_{(*)}0$$

$$\because 4(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{ in order to prove } (*),\text{ it suffices to prove :}$$
$$\text{LHS of } (*)\overset{?}{\geq}4(s^2-16Rr+5r^2)^2\Leftrightarrow$$
$$256R^4+256R^3r-928R^2r^2+656Rr^3-99r^4\underbrace{\overset{?}{\geq}}_{(**)}(64R^2-60Rr-r^2)s^2$$

$$\textit{Finally},(64R^2-60Rr-r^2)s^2\overset{\text{Gerretsen}}{\leq}(64R^2-60Rr-r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}$$
$$\text{LHS of } (**)\Leftrightarrow 20t^3-73t^2+70t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)(20t^2-33t+4)\overset{?}{\geq}0$$
$$\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow (**)\Rightarrow (*)\text{ is true}$$
$$\therefore \frac{r_a^3}{r_b^2-r_br_c+r_c^2}+\frac{r_b^3}{r_c^2-r_cr_a+r_a^2}+\frac{r_c^3}{r_a^2-r_ar_b+r_b^2}\geq 9r\ \forall\ ABC,$$
$$''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

4295. In any Δ ABC the following relationship holds :

$$\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}+\frac{\tan\frac{B}{2}}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{A}{2}\right)}+$$

$$+\frac{\tan\frac{C}{2}}{\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)}\geq\frac{54}{\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{r_a}=\sum_{cyc}\frac{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}{s\tan\frac{A}{2}}=\frac{4R}{s}\cdot\frac{4R+r}{2R}$$

$$\therefore\left(\sum_{cyc}\frac{a}{r_a}\right)^3\cdot\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}$$

$$=\frac{8(4R+r)^3}{s^3}\cdot s\sum_{cyc}\frac{r_a(r_b+r_c)}{(r_a+r_b)(r_b+r_c)(r_c+r_a)}=\frac{8(4R+r)^3}{s^2}\cdot\frac{2s^2}{\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)}$$

$$=\frac{16(4R+r)^3}{64R^3\cdot\frac{s^2}{16R^2}}=\frac{4(4R+r)^3}{Rs^2}\overset{\text{Gerretsen}}{\geq}\frac{4(4R+r)^3}{R(4R^2+4Rr+3r^2)}\overset{?}{\geq}54$$

$$\Leftrightarrow 20t^3-12t^2-57t+2\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(20t^2+28t-1)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\geq}2\therefore\sum_{cyc}\frac{\tan\frac{A}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)}\geq\frac{54}{\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)^3}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

4296. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}}\cdot(h_a+h_b+h_c)\geq\frac{g_bg_c}{r_a}+\frac{g_cg_a}{r_b}+\frac{g_ag_b}{r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x,$ $c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn)$, and now, $g_bg_c\overset{?}{\leq}r_aw_a \overset{\text{squaring and via Bogdan Fustei}}{\Leftrightarrow}$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right)\overset{?}{\leq}\frac{s(s-b)(s-c)}{s-a}\cdot\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2\overset{?}{\geq}(4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$

$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$
$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2\ \boxed{\overset{?}{\underset{(1)}{\geq}}}\ 0 \text{ and via}$$

set of relations "S", to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\ \boxed{\overset{?}{\underset{(2)}{\geq}}}\ 0$$

and it's trivially true when : $\Omega_2\geq 0$ and when : $\Omega_2<0$, then :

$$3m^3-5m^2-32m-28=(m-5)(3m^2+10m+18)+62<0$$
$$\Rightarrow (m-5)(3m^2+10m+18)<-62<0\Rightarrow m<5\to\text{(i)}$$

Now, LHS of (2) is a quadratic polynomial in "n" with discriminant, $\boxed{\delta}=$ $\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196)$ and when : $9m^3-75m^2+40m+196\leq 0$, then $\delta\leq 0\Rightarrow$ (2) is trivially true and when : $9m^3-75m^2+40m+196>0$, then : $9m^3-75m^2+40m+196$

$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$

$$\Rightarrow\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0 \text{ and } \because m\overset{\text{via (i)}}{\leq}5$$

$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0\therefore m<\frac{7}{3}\to\text{(ii) and so, when : }\delta>0,$$

$$\text{then it suffices to prove}: 2\Omega_1.n \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow \text{and} \because n \overset{AM-GM}{\leq} \frac{m^2}{4}$$

$$\therefore \text{suffices to prove}: \Omega_1.\frac{m^2}{2} \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$

$$\text{and} \because 3m^2-2m-14=\left(m+\frac{5}{3}\right)(3m-7)-\frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{it suffices to prove}:$$

$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0}$$

$$\therefore g_b g_c \leq r_a w_a \text{ and analogs} \to (*) \text{ and now}, (b+c)^2 \overset{?}{\geq} 32Rr.\cos^2\frac{A}{2}$$

$$=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}$$

$$=4a(b+c-a) \Leftrightarrow (b+c)^2+4a^2-4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \text{ and analogs} \to (**) \Rightarrow \frac{g_b g_c}{r_a} \overset{\text{via }(*)}{\leq} \frac{r_a w_a}{r_a}=h_a.\frac{w_a}{h_a}$$

$$=h_a.\frac{2R.2bc.\cos\frac{A}{2}}{(b+c)bc} \overset{\text{via }(**)}{\leq} h_a.\frac{4R.\cos\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}=h_a.\sqrt{\frac{R}{2r}}$$

$$\therefore \frac{g_b g_c}{r_a} \leq \sqrt{\frac{R}{2r}}.h_a \text{ and analogs} \Rightarrow \frac{g_b g_c}{r_a}+\frac{g_c g_a}{r_b}+\frac{g_a g_b}{r_c} \leq \sqrt{\frac{R}{2r}}.(h_a+h_b+h_c)$$

$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4297. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}}.(r_a+r_b+r_c) \geq \frac{g_b g_c}{h_a}+\frac{g_c g_a}{h_b}+\frac{g_a g_b}{h_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=s-a, y=s-b, z=s-c \text{ and then}: a=y+z, b=z+x,$$

$$c=x+y \text{ and } s=x+y+z \text{ and furthermore, we denote}: \frac{y+z}{x}=m \text{ and } \frac{yz}{x^2}=n$$

$$\text{and then, we have the following set } ''S'' \text{ of relations}: y^2+z^2=x^2(m^2-2n),$$

$$y^3+z^3=x^3(m^3-3nn), \text{and now}, g_b g_c \overset{?}{\leq} r_a w_a \overset{\substack{\text{squaring}\\ \text{and via Bogdan Fustei}}}{\Leftrightarrow}$$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\leq}$$

$$\frac{s(s-b)(s-c)}{s-a}\cdot\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\geq} (4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$
$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$
$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2 \boxed{\overset{?}{\underset{①}{\geq}}}\ 0 \text{ and via}$$

set of relations "S", to prove ①, it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\boxed{\overset{?}{\underset{②}{\geq}}}\ 0$$

and it's trivially true when : $\Omega_2 \geq 0$ and when : $\Omega_2 < 0$, then :
$$3m^3-5m^2-32m-28=(m-5)(3m^2+10m+18)+62<0$$
$$\Rightarrow (m-5)(3m^2+10m+18)<-62<0 \Rightarrow m<5 \to \text{(i)}$$
Now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\boxed{\delta} =$ $\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196)$ and when : $9m^3-75m^2+40m+196\leq 0$, then $\delta\leq 0 \Rightarrow$ ② is trivially true and when : $9m^3-75m^2+40m+196>0$, then : $9m^3-75m^2+40m+196$
$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$
$$\Rightarrow \left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0 \text{ and } \because m \overset{\text{via (i)}}{\leq} 5$$
$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0 \therefore m<\frac{7}{3}\to \text{(ii) and so, when : } \delta>0,$$
$$\text{then it suffices to prove : } 2\Omega_1.n \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow \text{ and } \because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$$
$$\therefore \text{ suffices to prove : } \Omega_1.\frac{m^2}{2} \overset{?}{\leq} -\Omega_2-\sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$
$$\text{and } \because 3m^2-2m-14=\left(m+\frac{5}{3}\right)(3m-7)-\frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{ it suffices to prove :}$$
$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$
$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0}$$
$$\therefore g_b g_c \leq r_a w_a \text{ and analogs} \to (*) \text{ and now, } (b+c)^2 \overset{?}{\geq} 32Rr.\cos^2\frac{A}{2}$$
$$=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}$$
$$=4a(b+c-a) \Leftrightarrow (b+c)^2+4a^2-4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true}$$
$$\therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \text{ and analogs} \to (**) \Rightarrow \frac{g_b g_c}{h_a} \overset{\text{via }(*)}{\leq} \frac{r_a w_a}{h_a}$$

$$= r_a.\frac{2R.2bc.\cos\frac{A}{2}}{(b+c)bc} \overset{via\ (**)}{\leq} r_a.\frac{4R.\cos\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}} = r_a.\sqrt{\frac{R}{2r}} \therefore \frac{g_b g_c}{h_a} \leq \sqrt{\frac{R}{2r}}.r_a \text{ and analogs}$$

$$\Rightarrow \frac{g_b g_c}{h_a} + \frac{g_c g_a}{h_b} + \frac{g_a g_b}{h_c} \leq \sqrt{\frac{R}{2r}}.(r_a + r_b + r_c)\ \forall\ \Delta\ ABC,$$

$$'' = ''\text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

4298.

In any $\Delta\ ABC$ with $a \geq b \geq c$ and $0 \leq d \leq e \leq f$ the following relationship holds :

$$d(r_b + r_c - 2m_a) + e(r_c + r_a - 2m_b) + f(r_a + r_b - 2m_c) \geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove : } b(s-b) \boxed{\underset{①}{\overset{?}{\geq}}} 2r.m_b \text{ and } c(s-c) \boxed{\underset{②}{\overset{?}{\geq}}} 2r.m_c;$$

$$\text{whenever : } a \geq b \geq c; \text{Let } x = s-a, y = s-b, z = s-c \text{ and then : } a = y+z,$$
$$b = z+x, c = x+y \text{ and } s = x+y+z \text{ and so, } a \geq b \geq c \Rightarrow$$
$$y+z \geq z+x \geq x+y \Rightarrow z \geq y \geq x \text{ and so, we can assign :}$$
$$\boxed{y = x+g} \text{ and } z = \boxed{x+g+h}\ (g,h \geq 0)$$
$$\text{Also, } d \leq e \leq f \Rightarrow \text{we can assign : } \boxed{e = d+m} \text{ and } \boxed{f = d+m+n}\ (m,n \geq 0)$$
$$\text{Now, } ① \Leftrightarrow (z+x)^2y^2 \overset{?}{\geq} \frac{xyz}{x+y+z}.(4y(x+y+z)+(z-x)^2)$$

$$\Leftrightarrow (x+g+h+x)^2(x+g)(x+x+g+x+g+h) \overset{?}{\geq}$$
$$x(x+g+h)(4(x+g)(x+x+g+x+g+h)+(g+h)^2)$$
$$\Leftrightarrow 2g^4+5g^3h+4g^3x+4g^2h^2+8g^2hx+2g^2x^2+gh^3+4gh^2x+4ghx^2+2h^2x^2$$
$$\overset{?}{\geq} 0 \rightarrow \text{true} \because g,h \geq 0 \Rightarrow \boxed{① \text{ is true}} \text{ and again, } ② \Leftrightarrow$$
$$(x+y)^2z^2 \overset{?}{\geq} \frac{xyz}{x+y+z}.(4z(x+y+z)+(x-y)^2)$$

$$\Leftrightarrow (x+x+g)^2(x+g+h)(x+x+g+x+g+h) \overset{?}{\geq}$$
$$x(x+g)(4(x+g+h)(x+x+g+x+g+h)+g^2)$$
$$\Leftrightarrow 2g^4+3g^3h+4g^3xg^2h^2+4g^2hx+2g^2x^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because g,h \geq 0 \Rightarrow \boxed{② \text{ is true}}$$
$$\text{We have : } r_b + r_c - 2m_a = \frac{rs.a(s-a)}{r^2s} - 2m_a = \frac{a(s-a)-2r.m_a}{r} \text{ and analogs}$$
$$\therefore \text{proposed inequality becomes : } d(a(s-a)-2r.m_a)+$$
$$(d+m)(b(s-b)-2r.m_b)+(d+m+n)(c(s-c)-2r.m_c) \overset{?}{\geq} 0$$
$$\Leftrightarrow d(x(y+z)-2r.m_a)+(d+m)(y(z+x)-2r.m_b)+$$

$$(\mathrm{d}+\mathrm{m}+\mathrm{n})(\mathrm{z}(x+\mathrm{y})-2\mathrm{r}.\mathrm{m_c})\overset{?}{\geq}0$$

$$\Leftrightarrow 2\mathrm{d}\left(\sum_{cyc}x\mathrm{y}-\mathrm{r}\left(\sum_{cyc}\mathrm{m}_a\right)\right)+\mathrm{m}\Big(\big((x\mathrm{y}+\mathrm{yz})-2\mathrm{r}.\mathrm{m_b}\big)+\big((x\mathrm{z}+\mathrm{yz})-2\mathrm{r}.\mathrm{m_c}\big)\Big)$$

$$+\mathrm{n}\big((x\mathrm{z}+\mathrm{yz})-2\mathrm{r}.\mathrm{m_c}\big)\overset{?}{\geq}0$$

$$\Leftrightarrow 2\mathrm{d}\left(\mathrm{r}(4\mathrm{R}+\mathrm{r})-\mathrm{r}\left(\sum_{cyc}\mathrm{m}_a\right)\right)+\mathrm{m}\big((\mathrm{b}(\mathrm{s}-\mathrm{b})-2\mathrm{r}.\mathrm{m_b})+(\mathrm{c}(\mathrm{s}-\mathrm{c})-2\mathrm{r}.\mathrm{m_c})\big)$$

$$+\mathrm{n}(\mathrm{c}(\mathrm{s}-\mathrm{c})-2\mathrm{r}.\mathrm{m_c})\overset{?}{\geq}0 \to \text{true via } ① \textit{ and } ② \text{ combined with :}$$

$$\sum_{cyc}\mathrm{m}_a\leq 4\mathrm{R}+\mathrm{r} \textit{ and } \because \mathrm{d},\mathrm{m},\mathrm{n}\geq 0 \textit{ and so,}$$

$$\mathrm{d}.(\mathrm{r_b}+\mathrm{r_c}-2\mathrm{m}_a)+\mathrm{e}.(\mathrm{r_c}+\mathrm{r}_a-2\mathrm{m_b})+\mathrm{f}.(\mathrm{r}_a+\mathrm{r_b}-2\mathrm{m_c})\geq 0$$

$$\forall\,\Delta\,\mathrm{ABC} \text{ with } a\geq \mathrm{b}\geq \mathrm{c} \textit{ and } 0\leq \mathrm{d}\leq \mathrm{e}\leq \mathrm{f},''='' \text{ iff } \Delta\,\mathrm{ABC} \text{ is equilateral (QED)}$$

4299.

In any Δ ABC with $a\geq \mathrm{b}\geq \mathrm{c}$ and $\mathrm{d}\geq \mathrm{e}\geq \mathrm{f}\geq 0$, the following relationship holds :

$$\mathrm{d}(3\mathrm{Rm}_a-2\mathrm{r_br_c})+\mathrm{e}(3\mathrm{Rm_b}-2\mathrm{r_cr}_a)+\mathrm{f}(3\mathrm{Rm_c}-2\mathrm{r}_a\mathrm{r_b})\geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } \mathrm{d}\geq \mathrm{e}\geq \mathrm{f}\geq 0 \therefore \text{ we can assign :}$$

$$\boxed{\mathrm{e}=\mathrm{f}+\mathrm{g}} \textit{ and } \boxed{\mathrm{d}=\mathrm{f}+\mathrm{g}+\mathrm{h}} \ (\mathrm{g},\mathrm{h}\geq 0)\ \&\ \text{via such substitutions,}$$

$$\mathrm{d}.(3\mathrm{Rm}_a-2\mathrm{r_br_c})+\mathrm{e}.(3\mathrm{Rm_b}-2\mathrm{r_cr}_a)+\mathrm{f}.(3\mathrm{Rm_c}-2\mathrm{r}_a\mathrm{r_b})\overset{?}{\geq}0$$

$$\Leftrightarrow \mathrm{f}.\left(3\mathrm{R}\sum_{cyc}\mathrm{m}_a-2\mathrm{s}\sum_{cyc}(\mathrm{s}-a)\right)+\mathrm{g}.\Big(3\mathrm{R}(\mathrm{m}_a+\mathrm{m_b})-2\mathrm{s}\big((\mathrm{s}-a)+(\mathrm{s}-\mathrm{b})\big)\Big)+$$

$$\mathrm{h}.\big(3\mathrm{Rm}_a-2\mathrm{s}(\mathrm{s}-a)\big)\boxed{\underset{(*)}{\overset{?}{\geq}}}0$$

$$\text{Firstly, } 3\mathrm{R}(\mathrm{m}_a+\mathrm{m_b})-2\mathrm{s}\big((\mathrm{s}-a)+(\mathrm{s}-\mathrm{b})\big)\overset{\text{Tereshin}}{\geq}\frac{3(\mathrm{b}^2+2\mathrm{c}^2+a^2)}{4}-2\mathrm{sc}\overset{?}{\geq}0$$

$$\Leftrightarrow 3\big((\mathrm{z}+x)^2+2(x+\mathrm{y})^2+(\mathrm{y}+\mathrm{z})^2\big)\overset{?}{\geq}8(x+\mathrm{y}+\mathrm{z})(x+\mathrm{y})$$

$$\left(\begin{matrix}x=\mathrm{s}-a,\mathrm{y}=\mathrm{s}-\mathrm{b},\mathrm{z}=\mathrm{s}-\mathrm{c}\Rightarrow a=\mathrm{y}+\mathrm{z},\mathrm{b}=\mathrm{z}+x,\mathrm{c}=x+\mathrm{y} \textit{ and}\\ \mathrm{s}=x+\mathrm{y}+\mathrm{z}\end{matrix}\right)$$

$$\Leftrightarrow x^2-4x\mathrm{y}-2\mathrm{z}x+\mathrm{y}^2-2\mathrm{yz}+6\mathrm{z}^2\overset{?}{\geq}0\Leftrightarrow(\mathrm{y}-\mathrm{z})^2+(\mathrm{z}-x)^2+4(\mathrm{z}^2-x\mathrm{y})\overset{?}{\geq}0$$

$$\to \text{true} \because \mathrm{z}^2\overset{?}{\geq}x\mathrm{y}(\because a\geq \mathrm{b}\geq \mathrm{c}\Rightarrow \mathrm{z}\geq \mathrm{y}\geq x)$$

$$\therefore 3R(m_a+m_b)-2s((s-a)+(s-b)) \boxed{\overset{①}{\geq}} 0$$

$$\text{Now, } (3Rm_a)^2 \overset{\substack{\text{Leibnitz}\\ \text{and}\\ \text{Lascu + AM-GM}}}{\geq} \left(\sum_{cyc} a^2\right) s(s-a) \overset{?}{\geq} 4s^2(s-a)^2$$

$$\Leftrightarrow (y+z)^2+(z+x)^2+(x+y)^2 \overset{?}{\geq} 4x(x+y+z)$$

$$\Leftrightarrow (y^2-x^2)+y(z-x)+z(z-x) \overset{?}{\geq} 0 \to \text{true} \because z \geq y \geq x$$

$$\therefore 3Rm_a-2s(s-a) \boxed{\overset{②}{\geq}} 0 \text{ and finally, } 3R\sum_{cyc} m_a - 2s\sum_{cyc}(s-a) \overset{\text{Tereshin}}{\geq}$$

$$\frac{3}{2}\cdot\sum_{cyc} a^2-2s^2 = 3\sum_{cyc} a^2-\left(\sum_{cyc} a\right)^2 \geq 0 \therefore 3R\sum_{cyc} m_a - 2s\sum_{cyc}(s-a) \boxed{\overset{③}{\geq}} 0$$

$\therefore$ ①, ② and ③ combined with $f, g, h \geq 0 \Rightarrow (*)$ is true

$\therefore d.(3Rm_a-2r_br_c)+e.(3Rm_b-2r_cr_a)+f.(3Rm_c-2r_ar_b) \geq 0 \ \forall \Delta ABC$ with $a \geq b \geq c$ and $d \geq e \geq f \geq 0$, " = " iff ΔABC is equilateral (QED)

4300.

In any ΔABC with $a \geq b \geq c$ and $d \geq e \geq f \geq 0$, the following relationships hold :

$$d(h_bh_c-3rh_a)+e(h_ch_a-3rh_b)+f(h_ah_b-3rh_c) \geq 0$$

$$f(3rm_a-h_bh_c)+e(3rm_b-h_ch_a)+d(3rm_c-h_ah_b) \geq 0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } a \geq b \geq c, \therefore h_bh_c \geq h_ch_a \geq h_ah_b \text{ and } -3rh_a \geq -3rh_b \geq -3rh_c$$

$$\therefore h_bh_c-3rh_a \geq h_ch_a-3rh_b \geq h_ah_b-3rh_c \text{ and } \because d \geq e \geq f \geq 0$$

$$\therefore d.(h_bh_c-3rh_a)+e.(h_ch_a-3rh_b)+f.(h_ah_b-3rh_c) \overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(h_bh_c-3rh_a)\right) = \frac{1}{3}(d+e+f)\left(\frac{2rs^2}{R}-\frac{3r}{2R}\cdot\sum_{cyc} ab\right)$$

$$= \frac{1}{3}(d+e+f)\left(\frac{r}{2R}\right)\left(\left(\sum_{cyc} a\right)^2-3\sum_{cyc} ab\right) \geq 0$$

$$(\because d \geq e \geq f \geq 0 \Rightarrow d+e+f \geq 0)$$

$$\therefore d.(h_bh_c-3rh_a)+e.(h_ch_a-3rh_b)+f.(h_ah_b-3rh_c) \geq 0$$

$$\text{Again, Since } a \geq b \geq c, \therefore -h_bh_c \leq -h_ch_a \leq -h_ah_b \text{ and } 3rm_a \leq 3rm_b \leq 3rm_c$$

$$\therefore 3rm_a-h_bh_c \leq 3rm_b-h_ch_a \leq 3rm_c-h_ah_b \text{ and } \because 0 \leq f \leq e \leq d$$

$$\therefore f.(3rm_a-h_bh_c)+e.(3rm_b-h_ch_a)+d.(3rm_c-h_ah_b) \overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3}(d+e+f)\left(\sum_{cyc}(3rm_a - h_b h_c)\right) \overset{\text{Tereshin}}{\geq} \frac{1}{3}(d+e+f)\left(3r\sum_{cyc}\frac{b^2+c^2}{4R} - \frac{2rs^2}{R}\right)$$

$$= \frac{1}{3}(d+e+f)\left(\frac{r}{2R}\right)\left(3\sum_{cyc}a^2 - \left(\sum_{cyc}a\right)^2\right) \geq 0$$

$$(\because d \geq e \geq f \geq 0 \Rightarrow d+e+f \geq 0)$$

$\therefore$ $f.(3rm_a - h_b h_c) + e.(3rm_b - h_c h_a) + d.(3rm_c - h_a h_b) \geq 0$ *and* so,
$d.(h_b h_c - 3rh_a) + e.(h_c h_a - 3rh_b) + f.(h_a h_b - 3rh_c) \geq 0$ *and*
$f.(3rm_a - h_b h_c) + e.(3rm_b - h_c h_a) + d.(3rm_c - h_a h_b) \geq 0 \ \forall \ \Delta ABC$ with
$a \geq b \geq c$ *and* $d \geq e \geq f \geq 0$, $'' = ''$ iff Δ ABC is equilateral (QED)

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru