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2801. Find:

$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+cosx)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ cosx = u\ then\ sinxdx = -du$$
$$and\ x = 0 \to u = 1\ , x = \frac{\pi}{6} \to u = \frac{\sqrt{3}}{2}$$
$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+cosx)}dx = -\int_1^{\frac{\sqrt{3}}{2}} \frac{du}{u(1+u)} = \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u(1+u)} = \int_{\frac{\sqrt{3}}{2}}^1 \frac{(1+u)-u}{u(1+u)}du =$$
$$= \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u} - \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{(1+u)} = (\ln u - \ln(1+u))_{\frac{\sqrt{3}}{2}}^1 = \ln\frac{2+\sqrt{3}}{2\sqrt{3}}$$

2802. Find:

$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+\sin^2 x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ cosx = u\ then\ sinxdx = -du$$
$$x = 0 \to u = 1, x = \frac{\pi}{6} \to u = \frac{\sqrt{3}}{2}$$
$$\int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(1+\sin^2 x)}dx = \int_0^{\frac{\pi}{6}} \frac{sinx}{cosx(2-\cos^2 x)}dx = -\int_1^{\frac{\sqrt{3}}{2}} \frac{du}{u(2-u^2)} =$$
$$= \int_{\frac{\sqrt{3}}{2}}^1 \frac{du}{u(2-u^2)} = \frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^1 \frac{u^2+(2-u^2)}{u(2-u^2)}du = \frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^1 \left(\frac{u}{(2-u^2)} + \frac{1}{u}\right)du =$$
$$= -\frac{1}{4}(\ln(2-u^2))_{\frac{\sqrt{3}}{2}}^1 + \frac{1}{2}(\ln(u))_{\frac{\sqrt{3}}{2}}^1 = \frac{1}{4}\ln\frac{5}{4} - \frac{1}{2}\ln\frac{\sqrt{3}}{2} = \frac{1}{4}\ln\frac{5}{4} - \frac{1}{4}\ln\frac{3}{4} = \frac{1}{4}\ln\frac{5}{3}$$

2803. Find:

$$\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx$$

Proposed by Neculai Stanciu-Romania

Solution by Mais Hasanov-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)\right)dx+\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx$$

$$I=I_1+I_2. \textit{Using King Rule } \int_b^a f(x)dx=\int_b^a f(a+b-x)\,dx$$

$$\textit{Substitution } x=\frac{\pi}{2}-t \quad dx=-dt$$

$$I_1=\int_0^{\frac{\pi}{2}} \sin^2(t)\cos(t)\sin^2\left(\frac{\pi}{2}\sin(t)\right)dt, \qquad I_2=\int_0^{\frac{\pi}{2}} \sin^3(t)\sin^2\left(\frac{\pi}{2}\cos(t)\right)dt$$

$$2I=\int_0^{\frac{\pi}{2}} \sin^2(x)\left(\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)+\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)\right)dx$$

$$+\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} \sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)dx+\int_0^{\frac{\pi}{2}} \cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)dx\ \left\{x\to\frac{\pi}{2}-x\right\}$$

$$I=\int_0^{\frac{\pi}{2}} \sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)dx\to\left\{\frac{\pi}{2}\cos(x)=t\ \ \frac{dt}{dx}=-\frac{\pi}{2}\sin(x)\right\}$$

$$I=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2(t)\,dt=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} dt-\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt=\frac{1}{2}-\frac{1}{\pi}\left[\frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}=\frac{1}{2}$$

$$\textit{Therefore } \int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=\frac{1}{2}$$

2804. Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{2(2sinx+3cosx)}{2cosx+3sinx}\,dx>\pi$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\boldsymbol{Lemma}: \forall\, x > 0, \ln(1+x) > \frac{x}{1+x}$$

$$\boldsymbol{Proof}: let\ f(x) = \ln(1+x) - \frac{x}{1+x}$$

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0\ so\ f(x)\ is\ increasing, f(0) = 0$$

$$and\ f(x) > f(0)\ or, \ln(1+x) - \frac{x}{1+x} > 0\ or, \ln(1+x) > \frac{x}{1+x}\ \ (1)$$

$$\ln\frac{3}{2} = \ln\left(1+\frac{1}{2}\right) \overset{(1)}{>} \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}\ (2)$$

$$clearly\ 10 > \frac{66}{7} = 3 \times \frac{22}{7} = 3\pi\ (3)$$

$$let\ 4sinx + 6cosx = m(2cosx + 3sinx) + n.\frac{d}{dx}(2cosx + 3sinx)$$

$$4sinx + 6cosx = m(2cosx + 3sinx) + n(3cosx - 2sinx)$$

$$4sinx + 6cosx = (3m - 2n)sinx + (2m + 3n)cosx$$

$$comparing\ the\ coeficient\ of\ sinx\ \&\ cosx\ we\ get$$

$$2m + 3n = 6, 3m - 2n = 4\ , solving\ we\ get\ m = \frac{24}{13}, n = \frac{10}{13}$$

$$\int_0^{\frac{\pi}{2}} \frac{2(2sinx + 3cosx)}{2cosx + 3sinx} dx = \int_0^{\frac{\pi}{2}} \frac{(4sinx + 6cosx)}{2cosx + 3sinx} dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{24}{13}(2cosx + 3sinx) + \frac{10}{13}(3cosx - 2sinx)}{2cosx + 3sinx} dx =$$

$$= \left(\frac{24}{13}x + \frac{10}{13}\ln|2cosx + 3sinx\,|\right)_0^{\frac{\pi}{2}} = \frac{12\pi}{13} + \frac{10}{13}\ln\frac{3}{2} \overset{(2)\&(3)}{>} \frac{12\pi}{13} + \frac{3\pi}{13}\cdot\frac{1}{3} = \frac{13\pi}{13} = \pi$$

2805. Find:

$$\lim_{n\to\infty} \sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} - \frac{\sqrt[n]{n!}}{n}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$S=\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=\lim_{n\to\infty}nS_n(S_{n+1}-S_n)$$

$$Asymptotik\ estimate\ of\ \sqrt[n]{n!}$$

$$\sqrt[n]{n!}=e^{\frac{1}{n}\ln(n!)}=e^{\frac{1}{n}\sum_{k=1}^{n}\ln(k)}\to\sum_{k=1}^{n}\ln(k)=n\ln(n)-n+\frac{1}{2}\ln(2\pi n)+O(1)$$

$$\sqrt[n]{n!}=e^{\ln(n)-1+\frac{1}{2n}\ln(2\pi n)+O\left(\frac{1}{n}\right)}=\frac{n}{e}\left(1+\frac{\ln(2\pi n)}{2n}+O\left(\frac{1}{n}\right)\right)$$

$$S_n=\frac{\sqrt[n]{n!}}{n}\sim\frac{1}{e}$$

$$Functional\ Asymptotik\ approach:$$

To evaluate the difference $S_{n+1}-S_n$ where $S_n=\frac{\sqrt[n]{n!}}{n}$, we consider the continuous function: $f(n)=\frac{1}{e}\left(1+\frac{\ln(2\pi n)}{2n}\right)$

$$Mean\ Value\ theorem: S_{n+1}-S_n\approx\frac{d}{dn}f(n)=\frac{1-\ln(2\pi n)}{2en^2}$$

$$S=\lim_{n\to\infty}nS_n(S_{n+1}-S_n)=\lim_{n\to\infty}\frac{n}{e}\left(\frac{1-\ln(2\pi n)}{2en^2}\right)=\frac{1}{2e^2}\lim_{n\to\infty}\left(\frac{1}{n}-\frac{\ln(2\pi)}{n}-\frac{\ln(n)}{n}\right)=$$

$$=\frac{1}{2e^2}\left(\lim_{n\to\infty}\frac{1}{n}-\lim_{n\to\infty}\frac{\ln(2\pi)}{n}-\lim_{n\to\infty}\frac{\ln(n)}{n}\right)=0$$

$$Therefore\ \ \lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=0$$

2806. **Find:**

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$S=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}$$

$$Stirling\ formula:\ n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\sqrt[n]{n!}=(n!)^{\frac{1}{n}}=e^{\frac{1}{n}\ln(n!)}=e^{\frac{1}{n}\sum_{k=1}^{n}\ln(k)}$$

$$\sum_{k=1}^{n}\ln(k)=n\ln(\mathrm{n})-\mathrm{n}+\frac{1}{2}\ln(2\pi n)+O(1)$$

$$\sqrt[n]{n!}\approx\frac{n}{e}\left(1+\frac{\ln(2\pi n)}{2\mathrm{n}}\right)\sim\frac{n}{e}\to\sqrt[n+1]{(n+1)!}\sim\frac{n+1}{e}$$

$$\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\sim\frac{n+1}{e}-\frac{n}{e}=\frac{1}{e}$$

$$\sqrt[n]{(2n-1)!!}=1\bullet3\bullet5\bullet\ldots\bullet(2n-1)=\frac{(2n)!}{2^n n!}$$

$$n!\approx\left(\frac{n}{e}\right)^n,(2n)!\approx\left(\frac{2n}{e}\right)^{2n}\to(2n-1)!!\approx\frac{\left(\frac{2n}{e}\right)^{2n}}{2^n\left(\frac{n}{e}\right)^n}\approx\left(\frac{2n}{e}\right)^n$$

$$\sqrt[n]{(2n-1)!!}\approx\frac{2n}{e}$$

$$\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)=\sin(a_n)\ \lim_{n\to\infty}a_n=0\to\sin(a_n)\sim a_n$$

$$\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)\sim\frac{\pi}{\sqrt[n]{n!}}\sim\frac{e\pi}{n}$$

$$S=\lim_{n\to\infty}\left(\frac{1}{e}\cdot\frac{2n}{e}\cdot\frac{e\pi}{n}\right)=\lim_{n\to\infty}\frac{2e\pi}{e^2}=\frac{2\pi}{e}$$

$$\textit{Therefore}\ \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)=\frac{2\pi}{e}$$

2807. Find:

$$\sum_{n=1}^{99}\ln\left(\frac{n(n+2)}{(n+1)^2}\right)$$

Proposed by Sonu Aarnav-India

Solution by Mais Hasanov-Azerbaijan

$$S=\sum_{n=1}^{99}\ln\left(\frac{n^2+2n}{(n+1)^2}\right)=\sum_{n=1}^{99}\ln\left(\frac{(n+1)^2-1}{(n+1)^2}\right)=$$

$$=\sum_{n=1}^{99}\ln\left(1-\frac{1}{(n+1)^2}\right)=\sum_{n=1}^{99}\ln\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right)=$$

$$=\ln\left(\prod_{n=1}^{99}\left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right)\right)=$$

$$=\ln\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\ldots\left(1-\frac{1}{100}\right)\left(1+\frac{1}{100}\right)=$$

$$=\ln\left(\frac{1}{2}*\frac{3}{2}*\frac{2}{3}*\frac{4}{3}*\dots*\frac{99}{100}*\frac{101}{100}\right)=\ln\left(\frac{1}{2}*\frac{101}{100}\right)=\ln\left(\frac{101}{200}\right)$$

$$Therefore\sum_{n=1}^{99}\ln\left(\frac{n(n+2)}{(n+1)^2}\right)=\ln\left(\frac{101}{200}\right)$$

2808. Find:

$$\int_0^\infty\frac{\cos(ln(x))}{1+x+x^2}dx$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$Euler\ formula\ e^{ik}=\cos(k)+i\sin(k)\ \ \cos(k)=e^{ik}-i\sin(k)$$
$$k=\ln(x)\to\cos(ln(x))=e^{i\ln x}-i\sin(ln(x))=x^i-i\sin(\ln(x))$$
$$I=\int_0^\infty\frac{\cos(ln(x))}{1+x+x^2}dx=\int_0^\infty\frac{x^i}{1+x+x^2}dx-i\int_0^\infty\frac{\sin(ln(x))}{1+x+x^2}dx=I_1-iI_2$$
$$I_2=\int_0^\infty\frac{\sin(ln(x))}{1+x+x^2}dx\overset{x\to\frac{1}{x}}{\cong}-\int_0^\infty\frac{\sin(ln(x))}{1+x+x^2}dx=-I_2\to 2I_2=0,\qquad I_2=0$$
$$I=\int_0^\infty\frac{x^i}{1+x+x^2}dx=\int_0^1\frac{x^i}{1+x+x^2}dx+\underbrace{\int_1^\infty\frac{x^i}{1+x+x^2}dx}_{x\to\frac{1}{x}}=$$
$$=\int_0^1\frac{x^i}{1+x+x^2}dx+\int_0^1\frac{x^{-i}}{1+x+x^2}dx$$
$$\to I(n)=\int_0^1\frac{x^n}{1+x+x^2}dx,\qquad \begin{cases}n_1=i\\n_2=-i\end{cases}\to I=I(n_1)+I(n_2)$$
$$I(n)=\int_0^1\frac{x^n}{1+x+x^2}dx=\int_0^1\frac{x^n(1-x)}{1-x^3}dx,\qquad substitution\ x^3=t,$$
$$\frac{dt}{dx}=3x^2=3t^{\frac{2}{3}}$$
$$I(n)=\frac{1}{3}\int_0^1\frac{t^{\frac{n}{3}}\left(1-t^{\frac{1}{3}}\right)}{(1-t)t^{\frac{2}{3}}}dt=\frac{1}{3}\left(\int_0^1\frac{t^{\frac{n-2}{3}}-t^{\frac{n-1}{3}}}{1-t}dt\right)=\frac{1}{3}\int_0^1\left(t^{\frac{n-2}{3}}-t^{\frac{n-1}{3}}\right)(1-t)^{-1}dt$$
$$Note: Beta\ function\ \int_0^1 t^{a-1}(1-t)^{b-1}dt=\beta(a;b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$
$$I(n)=\lim_{a\to0^+}\frac{1}{3}\int_0^1\left(t^{\frac{n+1}{3}-1}-t^{\frac{n+2}{3}-1}\right)(1-t)^{a-1}dt=$$
$$=\lim_{a\to0^+}\frac{1}{3}\left(\beta\left(\frac{n+1}{3};a\right)-\beta\left(\frac{n+2}{3};a\right)\right)=$$

$$=\lim_{a\to 0^+}\frac{1}{3}\left(\frac{\Gamma\left(\frac{n+1}{3}\right)\Gamma(a)}{\Gamma\left(\frac{n+1}{3}+a\right)}-\frac{\Gamma\left(\frac{n+2}{3}\right)\Gamma(a)}{\Gamma\left(\frac{n+2}{3}+a\right)}\right)$$

The Laurent series expansion of the Gamma function $\Gamma(a)$ about the simple pole at a=0 is given by: $\Gamma(a)=\frac{1}{a}-\gamma+\frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)a+O(a^2)\sim\frac{1}{a}+O(1)$

The Taylor series expansion of the ratio $\frac{\Gamma(b)}{\Gamma(a+b)}$ around a = 0 is obtained by using the definition of the Digamma function $\psi^{(0)}(z)$: $\frac{\Gamma(b)}{\Gamma(a+b)}=1-\psi^{(0)}(b)a+O(a^2)$

$$I(n)=\frac{1}{3}\lim_{a\to 0^+}\Gamma(a)\left(\frac{\Gamma\left(\frac{n+1}{3}\right)}{\Gamma\left(\frac{n+1}{3}+a\right)}-\frac{\Gamma\left(\frac{n+2}{3}\right)}{\Gamma\left(\frac{n+2}{3}+a\right)}\right)=$$

$$=\frac{1}{3}\lim_{a\to 0^+}\left(\frac{1}{a}+O(1)\right)\left(1-\psi^{(0)}\left(\frac{n+1}{3}\right)a-1+\psi^{(0)}\left(\frac{n+2}{3}\right)a\right)=\frac{\psi^{(0)}\left(\frac{n+2}{3}\right)-\psi^{(0)}\left(\frac{n+1}{3}\right)}{3}$$

$$I=I(n_1)+I(n_2)=I(i)+I(-i)=$$

$$=\frac{\psi^{(0)}\left(\frac{i+2}{3}\right)-\psi^{(0)}\left(\frac{i+1}{3}\right)}{3}+\frac{\psi^{(0)}\left(\frac{-i+2}{3}\right)-\psi^{(0)}\left(\frac{-i+1}{3}\right)}{3}=$$

$$=\frac{1}{3}\left(\psi^{(0)}\left(1-\frac{1-i}{3}\right)-\psi^{(0)}\left(\frac{1-i}{3}\right)+\psi^{(0)}\left(1-\frac{1+i}{3}\right)-\psi^{(0)}\left(\frac{1+i}{3}\right)\right)$$

Digamma reflection formula: $\psi^{(0)}(1-z)-\psi^{(0)}(z)=\pi\cot(\pi z)$

$$I=\frac{1}{3}\left(\pi\cot\left(\pi\left(\frac{1-i}{3}\right)\right)+\pi\cot\left(\pi\left(\frac{i+1}{3}\right)\right)\right)=\frac{\pi}{3}\left(\cot\left(\frac{\pi}{3}-\frac{\pi i}{3}\right)+\cot\left(\frac{\pi}{3}+\frac{\pi i}{3}\right)\right)$$

$$\cot(\mathrm{a}-\mathrm{b})+\cot(\mathrm{a}+\mathrm{b})=\frac{\sin(2a)}{\sin^2(a)-\sin^2(b)}$$

$$I=\frac{\pi}{3}\frac{\sin\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{\pi}{3}\right)-\sin^2\left(\frac{\pi i}{3}\right)}=\frac{\pi}{3}\frac{\frac{\sqrt{3}}{2}}{\frac{3}{4}-\left(i\sinh\left(\frac{\pi}{3}\right)\right)^2}=\frac{2\pi\sqrt{3}}{9+12\sinh^2\left(\frac{\pi}{3}\right)}=$$

$$=\frac{2\pi}{\sqrt{3}\left(3+4sinh^2\left(\frac{\pi}{3}\right)\right)}\to\left\{2\sinh^2\left(\frac{\pi}{3}\right)=\cosh\left(\frac{2\pi}{3}\right)-1\right\}$$

$$Therefore\ \int_0^\infty\frac{\cos(ln(x))}{1+x+x^2}dx=\frac{2\pi}{\sqrt{3}\left(1+2\cosh\left(\frac{2\pi}{3}\right)\right)}$$

2809. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cot^2(x)+\tan^2(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mais Hasanov-Azerbaijan

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cot^2(x)+\tan^2(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)\cos^2(x)}{\sin^4(\mathrm{x})+\cos^4(\mathrm{x})}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)\cos^2(x)}{1-2\sin^2(x)\cos^2(\mathrm{x})}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sin^2(2x)}{4}.}{1-\frac{\sin^2(2x)}{2}.}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(2x)}{2-\sin^2(2x)}dx$$

$$2x=t \quad dt=2dx \quad t\to\frac{2\pi}{3};\frac{\pi}{3}$$

$$I=\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin^2(t)}{2-\sin^2(t)}dx=\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1-\cos(2t)}{3+\cos(2t)}dt \quad 2t\to t$$

$$I=\frac{1}{8}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{4-(3+\cos(t))}{3+\cos(t)}dt=\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{3+\cos(t)}dt-\frac{1}{8}.\frac{2\pi}{3}=\frac{1}{2}K-\frac{\pi}{12}$$

$$K=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{3+\cos(t)}dt \quad \to \quad \cos(t)=\frac{1-tan^2\left(\frac{t}{2}\right)}{1+tan^2\left(\frac{t}{2}\right)}$$

The Weierstrass Substitution

$$\tan\left(\frac{\mathrm{t}}{2}\right)=\mathrm{u} \quad \frac{\mathrm{du}}{\mathrm{dt}}=\frac{1}{2}\sec^2\left(\frac{\mathrm{t}}{2}\right)=\frac{1}{1+\cos(t)}=\frac{1+\tan^2\left(\frac{t}{2}\right)}{2}=\frac{1+u^2}{2}$$

$$K(a,b)=\int\frac{1}{a+b\cos(t)}dt \overset{\tan\left(\frac{t}{2}\right)=\mathrm{u}}{\cong} \int\frac{1}{a+b\frac{1-u^2}{1+u^2}}\frac{2du}{u^2+1}$$

$$=2\int\frac{1}{a+au^2+b-bu^2}du=$$

$$=2\int\frac{1}{u^2(a-b)+a+b}du=\frac{2}{a-b}\int\frac{1}{u^2+\left(\frac{a+b}{a-b}\right)^2}du$$

$$=2\left[\frac{\arctan\left(\frac{u\sqrt{a-b}}{\sqrt{a+b}}\right)}{\sqrt{a^2-b^2}}\right]_{u=\tan\left(\frac{t}{2}\right)}+C$$

$$=\frac{2}{\sqrt{a^2-b^2}}\arctan\left(\frac{\tan\left(\frac{t}{2}\right)\sqrt{a-b}}{\sqrt{a+b}}\right)+C\to\ a=3\ b=1$$

$$K(3,1)=\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}\,tan\left(\frac{t}{2}\right)}{2}\right)+C\quad t\to\tan\left(\frac{2\pi}{3}\right)=-\sqrt{3},\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$K=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{\sqrt{2}\,tan\left(\frac{t}{2}\right)}{2}\right)\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}=\frac{1}{\sqrt{2}}\left(-\tan^{-1}\sqrt{\frac{3}{2}}-\tan^{-1}\sqrt{\frac{3}{2}}\right)$$

Note that t = π is a point of discontinuity for $tan\left(\frac{t}{2}\right)$ within the given range. Thus, the antiderivative must be adjusted for continuity, yielding:

$$K=\frac{1}{\sqrt{2}}\left(\pi-2\arctan\sqrt{\frac{3}{2}}\right)$$

$$Therefore\ \ I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\cot^2(x)+\tan^2(x)}dx=\frac{1}{2}K-\frac{\pi}{12}=\frac{1}{2\sqrt{2}}\left(\pi-2\arctan\sqrt{\frac{3}{2}}\right)-\frac{\pi}{12}$$

2810. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sec(x)+\cos^3(x)}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mais Hasanov-Azerbaijan

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\sec(x)+\cos^3(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos(x)}{1+\cos^4(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos(x)}{1+\left(1-\sin^2(x)\right)^2}dx$$

$$substitution\ \sin(x)=t\ \ \frac{dt}{dx}=\cos(x),\ \ t\to\frac{\sqrt{3}}{2};\frac{1}{2}$$

$$I=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{t^4-2t^2+2}dt\to x^4-2t^2+2=(x^2+a)(x^2+b)\to\begin{cases}a+b=-2\\ ab=2\end{cases}$$

$$a=-(1-i)\quad b=-(1+i)$$

$$\frac{1}{(x^2+a)(x^2+b)}=\frac{A}{x^2+a}+\frac{B}{x^2+b}\to\begin{cases}A+B=0\\ Ab+Ba=1\end{cases}\to\begin{cases}A=\dfrac{1}{b-a}\\ B=\dfrac{1}{a-b}\end{cases}$$

$$I_{a;b}=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{(x^2+a)(x^2+b)}dx=\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x^2+a}dx-\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x^2+b}dx\right)$$
$$=\frac{I(b)-I(a)}{a-b}$$

$$I(a)=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x^2+a}dx=\frac{1}{a}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\left(\frac{x}{\sqrt{a}}\right)^2+1}dx=\left[\frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$
$$=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{a}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{a}}\right)}{\sqrt{a}}$$

$$I(b)=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x^2+b}dx=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{b}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{b}}\right)}{\sqrt{b}}\quad a=i-1;\quad b=-i-1$$

$$I=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{-i-1}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{-i-1}}\right)}{2i\sqrt{-i-1}}-\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{i-1}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{i-1}}\right)}{2i\sqrt{i-1}}$$

2811. Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{secx+sinx}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\int\frac{dx}{secx+sinx}=\int\frac{cosx\,dx}{1+cosx.sinx}=\int\frac{2cosx\,dx}{2+sin2x}=$$

$$=\int\frac{(cosx+sinx)+(cosx-sinx)}{2+sin2x}dx=$$

$$=\int\frac{(cosx+sinx)}{2+sin2x}dx+\int\frac{(cosx-sinx)}{2+sin2x}dx=$$

$$=\int\frac{(cosx+sinx)}{3-(sinx-cosx)^2}dx+\int\frac{(cosx-sinx)}{1+(sinx+cosx)^2}dx=$$

$$=\int\frac{d(sinx-cosx)}{3-(sinx-cosx)^2}dx+\int\frac{d(sinx+cosx)}{1+(sinx+cosx)^2}dx=$$

$$=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+(sinx-cosx)}{\sqrt{3}-(sinx-cosx)}\right|+\arctan(sinx+cosx)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{secx+sinx}=\left(\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+(sinx-cosx)}{\sqrt{3}-(sinx-cosx)}\right|+\arctan(sinx+cosx)\right)_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$=\frac{1}{2\sqrt{3}}\left(\ln\left|\frac{3\sqrt{3}-1}{\sqrt{3}+1}\right|-\ln\left|\frac{\sqrt{3}+1}{3\sqrt{3}-1}\right|\right)=\frac{1}{\sqrt{3}}\ln\left|\frac{3\sqrt{3}-1}{\sqrt{3}+1}\right|=\frac{1}{\sqrt{3}}\ln|5-2\sqrt{3}|$$

2812. Solve for reals:

$$\int_0^1\left(\sqrt[9]{\left(x^{9x^4+9}+x^{9x^2}+x^{18}\right)}\right)^5dx=\frac{\sqrt[9]{4782969}}{5x^4+10x^2+18}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$We\ will\ show: a,b,c>0,\int_0^1\left(\sqrt[9]{(x^{9a}+x^{9b}+x^{9c})}\right)^5dx\ge\frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

we know that ,

if $a_1,a_2,\dots,a_n$ *be n positive real numbers and m be a rational number then,* $\frac{a_1^m+a_2^m+\cdots+a_n^m}{n}>\ or<\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m$

according as m does not or does lie between 0 and 1

(Reference: S.K.Mapa classical algebra book, page – 22)

$$\frac{\left((x^{9a})^{\frac{5}{9}}+(x^{9b})^{\frac{5}{9}}+(x^{9c})^{\frac{5}{9}}\right)}{3}\le\left(\frac{x^{9a}+x^{9b}+x^{9c}}{3}\right)^{\frac{5}{9}}\ as\ \frac{5}{9}\in(0,1)$$

$$\left(x^{9a}+x^{9b}+x^{9c}\right)^{\frac{5}{9}}\ge\frac{1}{3^{\frac{4}{9}}}\left(\left(x^{9a}\right)^{\frac{5}{9}}+\left(x^{9b}\right)^{\frac{5}{9}}+\left(x^{9c}\right)^{\frac{5}{9}}\right)$$

$$\left(\sqrt[9]{x^{9a}+x^{9b}+x^{9c}}\right)^{5}\ge\frac{1}{3^{\frac{4}{9}}}\left(x^{5a}+x^{5b}+x^{5c}\right)\ (1)$$

$$\int_0^1\left(\sqrt[9]{(x^{9a}+x^{9b}+x^{9c})}\right)^5 dx\overset{(1)}{\ge}\frac{1}{3^{\frac{4}{9}}}\int_0^1\left(x^{5a}+x^{5b}+x^{5c}\right)dx=$$

$$=\frac{1}{3^{\frac{4}{9}}}\left[\left(\frac{x^{5a+1}}{5a+1}\right)_0^1+\left(\frac{x^{5b+1}}{5b+1}\right)_0^1+\left(\frac{x^{5c+1}}{5c+1}\right)_0^1\right]=$$

$$\ge\frac{1}{3^{\frac{4}{9}}}\sum\frac{1}{5a+1}\overset{Bergstrom}{\ge}\frac{1}{3^{\frac{4}{9}}}\frac{9}{5(a+b+c)+3}=$$

$$=\frac{3^{\frac{14}{5}}}{5(a+b+c)+3}\ge\frac{\sqrt[9]{3^{14}}}{5(a+b+c)+3}=\frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

$$\textit{Equality holds for } a=b=c$$

$$\textit{now if we take } a=x^4+1, b=2x^2, c=2 \textit{ then}$$

$$\int_0^1\left(\sqrt[9]{\left(x^{9x^4+9}+x^{9x^2}+x^{18}\right)}\right)^5 dt\ge\frac{\sqrt[9]{4782969}}{5(x^4+1+2x^2+2)+3}=\frac{\sqrt[9]{4782969}}{5x^4+10x^2+18}$$

$$\textit{Equlity occurs when } a=b=c \textit{ or},\quad x^4+1=2x^2=2 \textit{ or}, x^2=1 \textit{or}, x=\pm1$$

2813. Find:

$$\int_{-1}^{1}\frac{dx}{\sqrt[5]{(1-x)^3(1+x)^2}}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Amin Hajiyev-Azerbaijan

$$I=\int_{-1}^{1}\frac{dx}{\sqrt[5]{(1-x)^3(1+x)^2}}\quad \textit{substutions } x=2t-1\ \ dx=2dt$$

$$I=2\int_0^1\frac{dt}{\sqrt[5]{\left(1-(2t-1)\right)^3\left(1+(2t-1)\right)^2}}=2\int_0^1\frac{dt}{\sqrt[5]{(2-2t)^3 8t^2}}=$$

$$=\int_0^1\frac{dt}{\sqrt[5]{(1-t)^3t^2}}=\int_0^1(1-t)^{-\frac{3}{5}}t^{-\frac{2}{5}}dt=\int_0^1(1-t)^{\frac{2}{5}-1}t^{\frac{3}{5}-1}dt=$$

$$=\beta\left(\frac{2}{5};\frac{3}{5}\right)=\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{\Gamma(1)}=\Gamma\left(1-\frac{3}{5}\right)\Gamma\left(\frac{3}{5}\right)=\frac{\pi}{\sin\left(\frac{3\pi}{5}\right)}=\frac{4\pi}{\sqrt{10+2\sqrt{5}}}$$

Note section:

- *The beta function* $\beta(x;y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\displaystyle\int_0^1(1-t)^{x-1}t^{y-1}dt\quad x,y>0$
- *Gamma reflection formula* $\Gamma(1-z)\Gamma(z)=\dfrac{\pi}{\sin(\pi z)}$
- $5\alpha=90^\circ\to\quad 2\alpha=90^\circ-3\alpha\to\sin(2\alpha)=\sin\left(\dfrac{\pi}{2}-3\alpha\right)$

$$\sin(2\alpha)=\cos(3\alpha)\to 2\sin(\alpha)\cos(\alpha)=4\cos^3(\alpha)-3\cos(\alpha)$$

$$4\sin^2(\alpha)+2\sin(\alpha)-1=0\to\sin\left(\frac{\pi}{10}\right)=\frac{\sqrt{5}-1}{4}$$

$$\cos\left(\frac{\pi}{10}\right)=\sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cos\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\sin\left(\frac{3\pi}{5}\right)$$

Solution 2 by Tapas Das-India

Let $x=2t-1$ *then* $dx=2dt$

when $x=-1\to t=0, x=1\to t=1$

using the above substitution we get

$$\Omega=\int_0^1\frac{2dt}{2^{\frac{3}{5}}(1-t)^{\frac{3}{5}}(2t)^{\frac{2}{5}}}=\int_0^1 t^{-\frac{2}{5}}(1-t)^{-\frac{3}{5}}dt=\int_0^1 t^{\frac{3}{5}-1}(1-t)^{\frac{2}{5}-1}dt$$

$$=\mathrm{B}\left(\frac{3}{5},\frac{2}{5}\right)=\frac{\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{2}{5}+\frac{3}{5}\right)}\overset{\Gamma(1)=1}{=}\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{2}{5}\right)=\Gamma\left(\frac{3}{5}\right)\Gamma\left(1-\frac{3}{5}\right)=\frac{\pi}{\sin\left(\frac{3\pi}{5}\right)}=\frac{4\pi}{\sqrt{10+2\sqrt{5}}}$$

$$\textit{Note}:\sin\left(\frac{3\pi}{5}\right)=\sin 108^\circ=\cos 18^\circ=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

2814.

If $(a_n)_{n\ge1},\quad a_1=1,\quad a_{n+1}=(n+1)!\,a_n,\forall n\in N^*$ *and* $(F_n)_{n\ge0},\quad F_0=0,\quad F_1=1,\quad F_{n+2}=F_{n+1}+F_n,\quad\forall n\in N$ *is Fibanocci's sequence, the compute*

$$\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$a_1=1,a_{n+1}=(n+1)!\,a_n\to\ a_2=2!\cdot1,a_3=2!\cdot3!\,,\dots\ a_n=\prod_{n=1}^{k}k!$$

By Binet's Formula, Fibonacci numbers exhibit asymptotic growth such that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\phi=\frac{1+\sqrt{5}}{2},\quad F_n\sim\frac{\phi^n}{\sqrt{5}}\ \ as\ \ n\to\infty$$

$$S=\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}}=\lim_{n\to\infty}\frac{(n!)^{\frac{1}{2n}}F_n^{\frac{1}{n}}}{a_n^{\frac{1}{n^2}}}\to\ln(S)=\lim_{n\to\infty}ln\left(\frac{(n!)^{\frac{1}{2n}}F_n^{\frac{1}{n}}}{a_n^{\frac{1}{n^2}}}\right)$$

$$\ln(S)=\lim_{n\to\infty}\left(\frac{\ln(\mathrm{n}!)}{2\mathrm{n}}+\frac{\ln(F_n)}{\mathrm{n}}-\frac{\ln(a_n)}{n^2}\right)$$

Stirling formula: $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,\ln(\mathrm{n}!)\sim\mathrm{n}\cdot\ln(n)-\mathrm{n}+\frac{1}{2}\ln(2\pi n)+O\left(\frac{1}{\mathrm{n}}\right)$

- $\ln(n!)\approx n\cdot\ln(n)-n$
- $F_n\sim\frac{\phi^n}{\sqrt{5}}\to\ \ln(F_n)\sim\mathrm{n}\ln(\phi)-\ln(\sqrt{5})$
- $a_n=\prod_{k=1}^{n}k!\to\ln(a_n)=\ln\left(\prod_{k=1}^{n}k!\right)=\sum_{k=1}^{n}\ln(k!)\approx\sum_{k=1}^{n}(k\ln(k)-k)$

Integral Approximation method: $\sum_{k=1}^{n}f(k)\sim\int_1^n f(x)dx\ \ \ as\ \ n\to\infty$

$$\sum_{k=1}^{n}(k\ln(\mathrm{k})-\mathrm{k})\sim\int_1^n(x\ln(x)-x)dx=\left[\frac{x^2}{2}\ln(x)-\frac{3x^2}{4}\right]_1^n=\frac{n^2}{2}\ln(n)-\frac{3n^2}{4}$$

$$\ln(S)=\lim_{n\to\infty}\left(\frac{1}{2}\ln(n)-\frac{1}{2}+\ln(\phi)-\frac{\ln(\sqrt{5})}{\mathrm{n}}-\frac{\ln(n)}{2}+\frac{3}{4}\right)=\ln(\phi)+\frac{1}{4}$$

$$Therefore\ \ S=\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}}=e^{\ln(\phi)+\frac{1}{4}}=\phi\, e^{\frac{1}{4}}=\frac{1+\sqrt{5}}{2}\sqrt[4]{e}$$

2815.

$$If\ \ (H_n)_{n\geq1},H_n=\sum_{k=1}^{n}\frac{1}{k}\to(Harmonik\ numbers)\ \ then\ prove$$

$$\lim_{n\to\infty}e^{-H_n}\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$I=\lim_{n\to\infty}e^{-H_n}\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}=\lim_{n\to\infty}\frac{1}{e^{H_n}}\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}=\lim_{n\to\infty}\frac{a_n}{b_n}$$

By the Stolz-Cesàro theorem: $I\overset{C-S}{=}\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

$$a_{n+1}-a_n=\sum_{k=1}^{n+1}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}-\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}=\frac{e^{H_{n+1}}}{\sqrt[n+1]{(2n+1)!!}}$$

$$b_{n+1}-b_n=e^{H_{n+1}}-e^{H_n}\overset{H_n=H_{n+1}-\frac{1}{n+1}}{\triangleq}e^{H_n}\left(e^{\frac{1}{n+1}}-1\right)$$

$$I=\lim_{n\to\infty}\frac{e^{H_{n+1}}}{e^{H_n}\left(e^{\frac{1}{n+1}}-1\right)\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{e^{H_{n+1}-H_n}}{\left(e^{\frac{1}{n+1}}-1\right)\sqrt[n+1]{(2n+1)!!}}$$

$$Stirling\ formula:$$

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,\qquad (2n-1)!!=\frac{(2n)!}{2^n n!}\sim\frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}=\sqrt{2}\left(\frac{2n}{e}\right)^n$$

$$(2n+1)!!\sim\sqrt{2}\left(\frac{2(n+1)}{e}\right)^{n+1}$$

$$e^x-1\sim x\ \ as\ x\to 0\qquad e^{\frac{1}{n+1}}-1\sim\frac{1}{n+1}\to\ lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0$$

$$I=\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}}{\frac{\left(e^{\frac{1}{n+1}}-1\right)2^{\frac{1}{2(n+1)}}2(n+1)}{e}}=\frac{e}{2}\frac{\lim\limits_{n\to\infty}e^{\frac{1}{n+1}}}{\lim\limits_{n\to\infty}\left(\frac{1}{n+1}2^{\frac{1}{2(n+1)}}(n+1)\right)}=\frac{e}{2}\cdot\frac{e^0}{2^0}=\frac{e}{2}$$

$$Therefore\ \ \lim_{n\to\infty}e^{-H_n}\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}=\frac{e}{2}$$

2816. Find:

$$\lim_{n\to\infty}n^{1-a}\cdot\sqrt[n]{\frac{((3n)!!)^a}{(3an)!!}},\qquad where\ a\in N^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania
Solution by Amin Hajiyev-Azerbaijan

According to the Ratio-to-Root limit theorem (Cauchy-D'Alembert criterion):

$$If\ \lim_{n\to\infty}\frac{x_{n+1}}{x_n}=L,\qquad then\quad \lim_{n\to\infty}\sqrt[n]{x_n}=L$$
$$Where\ L\in[0;\infty),\qquad \{x_n\}_{n\in N}$$

Stirling formula: $n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n,\sqrt[n]{(2n)!!}=\sqrt[n]{2^n n!}\sim\left(\frac{2n}{e}\right)\ as\ n\to\infty$

$$L=\lim_{n\to\infty} n^{1-a}\cdot\sqrt[n]{\frac{\left((3n)!!\right)^a}{(3an)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{n(1-a)}\left((3n)!!\right)^a}{(3an)!!}}=\lim_{n\to\infty}\sqrt[n]{x_n}$$

$$L=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{\dfrac{(n+1)^{(n+1)(1-a)}\left((3(n+1))!!\right)^a}{(3a(n+1))!!}}{\dfrac{n^{n(1-a)}\left((3n)!!\right)^a}{(3an)!!}}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{(n+1)(1-a)}\left((3(n+1))!!\right)^a(3an)!!}{n^{n(1-a)}(3a(n+1))!!\left((3n)!!\right)^a}$$

Derivation of the Asymptotic Relation: $(2n)!!=2^n n!\to\frac{(2n+2m)!!}{(2n)!!}=\frac{2^m(m+n)!}{n!}$

$$\frac{(n+m)!}{n!}=(n+1)(n+2)\dots(n+m)\sim n^m,\qquad n=\frac{c}{2},m=\frac{b}{2}$$

$$\frac{(c+b)!!}{c!!}\sim 2^{\frac{b}{2}}\left(\frac{c}{2}\right)^{\frac{b}{2}}=\frac{2^{\frac{b}{2}}c^{\frac{b}{2}}}{2^{\frac{b}{2}}}=c^{\frac{b}{2}}$$

$$\frac{\left((3n+3)!!\right)^a}{\left((3n)!!\right)^a}\sim(3n)^{\frac{3a}{2}},\qquad \frac{(3an)!!}{(3an+3a)!!}\sim\frac{1}{(3an)^{\frac{3a}{2}}}$$

$$L=\lim_{n\to\infty}\frac{(n+1)^{(n+1)(1-a)}n^{an}(3n)^{\frac{3a}{2}}}{n^{n(1-a)}(3n)^{\frac{3a}{2}}a^{\frac{3a}{2}}}=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}$$

Using the fundamental limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$, we obtain the following asymptotic behavior: $\left(\left(1+\frac{1}{n}\right)^n\right)^{1-a}(1+n)^{1-a}\sim e^{1-a}(1+n)^{1-a}\ \ as\ \ n\to\infty$

$$L=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}=\lim_{n\to\infty}e^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}$$

$$L=e^{1-a}a^{-\frac{3a}{2}}\lim_{n\to\infty}\frac{1+n}{(1+n)^a}=0\ \ as\ \forall a\in N,a>1$$

2817. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

$$where\ (a_n)_{n\geq1}>0, such\ that\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$(2n-1)!!=\frac{(2n)!}{2^n n!},\qquad Stirling\ formula,\qquad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\left((2n-1)!!\right)^{\frac{1}{n}}\sim\frac{2n}{e}\to\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}$$

• Stolz-Cesàro theorem: $(x_n)_{n\geq1}, (y_n)_{n\geq1} \to \lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \lim_{n\to\infty}\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right) = M$

$$S = \lim_{n\to\infty}\sqrt[n]{a_n}, \ln S = \lim_{n\to\infty}\left(\frac{\ln(a_n)}{n}\right) = \lim_{n\to\infty}\frac{\ln(a_{n+1}) - \ln(a_n)}{n+1-n}$$

$$S = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

• Cauchy-D'Alembert criterion: $\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\sqrt[n]{b_n} = K$

$$b_n = \frac{a_n}{n^n} \to L = \lim_{n\to\infty}\sqrt[n]{b_n} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{n^n}\right)} = \lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\frac{\frac{a_{n+1}}{(n+1)^{n+1}}}{\frac{a_n}{n^n}}$$

$$= \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\right)$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}} = a \to \textit{Asymptotik approach}\frac{a_{n+1}}{a_n}\sim a\sqrt[n]{(2n-1)!!}$$

$$L = \lim_{n\to\infty} a\sqrt[n]{(2n-1)!!}\cdot\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{2a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} =$$

$$= \frac{2a}{e}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{2a}{e}\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{2a}{e^2}$$

2818.

$$\textit{Find } \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}, \textit{where } x_n = \prod_{k=1}^{n} a_k^{\frac{1}{k}} \textit{ and } (a_n)_{n\geq1} > 0$$

$$\textit{such that } \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}} = a \in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

The limit form of Stirling's Formula: $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{n}\right) = \frac{a}{e}$$

Cauchy-D'Alembert criterion: $\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = y \to \lim_{n\to\infty}\sqrt[n]{b_n} = y$

$$b_n = \frac{a_n}{(n-1)!} \to \frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{\frac{n!}{(n-1)!}a_n} = \frac{a_{n+1}}{na_n}$$

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}=\frac{a}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}\cdot\frac{\sqrt[n]{(n-1)!}}{n}=\frac{a}{e}\cdot\frac{1}{e}=\frac{a}{e^2}$$

$$x_n=\prod_{k=1}^{n}a_k^{\frac{1}{k}}\to x_{n+1}=x_n\sqrt[n+1]{a_{n+1}}$$

$$L=\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\overset{Cauchy-D'Almbert}{\cong}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)^{n+1}}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{a}{e^2}\cdot\frac{1}{e}=\frac{a}{e^3}$$

$$Therefore\ \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\frac{a}{e^3}$$

2819.

$$Find\ \lim_{n\to\infty}\left(\frac{\pi^2}{4}-a_n^2\right)n\,,where\ a_n=\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\arctan\frac{1}{k^2-k+1}=\arctan\frac{k-(k-1)}{1+k(k-1)}=\arctan(k)-\arctan(k-1)$$

$$then,\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}$$

$$=(\arctan 1-\arctan 0)+(\arctan 2-\arctan 1)+\cdots+(\arctan(n)-\arctan(n-1))$$

$$=\arctan(n)-\arctan(0)=\arctan(n)$$

$$\lim_{n\to\infty}\left(\frac{\pi^2}{4}-a_n^2\right)n=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-(\arctan(n))^2\right)n=\lim_{z\to 0}\frac{\left(\frac{\pi^2}{4}-\left(\arctan\left(\frac{1}{z}\right)\right)^2\right)}{z}$$

$$\left(let\ n=\frac{1}{z}, when\ n\to\infty\ then\ z\to 0\right)$$

$$using\ L'Hospital\ Rule\ \left(\frac{0}{0}\right) from\ we\ get$$

$$\lim_{z\to 0}\frac{\left(-2\arctan\frac{1}{z}\right).\frac{z^2}{z^2+1}.\left(-\frac{1}{z^2}\right)}{1}=\lim_{z\to 0}\frac{\left(2\arctan\frac{1}{z}\right).\frac{1}{z^2+1}.}{1}.=2\times\frac{\pi}{2}=\pi$$

2820. Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right),\ \ where\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$Stirling\ formula\quad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(2n-1)!!=\frac{(2n)!}{2^n n!}\sim\left(\frac{2n}{e}\right)^n\to\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

Cauchy-D'Alembert criterion: $A=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\sqrt[n]{n!}$

$If\ \frac{a_{n+1}}{a_n}\sim n^k$This implies that the sequence a_n grows at an asymptotic

. rate of $(n!)^k$. $\frac{a_{n+1}}{a_n}\sim Cn^k\to a_n=a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_{n+1}}{a_n}$

$$a_n\sim a_1(C\cdot 1^k)(C\cdot 2^k)\ldots(C\cdot n^k)=C^n(n!)^k\sim C^n\left(\frac{n}{e}\right)^{nk}$$

$$Asyptotik\ approach\quad \sqrt[n]{a_n}\sim C\left(\frac{n}{e}\right)^k$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\to\frac{a_{n+1}}{a_n}\sim a\sqrt[n]{(2n-1)!!}\sim\frac{2an}{e}$$

$$C=\frac{2a}{e},k=1\to\sqrt[n]{n!}\sim\frac{2a}{e}\cdot\frac{n}{e}=\frac{2an}{e^2}$$

$$L=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\frac{2a(n+1)}{e^2}}-\frac{n^2}{\frac{2an}{e^2}}\right)=$$

$$=\frac{e^2}{2a}\lim_{n\to\infty}((n+1)-n))=\frac{e^2}{2a}$$

2821. **Find:**

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

$$where, \lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$Stirling\ formula\quad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(2n-1)!!=\frac{(2n)!}{2^n n!}\sim\left(\frac{2n}{e}\right)^n\to\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

Cauchy-D'Alembert criterion: $A=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{n!}$

$If\ \frac{a_{n+1}}{a_n}\sim n^k$This implies that the sequence x_n grows at an asymptotic

. rate of $(n!)^k$. $\frac{a_{n+1}}{a_n}\sim Cn^k\to a_n=a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_{n+1}}{a_n}$

$$a_n\sim a_1(C\cdot 1^k)(C\cdot 2^k)\ldots(C\cdot n^k)=C^n(n!)^k\sim C^n\left(\frac{n}{e}\right)^{nk}$$

$$Asyptotik\ approach\quad \sqrt[n]{a_n}\sim C\left(\frac{n}{e}\right)^k$$

$$\to\lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}=a,\frac{a_{n+1}}{na_n}\sim a\sqrt[n]{(2n-1)!!},\qquad\frac{a_{n+1}}{a_n}\sim\frac{2n^2a}{e}$$

$$C=\frac{2a}{e},k=2\to\sqrt[n]{a_n}\sim\frac{2a}{e}\cdot\frac{n^2}{e^2}=\frac{2an^2}{e^3}$$

$$L=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{2a(n+1)^2}{e^3}-\frac{2an^2}{e^3}\right)=$$

$$=\frac{2a}{e^3}\lim_{n\to\infty}\frac{1}{n}(n^2+2n+1-n^2)=\frac{2a}{e^3}\lim_{n\to\infty}\left(2+\frac{1}{n}\right)=\frac{4a}{e^3}$$

2822.

$$\text{Find } \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}, \text{where } x_n=\prod_{k=1}^{n} a_k^{\frac{1}{k}} \text{ and } (a_n)_{n\ge1}>0$$

$$\text{such that } \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}=a\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

The limit form of Stirling's Formula:

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},\quad \lim_{n\to\infty}\frac{a_{n+1}}{na_n}=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{n}\right)=\frac{a}{e}$$

Cauchy-D'Alembert criterion: $\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=y\to\lim_{n\to\infty}\sqrt[n]{b_n}=y$

$$b_n=\frac{a_n}{(n-1)!}\to\frac{b_{n+1}}{b_n}=\frac{a_{n+1}}{\frac{n!}{(n-1)!}a_n}=\frac{a_{n+1}}{na_n},\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}=\frac{a}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}\cdot\frac{\sqrt[n]{(n-1)!}}{n}=\frac{a}{e}\cdot\frac{1}{e}=\frac{a}{e^2}$$

$$x_n=\prod_{k=1}^{n}a_k^{\frac{1}{k}}\to x_{n+1}=x_n\sqrt[n+1]{a_{n+1}}$$

$$L=\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\overset{Cauchy-D'Alembert}{\cong}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)^{n+1}}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{a}{e^2}\cdot\frac{1}{e}=\frac{a}{e^3}$$

$$\text{Therefore } \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\frac{a}{e^3}$$

2823. ***Find:***

$$\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$L=\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}=\lim_{x\to 0}\frac{1-\prod_{k=1}^{n}\left(\cos(kx)\right)^{\frac{1}{k}}}{x^2}$$

$$L=\lim_{x\to 0}\frac{1-f(nx)}{x^2}\to f(nx)=\prod_{k=1}^{n}(\cos(kx))^{\frac{1}{k}}$$

Using the Maclaurin series expansion for $\cos(kx)$ $as\ x\to 0$:

$$\cos(kx)=\sum_{m=0}^{\infty}\frac{(-1)^m(kx)^{2m}}{(2m)!}\to \cos(\mathrm{kx})\sim 1-\frac{(kx)^2}{2}\ as\ x\to 0$$

The expression is evaluated by applying the generalized binomial expansion for |x| <<1. Specifically, we use the first-order linear approximation derived from Newton's binomial theorem: $(1+x\beta)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}\beta x^k\sim 1-\alpha\beta x,\ as\ \ x\to 0$

$$f(nx)=\prod_{k=1}^{n}(\cos(kx))^{\frac{1}{k}}\sim\prod_{k=1}^{n}\left(1-\frac{(kx)^2}{2}\right)^{\frac{1}{k}}\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)$$

$$f(nx)\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)=\left(1-\frac{x^2}{2}\right)(1-x^2)\left(1-\frac{3x^2}{2}\right)\ldots\left(1-\frac{nx^2}{2}\right)$$

Weierstrass Product Inequality: $\prod_{i=1}^{n}(1-a_i)\geq 1-\sum_{k=1}^{n}a_i$

$$f(nx)\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)\sim 1-\sum_{k=1}^{n}\frac{kx^2}{2}=1-\frac{x^2}{2}\sum_{k=1}^{n}k=1-\frac{x^2n(n+1)}{4}$$

$$L=\lim_{x\to 0}\frac{1-f(nx)}{x^2}=\lim_{x\to 0}\frac{1-\left(1-\frac{x^2n(n+1)}{4}\right)}{x^2}=\frac{n(n+1)}{4}$$

$$\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}=\frac{n(n+1)}{4}$$

2824. **Find:**

$$\Omega = \lim_{x\to\infty}\left[\int_x^{x+\frac{3}{x}} t^2 \arcsin\left(\frac{1}{t}\right)dt\right]$$

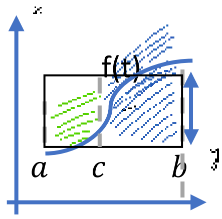
Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

Mean Value Theorem for Integrals:

If f is continuous on $[a;\ b]$, then there exists $c \in [a;\ b]$ such that:

$$\int_a^b f(t)dt = f(c)(b-a)$$



$h = f(c)$

$$f(t) = t^2\sin^{-1}\left(\frac{1}{t}\right) \quad a = x, b = x + \frac{3}{x}$$

$$\Omega = \lim_{x\to\infty}\left[\left(x + \frac{3}{x} - x\right)f(c)\right]$$

$$= 3\lim_{x\to\infty}\left[\frac{1}{x}\left(c^2\sin^{-1}\left(\frac{1}{c}\right)\right)\right]$$

$$\textit{Asymptotik analysis}:\quad x \le c \le x + \frac{3}{x} \to\ x\to\infty, \qquad x \approx c$$

Taylor series for: $\arcsin(y) = \sum_{n=0}^{\infty}\frac{(2n)!y^{2n+1}}{4^n(2n+1)(n!)^2} \sim y + O(y^3)\ as\ |y| \le 1$

$$x\to\infty, x\approx c \to \left|\frac{1}{c}\right| < 1 \quad \arcsin\left(\frac{1}{c}\right) \sim \frac{1}{c} + O\left(\frac{1}{c^3}\right)$$

$$\text{Therfore}\ \ \Omega = 3\lim_{x\to\infty}\left[\frac{1}{x}\cdot c^2\cdot\left(\frac{1}{c} + O\left(\frac{1}{c^3}\right)\right)\right] = 3\lim_{x\to\infty}\frac{c}{x} = 3$$

2825. **Find:**

$$\lim_{n\to\infty} n\left(\frac{4}{3} - \frac{a_n}{b_n}\right),\quad a_n = \sum_{k=1}^{n}\frac{1}{k^2}\ \textit{and}\ b_n = \sum_{k=1}^{n}\frac{1}{(2k-1)^2}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\textit{We know}\ \sum_{k=1}^{\infty}\frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}, \qquad H_n^{(s)} = \sum_{k=1}^{n}\frac{1}{k^s}$$

$$a_n=H_n^{(2)}=\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=n+1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}-\psi^{(1)}(1+n)$$

Trigamma function:

$$\psi^{(1)}(x)=\sum_{n=0}^{\infty}\frac{B_n}{x^{n+1}}=\frac{1}{x}+\frac{1}{2x^2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{x^{2n+1}}\sim\frac{1}{x}+\frac{1}{2x^2}+O\left(\frac{1}{x^3}\right)$$

Binomial theorem: $(1+\alpha)^{\beta}=1+\beta\alpha+\frac{\beta(\beta-1)}{2!}\alpha^2+\frac{\beta(\beta-1)(\beta-2)}{3!}\alpha^3+\cdots$

$$\frac{1}{n+1}=\frac{1}{n}\left(1+\frac{1}{n}\right)^{-1}\sim\frac{1}{n}\left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\sim\frac{1}{n}-\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)$$

$$\frac{1}{2(n+1)^2}=\frac{1}{2n^2}\left(1+\frac{1}{n}\right)^{-2}\sim\frac{1}{2n^2}\left(1-\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)\sim\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)$$

$$\psi^{(1)}(n+1)\sim\frac{1}{n+1}+\frac{1}{2(n+1)^2}+O\left(\frac{1}{n^3}\right)\sim\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)$$

$$a_n=H_n^{(2)}=\frac{\pi^2}{6}-\psi^{(1)}\sim\frac{\pi^2}{6}-\frac{1}{n}+\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)$$

We know $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\beta(2)=\frac{\pi^2}{8}$, $\quad b_n=\beta(2)-\sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^2}=$

$$=\frac{\pi^2}{8}-\frac{1}{4}\sum_{k=n+1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2}$$

$$=\frac{\pi^2}{8}-\frac{1}{4}\psi^{(1)}\left(n+\frac{1}{2}\right)\sim\frac{\pi^2}{8}-\frac{1}{4\left(n+\frac{1}{2}\right)}-\frac{1}{8\left(n+\frac{1}{2}\right)^2}-O\left(\frac{1}{n^3}\right)$$

$$\frac{1}{n+\frac{1}{2}}\sim\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right),\frac{1}{\left(n+\frac{1}{2}\right)^2}\sim\frac{1}{n^2}+O\left(\frac{1}{n^3}\right),\psi^{(1)}\left(n+\frac{1}{2}\right)\sim\frac{1}{n}+O\left(\frac{1}{n^3}\right)$$

$$b_n\sim\frac{\pi^2}{8}-\frac{1}{4n}-O\left(\frac{1}{n^3}\right)$$

$$\lim_{n\to\infty}n\left(\frac{4}{3}-\frac{a_n}{b_n}\right)=\lim_{n\to\infty}n\left(\frac{4}{3}-\frac{\frac{\pi^2}{6}-\frac{1}{n}+\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)}{\frac{\pi^2}{8}-\frac{1}{4n}+O\left(\frac{1}{n^3}\right)}\right)=$$

$$=\lim_{n\to\infty}n\left(\frac{4}{3}-\frac{\frac{\pi^2}{6}\left(1-\frac{6}{n\pi^2}+\frac{3}{\pi^2n^2}+O\left(\frac{1}{n^3}\right)\right)}{\frac{\pi^2}{8}\left(1-\frac{2}{\pi^2n}+O\left(\frac{1}{n^3}\right)\right)}\right),\ \frac{a_n}{b_n}\sim$$

$$\frac{4}{3}\left(1-\frac{6}{n\pi^2}+\frac{3}{n^2\pi^2}\right)\cdot\left(1-\frac{2}{\pi^2n}\right)^{-1}$$

$$\frac{a_n}{b_n} \sim \frac{4}{3}\left(1-\frac{6}{n\pi^2}+\frac{3}{n^2\pi^2}\right)\cdot\left(1+\frac{2}{\pi^2 n}+\frac{4}{\pi^4 n^2}+O\left(\frac{1}{n^3}\right)\right) \sim \frac{4}{3}\left(1+\left(-\frac{6}{\pi^2 n}\right)+\frac{2}{\pi^2 n}\right)$$

$$\frac{a_n}{b_n} \sim \frac{4}{3}\left(1-\frac{4}{\pi^2 n}\right) \sim \frac{4}{3}-\frac{16}{3\pi^2 n} \to \lim_{n\to\infty} n\left(\frac{4}{3}-\frac{4}{3}+\frac{16}{3n\pi^2}\right)=16/3\pi^2$$

$$Therefore \lim_{n\to\infty} n\left(\frac{4}{3}-\frac{a_n}{b_n}\right)=\frac{16}{3\pi^2}$$

2826. Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{"n"}-times}\right)(x)\right),\qquad f(x)=\frac{x}{\sqrt{1+x^2}}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Daniel Sitaru-Romania

$$(f\circ f)(x)=f(f(x))=f\left(\frac{x}{\sqrt{1+x^2}}\right)=\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}=\frac{x}{\sqrt{1+2x^2}}$$

$$(f\circ f\circ f)(x)=f\big((f\circ f)(x)\big)=f\left(\frac{x}{\sqrt{1+2x^2}}\right)=\frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}=\frac{x}{\sqrt{1+3x^2}}$$

$$P(n):\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{n}-times}\right)(x)=\frac{x}{\sqrt{1+nx^2}}-suppose\ true$$

$$P(n+1):\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{n+1}-times}\right)(x)=\frac{x}{\sqrt{1+(n+1)x^2}}-to\ prove$$

$$\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{n+1}-times}\right)(x)=f\left(\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{n}-times}\right)(x)\right)=f\left(\frac{x}{\sqrt{1+nx^2}}\right)=$$

$$=\frac{\frac{x}{\sqrt{1+nx^2}}}{\sqrt{1+\frac{x^2}{1+nx^2}}}=\frac{x}{\sqrt{1+(n+1)x^2}}$$

$$P(n)\to P(n+1)$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\underbrace{f\circ f\circ\ldots\circ f}_{\text{"n"}-times}\right)(x)\right)=\lim_{n\to\infty}\left(\lim_{x\to\infty}\left(\frac{x}{\sqrt{1+nx^2}}\right)\right)=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\right)=0$$

2827.

$$If\ x_n=\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right), n\in N^*\ Find$$

$$\lim_{n\to\infty}\left(\left(1-\frac{1}{3}\right)x_n\right)^{3^{2^{n+1}}}$$

Proposed by Marin Chirciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$L=\lim_{n\to\infty}\left(\left(1-\frac{1}{3}\right)x_n\right)^{3^{2^{n+1}}}=\lim_{n\to\infty}(I_n)^{3^{2^{n+1}}}$$

$$I_n=\left(1-\frac{1}{3}\right)x_n=\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)$$

$$I_n=\underbrace{\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}_{Teleskopic}=1-\frac{1}{3^{2^{n+1}}}$$

$$L=\lim_{n\to\infty}\left(1-\frac{1}{3^{2^{n+1}}}\right)^{3^{2^{n+1}}}\to 3^{2^{n+1}}=u\ \ u\to\infty$$

$$L=\lim_{u\to\infty}\left(1-\frac{1}{u}\right)^u=\lim_{u\to\infty}\left[\left(1+\frac{1}{-u}\right)^{-u}\right]^{-1}=e^{-1}=\frac{1}{e}$$

2828. Prove that:

$$\int_0^1\int_0^1\int_0^1\frac{x^2z\ln(xy)}{x^3yz+xyz+x^3y+x^2z+xy+x^2+z+1}dxdydz$$
$$=\ln\left(\frac{e}{2}\right)\left(\frac{\pi}{4}G-\frac{\pi^2}{96}\ln(2)-\frac{3}{4}\zeta(3)\right)$$
$$G-Catalan's\ constant$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^1\int_0^1\int_0^1\frac{f(x,y,z)}{g(x,y,z)}dxdydz$$

$$g(x,y,z)=x^3yz+xyz+x^3y+x^2z+xy+x^2+z+1=$$
$$=(x^3yz+x^3y)+(xyz+xy)+(x^2z+x^2)+(z+1)=$$
$$=(z+1)(x^3y+xy+x^2+1)=(z+1)(x^2+1)(xy+1)$$

$$I=\int_0^1\int_0^1\int_0^1\frac{x^2z\ln(x)+x^2z\ln(y)}{(1+z)(1+x^2)(1+xy)}dxdydz=I_1+I_2$$

$$I_1=\int_0^1\frac{z}{1+z}\left(\int_0^1\frac{x^2\ln(x)}{1+x^2}\left(\int_0^1\frac{1}{1+xy}dy\right)dx\right)dz=$$

$$=\int_0^1\frac{z}{1+z}dz\int_0^1\frac{x^2\ln(x)}{1+x^2}\left[\frac{\ln(1+xy)}{\mathrm{x}}\right]_0^1dx=(1-\ln(2))\underbrace{\int_0^1\frac{x\ln(x)\ln(1+x)}{1+x^2}dx}_{=K}$$

$$K=\int_0^1\frac{x\ln(x)\ln(1+x)}{1+x^2}dx\to\ K(a)=\int_0^1\frac{x\ln(x)\ln(1+ax)}{1+x^2}dx$$

$$\frac{\partial}{\partial a}K(a)=\int_0^1\frac{x^2\ln(x)}{(1+x^2)(1+ax)}dx$$

$$=\frac{1}{1+a^2}\left(a\int_0^1\frac{x\ln(x)}{1+x^2}dx-\int_0^1\frac{\ln(x)}{1+x^2}dx+\int_0^1\frac{\ln(x)}{1+ax}dx\right)$$

$$K_1=\int_0^1\frac{x\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n+1}\ln(x)\,dx\overset{IBP}{\cong}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2}=-\frac{1}{4}\eta(2)$$

$$=-\frac{\pi^2}{48}$$

$$K_2=\int_0^1\frac{\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}\ln(x)\,dx\overset{IBP}{\cong}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-\beta(2)=-G$$

$$K_3(a)=\int_0^1\frac{\ln(x)}{1+ax}dx,0\le a\le1\to\ K_3(a)$$

$$=\sum_{n=0}^{\infty}(-1)^n a^n\int_0^1x^n\ln(x)\overset{IBP}{\cong}-\sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(n+1)^2}$$

$$=\frac{1}{a}\sum_{n=1}^{\infty}\frac{(-1)^n a^n}{n^2}=\frac{Li_2(-a)}{a}$$

$$\frac{\partial}{\partial a}K(a)=\frac{1}{1+a^2}\big(aK_1-K_2+K_3(a)\big)\to\ K(1)=K,K(0)=0$$

$$K=\int_0^1\frac{1}{1+a^2}\big(aK_1-K_2+K_3(a)\big)da=$$

$$=-\frac{\pi^2}{48}\int_0^1\frac{a}{1+a^2}da+G\int_0^1\frac{1}{1+a^2}da+\int_0^1\frac{Li_2(-a)}{a(1+a^2)}da$$

$$=-\frac{\pi^2}{96}\ln(2)+\frac{\pi}{4}\mathrm{G}+\underbrace{\int_0^1\frac{Li_2(-a)}{a(1+a^2)}da}_{\mathrm{P}}\quad a\to x,P=\int_0^1\frac{Li_2(-x)}{x(1+x^2)}dx$$

$$=\int_0^1\frac{Li_2(-x)}{x}dx-\int_0^1\frac{x}{1+x^2}Li_2(-x)dx=P_1-P_2$$

$$P_1=\int_0^1\frac{Li_2(-x)}{x}dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1x^{n-1}\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\eta(3)=-\frac{3}{4}\zeta(3)$$

$$P_2=\int_0^1\frac{x\,Li_2(-x)}{1+x^2}dx\ \ IBP\to\begin{cases}u=Li_2(-x),\dfrac{du}{dx}=\dfrac{\ln(1+x)}{x}\\ \displaystyle\int\frac{x}{1+x^2}dx=\frac{1}{2}\ln(1+x^2)\end{cases}\ \ Li_s(-1)=-\eta(s)$$

$$P_2=\int_0^1\frac{x\,Li_2(-x)}{1+x^2}dx\overset{IBP}{=}-\frac{\eta(2)}{2}\ln(2)+\frac{1}{2}\int_0^1\frac{\ln(1+x^2)\ln(1+x)}{x}dx$$

$$=-\frac{\pi^2}{24}\ln(2)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{2n-1}\ln(1+x)\,dx$$

$$Note:\ \int_0^1 x^{n-1}\ln(1+x)dx=\frac{H_n-H_{\frac{n}{2}}}{n}$$

$$P_2=-\frac{\pi^2}{24}\ln(2)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(H_n-H_{2n})=$$

$$=-\frac{\pi^2}{24}\ln(2)+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{n^2}=-\frac{\pi^2}{24}\ln(2)+\frac{1}{4}(W_1-W_2)$$

$$Note:\ q\in Z^+\quad\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^{2q}}=-\frac{1}{2}(2q+1)\eta(2q+1)+\sum_{k=0}^{q-1}\zeta(2q-2k+1)\,\eta(2k)$$

$$|x|\le 1$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}x^n=Li_3(x)-Li_3(1-x)+\ln(1-x)Li_2(1-x)+\frac{1}{2}\ln(x)\ln^2(1-x)+\zeta(3)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{n^2}=4\sum_{n=1}^{\infty}\frac{i^{2n}H_{2n}}{(2n)^2}=4Re\left\{\sum_{n=1}^{\infty}\frac{i^nH_n}{n^2}\right\}=$$

$$=4Re\left\{Li_3(i)-Li_3(1-i)+\ln(1-i)Li_2(1-i)+\frac{1}{2}\ln(i)\ln^2(1-i)+\zeta(3)\right\}$$

$$Re\{ln(i)\}=0,Re\{ln(1-i)\}=\frac{ln(2)}{2}$$

$$Li_a(i)=(2^{1-2a}-2^{-a})\zeta(a)+i\beta(a)$$

$$Li_2(i)=-\frac{\pi^2}{48}+iG,Li_2(1-i)=\frac{3}{8}\zeta(2)-\left(\frac{\pi}{4}ln(2)+G\right)i$$

$$Li_3(i)=-\frac{3}{32}\zeta(3)+i\beta(3),Re\{Li_3(1-i)\}=Re\{Li_3(1+i)\}=\frac{3}{16}ln(2)\zeta(2)+\frac{35\zeta(3)}{64}$$

$$Therfore\ \ W_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}=-\frac{5}{8}\zeta(3),\qquad W_2=\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{n^2}=\frac{23}{16}\zeta(3)-\pi G$$

$$P_2=-\frac{\pi^2}{24}\ln(2)+\frac{1}{4}(W_1-W_2)=-\frac{\pi^2}{24}\ln(2)+\frac{1}{4}\left(-\frac{33}{16}\zeta(3)+\pi G\right)$$

$$=-\frac{\pi^2}{24}\ln(2)+\frac{\pi G}{4}-\frac{33}{64}\zeta(3)$$

$$P=P_1-P_2=-\frac{3}{4}\zeta(3)-\frac{\pi G}{4}+\frac{33}{64}\zeta(3)+\frac{\pi^2}{24}\ln(2)=-\frac{15}{64}\zeta(3)-\frac{\pi G}{4}+\frac{\pi^2}{24}\ln(2)$$

$$K=-\frac{\pi^2}{96}ln(2)+\frac{\pi G}{4}+P=-\frac{\pi^2}{96}ln(2)+\frac{\pi G}{4}+\left(-\frac{15}{64}\zeta(3)-\frac{\pi G}{4}+\frac{\pi^2}{24}\ln(2)\right)$$

$$=\frac{\pi^2}{32}\ln(2)-\frac{15}{64}\zeta(3)$$

$$I_1=(1-\ln(2))K=\ln\left(\frac{e}{2}\right)\left(\frac{\pi^2}{32}\ln(2)-\frac{15}{64}\zeta(3)\right)$$

$$I_2=\int_0^1\frac{z}{1+z}\left(\int_0^1\frac{x^2}{1+x^2}\left(\int_0^1\frac{\ln(y)}{1+xy}dy\right)dx\right)dz$$

$$Where\int_0^1\frac{\ln(y)}{1+xy}dy=\sum_{n=0}^{\infty}(-1)^n x^n\int_0^1 y^n\ln(y)\,dy\overset{IBP}{=}-\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)^2}=\frac{Li_2(-x)}{x}$$

$$I_2=(1-\ln(2))\underbrace{\int_0^1\frac{xLi_2(-x)}{1+x^2}dx}_{P_2}=\left(\ln\left(\frac{e}{2}\right)\right)\left(\frac{\pi G}{4}-\frac{\pi^2}{24}\ln(2)-\frac{33}{64}\zeta(3)\right)$$

$$I=(I_1+I_2)=\ln\left(\frac{e}{2}\right)\left(\frac{\pi^2}{32}\ln(2)-\frac{15}{64}\zeta(3)\right)+\ln\left(\frac{e}{2}\right)\left(\frac{\pi G}{4}-\frac{\pi^2}{24}\ln(2)-\frac{33}{64}\zeta(3)\right)=$$

$$=\ln\left(\frac{e}{2}\right)\left(\frac{\pi}{4}G-\frac{\pi^2}{96}\ln(2)-\frac{3}{4}\zeta(3)\right)$$

Therefore

$$\int_0^1\int_0^1\int_0^1\frac{x^2z\ln(xy)}{x^3yz+xyz+x^3y+x^2z+xy+x^2+z+1}dxdydz$$

$$=\ln\left(\frac{e}{2}\right)\left(\frac{\pi}{4}G-\frac{\pi^2}{96}\ln(2)-\frac{3}{4}\zeta(3)\right)$$

2829. ***Prove the closed form***

$$\int_0^\infty e^{-x}\ln(x)\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}x^n dx=4\ln(2)+8-5\gamma$$

$$where\ \gamma\ \ Euler-Mascheroni\ constant$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^\infty e^{-x}\ln(x)\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}x^n dx=\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}\int_0^\infty e^{-x}\ln(x)\,x^n dx$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}I_n\to\ I_n=\int_0^{\infty}e^{-x}x^n\ln(x)\,dx\to\ I_n(a)=\int_0^{\infty}e^{-x}x^{n+a-1}dx$$

$$I_n=\lim_{a\to1}\frac{\partial}{\partial a}I_n(a)=lim_{a\to1}\frac{\partial}{\partial a}\int_0^{\infty}e^{-x}\,x^{n+a-1}dx$$

$$\left\{Gamma\ function\ integral:\ \int_0^{\infty}e^{-x}x^{s-1}dx=\Gamma(s),\qquad Re\{s\}>0\right\}$$

$$I_n=\lim_{a\to1}\frac{\partial}{\partial a}\Gamma(a+n)=\lim_{a\to1}\Gamma(n+a)\psi^{(0)}(n+a)=\Gamma(n+1)\psi^{(0)}(n+1)$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}I_n=\sum_{n=1}^{\infty}\frac{(n^2-1)\Gamma(n+1)\psi^{(0)}(n+1)}{2^n\Gamma(n+1)}=\sum_{n=1}^{\infty}\frac{(n^2-1)\psi^{(0)}(n+1)}{2^n}$$

Relation between Digamma Function and Harmonic Numbers

$$\{\psi^{(0)}(n+1)=H_n-\gamma,\qquad Where\ \gamma\ \ Euler-Mascheroni\ constant\}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(n^2-1)(H_n-\gamma)}{2^n}=\sum_{n=1}^{\infty}\frac{n^2H_n}{2^n}-\sum_{n=1}^{\infty}\frac{H_n}{2^n}-\gamma\sum_{n=1}^{\infty}\frac{n^2}{2^n}+\gamma\sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$\Omega=\Omega_1-\Omega_2-\gamma\Omega_3+\gamma\Omega_4$$

$$\Omega_1=\sum_{n=1}^{\infty}\frac{n^2H_n}{2^n}=\sum_{n=1}^{\infty}n^2H_n\left(\frac{1}{2}\right)^n\ \ we\ know\ \left\{\sum_{n=1}^{\infty}x^nH_n=-\frac{\ln(1-x)}{1-x},|x|<1\right\}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}x^nH_n=\sum_{n=1}^{\infty}nx^{n-1}H_n=\frac{1-\ln(1-x)}{(1-x)^2}\to\sum_{n=1}^{\infty}nx^nH_n=\frac{x-xln(1-x)}{(1-x)^2}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}nx^nH_n=\sum_{n=1}^{\infty}n^2x^{n-1}H_n=\frac{2x-(x+1)\ln(1-x)+1}{(1-x)^3}$$

$$\sum_{n=1}^{\infty}n^2x^nH_n=\frac{2x^2-x(x+1)\ln(1-x)+x}{(1-x)^3}\to\ x=\frac{1}{2}$$

$$\Omega_1=\frac{2\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{3}{2}\ln\left(\frac{1}{2}\right)+\frac{1}{2}}{\frac{1}{8}}=8+6\ln(2)$$

$$\Omega_2=\sum_{n=1}^{\infty}\frac{H_n}{2^n}=\sum_{n=1}^{\infty}H_n\left(\frac{1}{2}\right)^n=-\frac{\ln\left(1-\frac{1}{2}\right)}{1-\frac{1}{2}}=2\ln(2)$$

$$\Omega_3=\sum_{n=1}^{\infty}\frac{n^2}{2^n}\ \ We\ know\left\{\sum_{n=0}^{\infty}x^n=\frac{1}{1-x},\sum_{n=1}^{\infty}x^n=\frac{x}{1-x},|x|\le1\right\}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}x^n=\sum_{n=1}^{\infty}nx^{n-1}=\frac{1+x}{(1-x)^2}\ \ \sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}nx^n=\sum_{n=1}^{\infty}n^2x^{n-1}=\frac{x+1}{(1-x)^3}\to\sum_{n=1}^{\infty}n^2x^n=\frac{x(x+1)}{(1-x)^3}\ as\ |x|\le1$$

$$x=\frac{1}{2}\to\ \Omega_3=\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\frac{\frac{1}{2}\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^3}=\frac{3}{4}\cdot 8=6$$

$$\Omega_4=\sum_{n=1}^{\infty}\frac{1}{2^n}\text{ ,}We\ know\ \sum_{n=1}^{\infty}x^n=\frac{x}{1-x},|x|\le 1\to\ \Omega_4=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

$$\Omega=\Omega_1-\Omega_2-\gamma\Omega_3+\gamma\Omega_4=8+6\ln(2)-2\ln(2)-6\gamma+\gamma=8+4\ln(2)-5\gamma$$

$$Therefore\ \int_0^{\infty}e^{-x}\ln(x)\sum_{n=1}^{\infty}\frac{(n^2-1)}{2^n\Gamma(n+1)}x^n dx=4\ln(2)+8-5\gamma$$

2830. *Find*:

$$\int_0^{\infty}\frac{x\ln(x)\tan^{-1}x}{(1+x^2)(1+x)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x^2)(1+x)^2}dx=\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx+\underbrace{\int_1^{\infty}\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx}_{substitutions\ x\to\frac{1}{x}}=$$

$$=\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx-\int_0^1\frac{x\ln(x)\tan^{-1}\left(\frac{1}{x}\right)}{(1+x)^2(1+x^2)}dx=$$

$$=\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx-\int_0^1\frac{x\ln(x)\left(\frac{\pi}{2}-\tan^{-1}(x)\right)}{(1+x)^2(1+x^2)}dx$$

$$=2\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx-\frac{\pi}{2}\int_0^1\frac{x\ln(x)}{(1+x)^2(1+x^2)}dx=2\Omega_1-\frac{\pi}{2}\Omega_2$$

$$\Omega_1=\int_0^1\frac{x\ln(x)\tan^{-1}(x)}{(1+x)^2(1+x^2)}dx=\frac{1}{2}\underbrace{\left(\int_0^1\frac{\ln(x)\tan^{-1}(x)}{1+x^2}dx-\int_0^1\frac{\ln(x)\tan^{-1}(x)}{(1+x)^2}dx\right)}_{M-K}$$

$$M=\int_0^1\frac{\ln(x)\tan^{-1}(x)}{1+x^2}dx\to substituton\text{: }\tan^{-1}(x)=t\text{ ,}\frac{dt}{dx}=\frac{1}{1+x^2}$$

$$M=\int_0^{\frac{\pi}{4}}t\cdot\ln(\tan(t))dt$$

Fourier series of $\ln(tanx):0<x<\frac{\pi}{2}$, *we have*

$$\ln(\tan(x))=-2\sum_{n=0}^{\infty}\frac{\cos(x(4n+2))}{2n+1}.$$

$$M=-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\underbrace{\int_0^{\frac{\pi}{4}}t\cdot\cos\big((4n+2)t\big)\,dt}_{IBP}$$

$$=-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{t\cdot\sin\big((4n+2)t\big)}{4n+2}+\frac{\cos\big((4n+2)t\big)}{(4n+2)^2}\right]_0^{\frac{\pi}{4}}=$$

$$=\underbrace{\frac{1}{2}\sum_{n=0}^{\infty}\frac{\sin(\pi n)}{(2n+1)^3}}_{\sin(\pi n)=0,n\in Z^+}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}-\frac{\pi}{4}\underbrace{\sum_{n=0}^{\infty}\frac{\cos(\pi n)}{(2n+1)^2}}_{\cos(\pi z)=(-1)^n,n\in Z^+}=$$

$$=\underbrace{\frac{1}{4}\left(\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\right)}_{2\sum_{n=0}^{\infty}a_{2n+1}=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^n a_n}-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{4}\big(\zeta(3)+\eta(3)\big)-\frac{\pi}{4}\beta(2)=$$

$$=\frac{7}{16}\zeta(3)-\frac{\pi G}{4}$$

$$K=\int_0^1\frac{\ln(x)\tan^{-1}(x)}{(1+x)^2}dx\to IBP\quad\begin{cases}u=\ln(x)\tan^{-1}(x),\dfrac{du}{dx}=\dfrac{\ln(x)}{1+x^2}+\dfrac{\tan^{-1}(x)}{x}\\ v=\displaystyle\int\frac{dx}{(1+x)^2},v=-\frac{1}{1+x}\end{cases}$$

$$K=\left[-\frac{\ln(x)\tan^{-1}(x)}{1+x}\right]_0^1+\int_0^1\frac{\ln(x)}{(1+x)(1+x^2)}dx+\int_0^1\frac{\tan^{-1}(x)}{x(1+x)}dx=$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{x\ln(x)}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x}dx+\int_0^1\frac{\tan^{-1}(x)}{x}dx$$
$$-\int_0^1\frac{\tan^{-1}(x)}{1+x}dx$$

$$K=\frac{1}{2}K_1-\frac{1}{2}K_2+\frac{1}{2}K_3+K_4-K_5$$

$$K_1=\int_0^1\frac{\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n}\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-\beta(2)=-G$$

$$K_2=\int_0^1\frac{x\ln(x)}{1+x^2}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{2n+1}\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2}=-\frac{\eta(2)}{4}=-\frac{\pi^2}{48}$$

$$K_3=\int_0^1\frac{\ln(x)}{1+x}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^1x^n\ln(x)\,dx=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=-\eta(2)=-\frac{\pi^2}{12}$$

$$K_4=\int_0^1\frac{\tan^{-1}(x)}{x}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1x^{2n}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$$

$$K_5=\int_0^1\frac{\tan^{-1}(x)}{1+x}dx=\underbrace{\int_0^1\frac{\frac{\pi}{4}-\tan^{-1}\left(\frac{1-x}{1+x}\right)}{1+x}dx}_{\tan^{-1}x=\frac{\pi}{4}-\tan^{-1}\frac{1-x}{1+x}}=\frac{\pi}{4}\ln(2)-\int_0^1\frac{\tan^{-1}\frac{1-x}{1+x}}{1+x}dx$$

$$substitutions:\ t=\frac{1-x}{1+x},\frac{dt}{dx}=-\frac{2}{(1+x)^2}\to\ 0\le t\le 1$$

$$K_5=\frac{\pi}{2}\ln(2)-\int_0^1\frac{\tan^{-1}(x)}{1+x}dx=\frac{\pi}{2}\ln(2)-K_5\to\ K_5=\frac{\pi}{8}\ln(2)$$

$$K=\frac{1}{2}K_1-\frac{1}{2}K_2+\frac{1}{2}K_3+K_4-K_5$$

$$K=-\frac{G}{2}+\frac{\pi^2}{96}-\frac{\pi^2}{24}+G-\frac{\pi}{8}\ln(2)=\frac{G}{2}-\frac{\pi^2}{32}-\frac{\pi}{8}\ln(2)$$

$$\Omega_1=\frac{1}{2}(M-K)=\frac{1}{2}\left(\frac{7}{16}\zeta(3)-\frac{\pi G}{4}-\frac{G}{2}+\frac{\pi^2}{32}+\frac{\pi}{8}\ln(2)\right)$$

$$\Omega_2=\int_0^1\frac{x\ln(x)}{(1+x)^2(1+x^2)}dx=\frac{1}{2}\left(\underbrace{\int_0^1\frac{\ln(x)}{1+x^2}dx}_{K_1}-\underbrace{\int_0^1\frac{\ln(x)}{(1+x)^2}dx}_{P}\right)=$$

$$=\frac{1}{2}\left(-G-\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^1x^n\ln(x)\,dx\right)=\frac{1}{2}\left(-G+\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\right)=$$

$$=-\frac{G}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\frac{G}{2}+\frac{\ln(2)}{2}$$

$$\Omega=2\Omega_1-\frac{\pi}{2}\Omega_2=\frac{7}{16}\zeta(3)-\frac{\pi G}{4}-\frac{G}{2}+\frac{\pi^2}{32}+\frac{\pi}{8}\ln(2)+\frac{\pi G}{4}-\frac{\pi}{4}\ln(2)=$$

$$=\frac{7}{16}\zeta(3)-\frac{G}{2}+\frac{\pi^2}{32}-\frac{\pi}{8}\ln(2)$$

$$Therefore\ \int_0^{\infty}\frac{x\ln(x)\tan^{-1}x}{(1+x^2)(1+x)^2}dx=\frac{7}{16}\zeta(3)-\frac{G}{2}+\frac{\pi^2}{32}-\frac{\pi}{8}\ln(2)$$

2831. ***Find***:

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{77n^4-86n^2-51}$$

Proposed by Sakthi Vel-India

Solution by Amin Hajiyev-Azerbaijan

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{77n^4-86n^2-51}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(7n^2+3)(11n^2-17)}=$$

$$=\frac{1}{152}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\frac{17}{11}}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+\frac{3}{7}}\right)$$

$$For\quad \alpha\in\ C\setminus Z\ we\ have:\rightarrow\quad S=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}\quad f(z)=\frac{1}{z^2-\alpha^2}$$

$$n\in Z\rightarrow\cos(\pi n)=(-1)^n\quad p(z)=\frac{\pi}{\sin(\pi z)}\quad Res(p(z),n)=\frac{\pi}{\frac{d}{dz}\sin(\pi z)}=\frac{1}{\cos(\pi n)}$$

$$=(-1)^n,\ \ g(z)=p(z)f(z)$$

$$g(z)=\frac{\pi}{\sin(\pi z)\,(z^2-\alpha^2)}\rightarrow\begin{cases}Integer\ poles\ z=n\in Z\\ Function\ poles\ z=\pm\alpha\end{cases}$$

Consider the function $g(z)=f(z)\cdot\frac{\pi}{sin(\pi z)}$. Since $g(z)=O(z^{-2})$ as $|z|\rightarrow\infty$, the integral over a large contour C_N (centered at the origin with radius $R=N+\frac{1}{2}$) vanishes as $N\rightarrow\infty$. By the Residue Theorem, the sum of all residues must be zero:

$$\sum_{n=-\infty}^{\infty}(Res(g,n))+Res\{g,-\alpha\}+Res\{g,\alpha\}=0$$

$$Integer\ poles:$$

$$Res(g,n)=\lim_{z\rightarrow n}(z-n)\cdot\frac{\pi}{(z^2-\alpha^2)\sin(\pi z)}=\frac{1}{n^2-\alpha^2}\cdot Res\left(\frac{\pi}{\sin(\pi z)},n\right)=\frac{(-1)^n}{n^2-\alpha^2}$$

$$z=\pm\alpha:$$

$$z=\alpha\rightarrow\ Res(g,\alpha)=\lim_{z\rightarrow\alpha}(z-\alpha)\cdot\frac{\pi}{(z^2-\alpha^2)\sin(\pi z)}=\frac{\pi}{2\alpha\sin(\pi\alpha)}$$

$$z=-\alpha\rightarrow\ Res(g,-\alpha)=\lim_{z\rightarrow-\alpha}(z+\alpha)\cdot\frac{\pi}{(z^2-\alpha^2)\sin(\pi z)}=\frac{\pi}{2\alpha\sin(\pi\alpha)}$$

$$\frac{\pi}{2\alpha\sin(\pi\alpha)}+\frac{\pi}{2\alpha\sin(\pi\alpha)}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}=0\rightarrow\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}=-\frac{\pi}{\alpha\sin(\pi\alpha)}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}+\sum_{n=-\infty}^{n=0}\frac{(-1)^n}{n^2-\alpha^2}=-\frac{\pi}{\alpha\sin(\pi\alpha)}\rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}$$

$$=-\frac{\pi}{\alpha\sin(\pi\alpha)}$$

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}-\frac{1}{\alpha^2}=-\frac{\pi}{\alpha\sin(\pi\alpha)}\rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\alpha^2}=\frac{1}{2\alpha^2}-\frac{\pi}{2\alpha\sin(\pi\alpha)}$$

$$I=\frac{1}{152}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\left(\sqrt{\frac{17}{11}}\right)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-\left(i\sqrt{\frac{3}{7}}\right)^2}\right)$$

$$=\frac{1}{152}\left(\frac{11}{34}-\frac{\pi\sqrt{11}}{2\sqrt{17}\sin\left(\pi\sqrt{\frac{17}{11}}\right)}+\frac{7}{6}+\frac{\pi\sqrt{7}}{2i\sqrt{3}\sin\left(\pi i\sqrt{\frac{3}{7}}\right)}\right)=$$

$$=\frac{1}{152}\left(\frac{76}{51}-\frac{\pi}{2}\sqrt{\frac{11}{17}}\csc\left(\pi\sqrt{\frac{17}{11}}\right)-\frac{\pi}{2}\sqrt{\frac{7}{3}}\operatorname{csch}\left(\pi\sqrt{\frac{3}{7}}\right)\right)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{77n^4-86n^2-51}=\frac{1}{152}\left(\frac{76}{51}-\frac{\pi}{2}\sqrt{\frac{11}{17}}\csc\left(\pi\sqrt{\frac{17}{11}}\right)-\frac{\pi}{2}\sqrt{\frac{7}{3}}\operatorname{csch}\left(\pi\sqrt{\frac{3}{7}}\right)\right)$$

2832. ***Find the closed form:***

$$\int_0^{\pi}\left(\frac{\sin(x)}{1+k\cdot\cos(x)}\right)^{a-1}\frac{dx}{1+k\cdot\cos(x)}$$

$$where\quad a>0 \text{ and } 0<k<1$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

Weierstrass Substitution: $\tan\left(\frac{x}{2}\right)=t,\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)=\frac{1}{1+\cos(x)}\quad 0\le t\le\infty$

$$We\ know\ \sin(x)=\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}=\frac{2t}{1+t^2},\ \cos(x)=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}=\frac{1-t^2}{1+t^2}$$

$$\frac{dt}{dx}=\frac{1}{1+\frac{1-t^2}{1+t^2}}=\frac{1+t^2}{2}\to dx=\frac{2dt}{1+t^2}$$

$$I=\int_0^{\infty}\left(\frac{\frac{2t}{1+t^2}}{1+k\cdot\frac{1-t^2}{1+t^2}}\right)^{a-1}\cdot\frac{\frac{2dt}{1+t^2}}{1+k\cdot\frac{1-t^2}{1+t^2}}=2^a\int_0^{\infty}\frac{t^{a-1}}{(1+k+(1-k)t^2)^a}dt=$$

$$=\frac{2^a}{(1+k)^a}\int_0^{\infty}\frac{t^{a-1}}{\left(1+\left(\frac{1-k}{1+k}\right)t^2\right)^a}dt$$

$$Substitution\ \frac{1-k}{1+k}\cdot t^2=u, \frac{du}{dt}=2\cdot\frac{1-k}{1+k}\cdot t=2\left(\frac{1-k}{1+k}\right)\sqrt{\frac{u(1+k)}{1-k}}=2\sqrt{\frac{u(1-k)}{1+k}}$$

$$I=\frac{2^a}{(1+k)^a}\int_0^\infty\frac{\left(\sqrt{\frac{u(1+k)}{1-k}}\right)^{a-1}}{(1+u)^a}\cdot\frac{1}{2\sqrt{u}}\sqrt{\frac{1+k}{1-k}}=\frac{2^{a-1}}{(1+k)^a}\cdot\left(\frac{1+k}{1-k}\right)^{\frac{a}{2}}\int_0^\infty\frac{u^{\frac{a}{2}-1}}{(1+u)^a}du$$

$$Beta\ function\ integral:\ \beta(a;\ b)=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt$$

$$I=\frac{2^{a-1}}{(1-k^2)^{\frac{a}{2}}}\int_0^\infty\frac{u^{\frac{a}{2}-1}}{(1+u)^{\frac{a}{2}+\frac{a}{2}}}du=\frac{2^{a-1}}{(1-k^2)^{\frac{a}{2}}}\beta\left(\frac{a}{2};\frac{a}{2}\right)=\frac{2^{a-1}}{(1-k^2)^{\frac{a}{2}}}\cdot\frac{\Gamma^2\left(\frac{a}{2}\right)}{\Gamma(a)}$$

Legendre Duplication Formula: $\Gamma(2b)=\frac{\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}{2^{1-2b}\sqrt{\pi}}\to b=\frac{a}{2}\to\Gamma(a)=\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{2^{1-a}\sqrt{\pi}}$

$$I=\frac{2^{a-1}}{(1-k^2)^{\frac{a}{2}}}\cdot\frac{\Gamma^2\left(\frac{a}{2}\right)}{\Gamma(a)}=\frac{2^{a-1}}{(1-k^2)^{\frac{a}{2}}}\cdot\frac{\Gamma^2\left(\frac{a}{2}\right)}{\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}{2^{1-a}\sqrt{\pi}}}=\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{a}{2}\right)}{(1-k^2)^{\frac{a}{2}}\Gamma\left(\frac{a+1}{2}\right)}$$

$$Therefore\ \int_0^\pi\left(\frac{\sin(x)}{1+k\cdot\cos(x)}\right)^{a-1}\frac{dx}{1+k\cdot\cos(x)}=\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{a}{2}\right)}{(1-k^2)^{\frac{a}{2}}\Gamma\left(\frac{a+1}{2}\right)}$$

2833. Find:

$$I=\int_0^1 sgn(\sin(\ln(x)))dx$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^1 sgn(\sin(\ln(x)))dx\ substitution\ e^{-t}=x, \frac{dx}{dt}=-e^{-t}$$

$$I=-\int_0^\infty sgn(\sin(ln(e^{-t})))\,e^{-t}dt=-\int_0^\infty sgn(\sin(-t))e^{-t}dt$$

$$I=\int_0^\infty sgn(\sin(t))e^{-t}dt$$

The sign of $\sin(t)$ alternates every π interval,
such that $sgn(\sin(\mathrm{t}))=(-1)^{\mathrm{n}}$ for $t\in[\pi n,\pi(n+1)]$.

$$I=\sum_{n=0}^\infty\int_{\pi n}^{\pi(n+1)}(-1)^n e^{-t}dt=\sum_{n=0}^\infty(-1)^n\left[-e^{-t}\right]_{\pi n}^{\pi(n+1)}=$$

$$=\sum_{n=0}^\infty(-1)^n\left(e^{-\pi n}-e^{-\pi(n+1)}\right)=(1-e^{-\pi})\sum_{n=0}^\infty(-1)^n e^{-\pi n}=$$

$$= (1-e^{-\pi})\cdot\frac{1}{1+e^{-\pi}} = \underbrace{\frac{e^{\pi}-1}{e^{\pi}+1}}_{\tanh(x)=\frac{e^{2x}-1}{e^{2x}+1}} = \tanh\left(\frac{\pi}{2}\right)$$

$$Therefore\ \int_0^1 sgn(\sin(\ln(x)))dx = \tanh\left(\frac{\pi}{2}\right)$$

2834.

$$If\ \ f(x)=y,\ \ \frac{dy}{dx}+2xy = x\cdot e^{-x^2}\ \ ,\ \ y(0)=1\ \ prove\ then:$$

$$\int_0^{\infty} x\cdot\ln(x)\cdot f(x)dx = \frac{1-3\gamma}{8}$$

$$where\ \gamma\ \ is\ the\ Euler-Mascheroni\ constant$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\frac{dy}{dx}+2xy = x\cdot e^{-x^2}\quad IF\ method\to\ \mu(x) = e^{\int 2xdx} = e^{x^2}$$

$$\mu(x)\cdot\left(\frac{dy}{dx}+2xy\right) = x\cdot e^{-x^2}\mu(x)\to \underbrace{\mu(x)\left(\frac{dy}{dx}+2xy\right)}_{\frac{d}{dx}(\mu(x)y)} = x$$

$$\frac{d}{dx}\left(e^{x^2}y\right) = x\to\ e^{x^2}y = \int xdx = \frac{x^2}{2}+C\to\ y = \frac{x^2}{2}e^{-x^2}+C\cdot e^{-x^2}$$

$$y(0)=1\to\ y(0)=C=1\ \ C=1$$

$$y = f(x) = \frac{x^2}{2}e^{-x^2}+e^{-x^2}$$

$$I = \int_0^{\infty} x\ln(x)\cdot f(x)dx = \int_0^{\infty}\left(\frac{x^3}{2}e^{-x^2}+xe^{-x^2}\right)\ln(x)\,dx \overset{x^2=t}{=}$$

$$= \frac{1}{8}\int_0^{\infty} t\cdot e^{-t}\ln(t)\,dt+\frac{1}{4}\int_0^{\infty} e^{-t}\ln(t)\,dt = \frac{I_1}{8}+\frac{I_2}{4}$$

$$Note: Gamma\ function\ integral\quad \int_0^{\infty} x^{n-1}e^{-x}dx = \Gamma(n)$$

$$\int_0^{\infty} x^{n-1}e^{-x}\ln(x)\,dx = \lim_{a\to n}\frac{\partial}{\partial a}\int_0^{\infty} x^{a-1}e^{-x}dx = \Gamma'(n)$$

$$I_1 = \int_0^{\infty} t\cdot e^{-t}\ln(t)\,dt = \Gamma'(2) = 1-\gamma,\qquad I_2 = \int_0^{\infty} e^{-t}\ln(t)\,dt = \Gamma'(1) = -\gamma$$

$$\text{Note: }\ \Gamma(n+1) = n\cdot\Gamma(n),\qquad \Gamma'(1) = -\gamma,\qquad \Gamma'(2) = 1+\Gamma'(1) = 1-\gamma$$

$$I = \frac{1}{8}I_1+\frac{1}{4}I_2 = \frac{1}{8}(1-\gamma)-\frac{1}{4}\gamma = \frac{1}{8}-\frac{3\gamma}{4} = \frac{1-3\gamma}{8}$$

$$\textit{Therefore } \int_0^{\infty} x\cdot\ln(x)\cdot f(x)dx=\frac{1-3\gamma}{8}$$

2835. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt{2}\,.\,\sqrt[4]{4}.\,\sqrt[8]{8}.\,\ldots.\,\sqrt[2^n]{2^n}}{n}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\sqrt{2}\,.\,\sqrt[4]{4}.\,\sqrt[8]{8}.\,\ldots.\,\sqrt[2^n]{2^n}=2^{\sum_{m=1}^{n}\frac{m}{2^m}}$$

$$\therefore\ \sum_{m=0}^{\infty}x^m=\frac{1}{1-x},\ \sum_{m=1}^{\infty}mx^{m-1}=\frac{1}{(1-x)^2}\to\sum_{m=1}^{\infty}mx^m=\frac{x}{(1-x)^2}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt{2}\,.\,\sqrt[4]{4}.\,\sqrt[8]{8}.\,\ldots.\,\sqrt[2^n]{2^n}}{n}\right)=\lim_{n\to\infty}\left(\frac{2^{\sum_{m=1}^{n}\frac{m}{2^m}}}{n}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{2^{\sum_{m=1}^{\infty}\frac{m}{2^m}}}{n}\right)=\lim_{n\to\infty}\left(\frac{2^{\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}}}{n}\right)=\lim_{n\to\infty}\left(\frac{2^2}{n}\right)=0$$

2836. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{k!+(k-1)!+(k-2)!}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{k(k-1)(k-2)!+(k-1)(k-2)!+(k-2)!}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{(k(k-1)+(k-1)+1)\,.\,(k-2)!}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{(k-2)!\,(k^2-k+k-1+1)}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{(k-2)!\,.\,k^2}\right)=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k-1}{k(k-1)(k-2)!}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k-1}{k!}\right)=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{k}{k!}-\sum_{k=3}^{n}\frac{1}{k!}\right)=$$

$$=\lim_{n\to\infty}\left(\sum_{k=3}^{n}\frac{1}{(k-1)!}-\sum_{k=3}^{n}\frac{1}{k!}\right)=$$

$$=\sum_{k=3}^{\infty}\frac{1}{(k-1)!}-\sum_{k=3}^{\infty}\frac{1}{k!}=(e-1-1)-\left(e-1-1-\frac{1}{2}\right)=\frac{1}{2}$$

2837. Find :

$$\Omega=\lim_{n\to\infty}\left[n\int_0^1\ln\left(1+e^{-nx}\right)dx\right]$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\lim_{n\to\infty}\left[n\int_0^1\ln\left(1+e^{-nx}\right)dx\right]\to\ substitution\ nx=t,\frac{dt}{dx}=n$$

$$n\to\infty\quad x=\frac{t}{n}\to 0,\qquad t=0,\ \ t=n$$

$$\Omega=\lim_{n\to\infty}\left[\int_0^n\ln\left(1+e^{-t}\right)dt\right]=-\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^n e^{-kt}dt=$$

$$=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\infty}e^{-kt}dt=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left[-\frac{e^{-kt}}{k}\right]_0^{\infty}=-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=\eta(2)=\frac{1}{2}\zeta(2)=\frac{\pi^2}{12}$$

2838. Prove that:

$$\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\left(K(k)-E(k)\right)-K(k)\right)dk=1-\frac{\pi}{4}$$

Proposed by Shobhit Jain-India

Solution by Pratham Prasad-India

$$We\ Know:$$

$$\frac{d}{dk}K(k)=\frac{1}{k}\left(\frac{E(k)}{1-k^2}-K(k)\right)$$

$$\frac{d}{dk}E(k)=\frac{1}{k}\left(E(k)-K(k)\right)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}\left(K(k)-E(k)\right)+\frac{1}{k^2}\left(\frac{1}{k}\left(E(k)-K(k)\right)-\frac{1}{k}\left(\frac{E(k)}{1-k^2}-K(k)\right)\right)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}\left(K(k)-E(k)\right)-\frac{1}{k}\left(\frac{E(k)}{1-k^2}\right)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))-\left(\frac{1}{k}\left(\frac{E(k)}{1-k^2}\right)-\frac{1}{k}K(k)\right)-\frac{1}{k}K(k)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))-\frac{d}{dk}K(k)-\frac{1}{k}K(k)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)=\frac{2}{k^3}(K(k)-E(k))-\frac{1}{k}K(k)$$

$$\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)\Bigg|_0^1=\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk$$

$$I=\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk=E(1)-\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)$$

$$\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk=1-\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right)=1-L-\frac{\pi}{2}$$

$$L=\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right)=\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right)$$

$$=\lim_{k\to 0}\left(\frac{\frac{1}{k}\left(E(k)-K(k)-\frac{E(k)}{1-k^2}+K(k)\right)}{2k}\right)$$

$$L=-\lim_{k\to 0}\left(\frac{\big(E(k)\big)}{2(1-k^2)}\right)=\frac{-E(0)}{2}=-\frac{\pi}{4}$$

$$I=1-L-\frac{\pi}{2}=1-\frac{\pi}{4}$$

$$I=\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk=1-\frac{\pi}{4}$$

2839. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\cdot\ldots\cdot\sqrt{n!}}}}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$A_n=1!+\sqrt{2!+\sqrt{3!+\sqrt{n!}}},\quad B_n=1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}$$

$$\frac{A_n}{B_n}=\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}}=\frac{1!}{B_n}+\frac{\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}}$$

Note: $1!\cdot 2!\cdot\sqrt{\sqrt{3!\ldots}}=1!\cdot\sqrt{2!}\cdot B_n'$, $\frac{A_n}{B_n}=\frac{1}{B_n}+\frac{1}{1!}\cdot\sqrt{\frac{1}{2!}+\frac{\sqrt{3!+\cdots}}{2!\cdot\sqrt{3!\ldots}}}$

$$n\to\infty\Rightarrow 1<\frac{A_n}{B_n}<1+\frac{C}{2^n}\quad (C=\text{constant})1^n<\left(\frac{A_n}{B_n}\right)^n<\left(1+\frac{C}{2^n}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{C}{2^n}\right)^n=\lim_{n\to\infty}e^{\frac{n\cdot C}{2^n}}=e^0=1\Rightarrow\Omega=1$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$A_n=1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}},\quad B_n=1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}=\prod_{k=1}^{n}(k!)^{\frac{1}{2^{k-1}}}$$

$$\frac{A_n}{B_n}=\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{\prod_{k=1}^{n}(k!)^{\frac{1}{2^{k-1}}}}=1+O\left(\frac{1}{2^n}\right),$$

$$\Omega=\lim_{n\to\infty}\left(1+\left(\frac{A_n}{B_n}-1\right)\right)^n=e^{\lim_{n\to\infty}n\cdot\left(\frac{A_n}{B_n}-1\right)}$$

$$\lim_{n\to\infty}n\cdot O\left(\frac{1}{2^n}\right)=\lim_{n\to\infty}\frac{n}{2^n}=0$$

$$\Omega=e^0=1\Rightarrow\Omega=1$$

2840. ***Prove that :***

$$\frac{1}{1\times2\times3}\left(\frac{1}{2}\right)+\frac{1}{2\times3\times4}\left(\frac{1}{2}\right)^2+\frac{1}{3\times4\times5}\left(\frac{1}{2}\right)^3+\cdots=\frac{1}{4}\ln\left(\frac{4}{e}\right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\frac{1}{1\times2\times3}\left(\frac{1}{2}\right)+\frac{1}{2\times3\times4}\left(\frac{1}{2}\right)^2+\frac{1}{3\times4\times5}\left(\frac{1}{2}\right)^3+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}\left(\frac{1}{2}\right)^n=$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)\right)\left(\frac{1}{2}\right)^n=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{1}{n+2}-\frac{2}{n+1}\right)\left(\frac{1}{2}\right)^n=$$

$$=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}\frac{1}{n+2}\left(\frac{1}{2}\right)^n-\sum_{n=1}^{\infty}\frac{2}{n+1}\left(\frac{1}{2}\right)^n\right)=$$

$$=\frac{1}{2}\left(-\ln\left(1-\frac{1}{2}\right)+\sum_{k=3}^{\infty}\frac{1}{k}\left(\frac{1}{2}\right)^{k-2}-\sum_{k=2}^{\infty}\frac{2}{k}\left(\frac{1}{2}\right)^{k-1}\right)=$$

$$\frac{1}{2}\left(-\ln\left(\frac{1}{2}\right)+\sum_{k=3}^{\infty}\frac{4}{k}\left(\frac{1}{2}\right)^{k}-2\sum_{k=2}^{\infty}\frac{2}{k}\left(\frac{1}{2}\right)^{k}\right)$$

$$=\frac{1}{2}\left(\ln(2)+4\left(\ln(2)-\frac{5}{8}\right)-2(2\ln(2)-1)\right)=$$

$$=\frac{1}{2}\left(\ln(2)+4\ln(2)-\frac{5}{2}-4\ln(2)+2\right)=\frac{1}{2}\left(\ln(2)-\frac{1}{2}\right)=\frac{1}{4}(2\ln(2)-1)=\frac{1}{4}\ln\left(\frac{4}{e}\right)$$

2841. Find:

$$\Omega=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n(2n)!!}$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\therefore\ (2n-1)!!=\frac{1\times 2\times 3\ldots\times(2n-1)\times 2n}{2\times 4\times 6\times\ldots\times 2n}=\frac{(2n)!}{2^n\times n!}$$

$$\therefore\ (2n)!!=2^n\times(1\times 2\times 3\times\ldots n)=2^n\times n!$$

$$\therefore\ \frac{(2n)!}{n!\times n!}=C_{2n}^{n}=\binom{2n}{n}\ ,\qquad \sum_{n=0}^{\infty}\binom{2n}{n}\times x^n=\frac{1}{\sqrt{1-4x}},\qquad |x|<\frac{1}{4}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n(2n)!!}=\sum_{n=1}^{\infty}\frac{\frac{(2n)!}{2^n\times n!}}{2^n\times n!}=\sum_{n=1}^{\infty}\left(\frac{(2n)!}{2^n\times n!}\times\frac{1}{2^n\times n!}\right)=$$

$$=\sum_{n=1}^{\infty}\frac{(2n)!}{2^{3n}\times(n!)^2}=\sum_{n=1}^{\infty}\binom{2n}{n}\times\left(\frac{1}{8}\right)^n=\left(\sum_{n=0}^{\infty}\binom{2n}{n}\times\left(\frac{1}{8}\right)^n\right)-\binom{0}{0}\times\left(\frac{1}{8}\right)^0=$$

$$=\frac{1}{\sqrt{1-4\times\frac{1}{8}}}-1=\sqrt{2}-1$$

2842. If $z\in\mathbb{C}, n\in\mathbb{N}$ then:

$$\cos^n z=\frac{1}{(\cos^2 a+sh^2 b)^{\frac{n}{2}}}e^{-inarctg(tgathb)}$$

Proposed by Samir Cabiyev-Azerbaijan

Solution by proposer

$$\cos^n z=\left(\frac{e^{iz}+e^{-iz}}{2}\right)^n=\left(\frac{e^{i(a+ib)}+e^{-i(a+ib)}}{2}\right)^n=$$

$$=\left(\frac{e^{ia}e^{-b}+e^{-ia}e^{b}}{2}\right)^n=\left[\frac{e^{-b}}{2}(\cos a+i\sin a)+\frac{e^{b}}{2}(\cos a-i\sin a)\right]^n=$$

$$=\left[\cos a\left(\frac{e^b+e^{-b}}{2}\right)-i\sin a\left(\frac{e^b-e^{-b}}{2}\right)\right]^n=(\cos a\,ch b-i\sin a\,sh b)^n$$

Remark:

$$\mathrm{A}-\mathrm{iB}=\frac{1}{\sqrt{A^2+B^2}}\left(\cos\left(arctg\frac{B}{A}\right)-i\sin\left(arctg\frac{B}{A}\right)\right)$$

$$(\cos a\,ch b-i\sin a\,sh b)^n=$$

$$=\frac{1}{\left(\sqrt{\cos^2 a\,ch^2 b+\sin^2 a\,sh^2 b}\right)^n}(\cos(arctg\,tg a\,th b)--i\sin(arctg(tg a\,th b)^n$$

Remark:

$$\cos^2 a\,ch^2 b+\sin^2 a\,sh^2 b=\cos^2 a\,ch^2 b+(1-\cos^2 a)sh^2 b=$$

$$=\cos^2 a(ch^2 b-sh^2 b)+sh^2 b=\cos^2 a+sh^2 b$$

$$\cos^n z=\frac{1}{(\cos^2 a+sh^2 b)^{\frac{n}{2}}}(\cos(arctg\,tg a\,th b)-i\sin(arctg(tg a\,th b)^n=$$

$$=\frac{1}{(\cos^2 a+sh^2 b)^{\frac{n}{2}}}e^{-in\,arctg(tg a\,th b)}$$

2843. If $f:\mathbb{R}\to\mathbb{R}$, $f(f(x))+f(x)=-x,\forall x\in\mathbb{R}$ then find:

$$\Omega=\int_0^{\pi}f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)dx$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\begin{cases}f(f(x))+f(x)=-x\\ f(f(x))=-x-f(x)\end{cases}$$

$$f\left(f(f(x))\right)+f(f(x))=-f(x),\qquad f\left(f(f(x))\right)+(-x-f(x))=-f(x)$$

$$f\left(f(f(x))\right)-x-f(x)+f(x)=0,\ f\left(f(f(x))\right)-x=0,\qquad f^3(x)=x$$

$$Identity\ \ function\ :\ x\overset{f}{\rightrightarrows}f(x)\overset{f}{\rightrightarrows}f(f(x))\overset{f}{\rightrightarrows}f\left(f(f(x))\right)\overset{f}{\rightrightarrows}x$$

$$\Omega=\int_0^{\pi} f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)dx=\int_0^{\pi}\frac{1}{2+\cos x}dx \overset{x=\pi-y}{\cong}$$

$$=\int_{-\pi}^{\pi}\frac{1}{2+\cos(\pi-y)}dy=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{-\pi-\varepsilon}^{\pi+\varepsilon}\frac{1}{2+\cos(\pi-y)}dy$$

$$=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{-\pi-\varepsilon}^{\pi+\varepsilon}\frac{1}{2-\cos y}dy \overset{y=2\arctan z}{\cong} \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{2-\frac{1-z^2}{1+z^2}}\cdot\frac{2}{1+z^2}dz=$$

$$=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{2}{2+2z^2-1+z^2}dz=2\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{1+3z^2}dz=$$

$$=\frac{2}{3}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{\frac{1}{3}+z^2}dz=$$

$$=\frac{2\sqrt{3}}{3}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\arctan\left(\sqrt{3}\cdot\tan\frac{\pi-\varepsilon}{2}\right)-\arctan\left(\sqrt{3}\cdot\tan\frac{\pi-\varepsilon}{2}\right)\right)=$$

$$=\frac{2\sqrt{3}}{3}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{2\pi\sqrt{3}}{3}$$

2844. Prove that:

$$\int_0^1 \frac{\frac{\ln(x)}{(1-x)^2}}{\frac{(1+x)^2}{\ln(x)}}dx=\frac{\pi^2+7\zeta(3)}{8}$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\int_0^1 \frac{\frac{\ln(x)}{(1-x)^2}}{\frac{(1+x)^2}{\ln(x)}}dx=\int_0^1\left(\frac{\ln(x)}{(1-x)^2}\div\frac{(1+x)^2}{\ln(x)}\right)dx=\int_0^1\frac{ln^2(x)}{(1-x)^2(1+x)^2}dx=$$

$$=\int_0^1\frac{ln^2(x)}{((1-x)^2)^2}dx=\sum_{j=0}^{\infty}(j+1)\int_0^1 x^2\,ln^2(x)dx=\sum_{j=0}^{\infty}\left((j+1)\times\frac{2}{(2j+1)^3}\right)=$$

$$=2\sum_{j=0}^{\infty}\frac{j+1}{(2j+1)^3}=\sum_{j=0}^{\infty}\left(\frac{1}{(2j+1)^2}+\frac{1}{(2j+1)^3}\right)=\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}+\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}=$$

$$=\left(1-\frac{1}{4}\right)\zeta(2)+\left(1-\frac{1}{8}\right)\zeta(3)=$$

$$=\frac{3}{4}\zeta(2)+\frac{7}{8}\zeta(3)=\frac{6\zeta(2)+7\zeta(3)}{8}=\frac{6\times\frac{\pi^2}{6}+7\zeta(3)}{8}=\frac{\pi^2+7\zeta(3)}{8}$$

$$\therefore\sum_{j=0}^{\infty}\frac{1}{(2j+1)^t}=(1-2^{-t})\zeta(t)\quad\therefore\int_0^1 x^m ln^2(x)dx=\frac{2}{(m+1)^3}$$

2845. ***Prove that :***

$$\Omega=\int_0^1\frac{\ln\left(ln\left(\frac{1}{x}\right)\right)}{\sqrt{ln\left(\frac{1}{x}\right)}}dx=-(\gamma+2\ln(2))\sqrt{\pi}$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\int_0^1\frac{\ln\left(ln\left(\frac{1}{x}\right)\right)}{\sqrt{ln\left(\frac{1}{x}\right)}}dx\quad\begin{cases}\ln\left(\frac{1}{x}\right)=y\\dx=-e^{-y}dy\end{cases}$$

$$\Omega=\int_\infty^0\frac{\ln(y)}{\sqrt{y}}.(-e^{-y}dy)=\int_0^\infty e^{-y}\frac{\ln(y)}{\sqrt{y}}dy=\int_0^\infty e^{-y}.y^{-\frac{1}{2}}\ln(y)\,dy=$$

$$=\Gamma'\left(\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right).\psi\left(\frac{1}{2}\right)=\sqrt{\pi}.(-\gamma-2\ln(2))=-(\gamma+2\ln(2))\sqrt{\pi}$$

Note :

$$\therefore\Gamma(z)=\int_0^\infty t^{z-1}.e^{-t}dt,\qquad\Gamma'(z)=\int_0^\infty t^{z-1}.e^{-t}\ln(t)dt$$

$$\therefore\psi(z)=\frac{d}{dz}ln\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)},\qquad\Gamma'(z)=\Gamma(z).\psi(z)$$

$$\therefore\psi\left(k+\frac{1}{2}\right)=-\gamma-2\ln(2)+\sum_{n=1}^{k}\frac{2}{2k-1}$$

$$\therefore\psi\left(\frac{1}{2}\right)=-\gamma-2\ln(2),\qquad\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

2846. Prove that:

$$\int_0^\infty ln\left(\frac{x^2+2026^2}{x^2+2023^2}\right)dx=23\pi$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Adgozel Adgozalov-Azerbaijan

$$I(a,b)=\int_0^\infty \ln\left(\frac{x^2+a^2}{x^2+b^2}\right)dx$$

$$I(b,b)=\int_0^\infty \ln\left(\frac{x^2+b^2}{x^2+b^2}\right)dx=\int_0^\infty \ln(1)dx=0$$

$$\frac{dI}{da}=\int_0^\infty \frac{d}{da}\left(\ln(x^2+a^2)-\ln(x^2+b^2)\right)dx$$

$$\frac{dI}{da}=\int_0^\infty \frac{2a}{x^2+a^2}dx$$

$$\frac{dI}{da}=2a\cdot\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right]_0^\infty$$

$$\frac{dI}{da}=2\left(\lim_{x\to\infty}\arctan\left(\frac{x}{a}\right)-\arctan(0)\right)$$

$$\frac{dI}{da}=2\left(\frac{\pi}{2}-0\right)=\pi$$

$$I(a,b)=\int \pi\,da=\pi a+C(b)$$

$$I(b,b)=\pi b+C(b)=0\implies C(b)=-\pi b$$

$I(a,b)=\pi(a-b)$

$$a=2026,\quad b=2003$$

$$\int_0^\infty \ln\left(\frac{x^2+2026^2}{x^2+2003^2}\right)dx=\pi(2026-2003)=23\pi$$

2847. Find:

$$\boldsymbol{\Omega=\int_0^\infty \frac{dx}{x\sqrt{x}+1}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\Omega=\int_0^\infty \frac{dx}{x\sqrt{x}+1}\overset{(x=y^2)}{=}\int_0^\infty \frac{2ydy}{y^3+1}$$

$$\frac{2y}{(y+1)(y^2-y+1)}=\frac{A}{y+1}+\frac{By+C}{y^2-y+1},\qquad 2y=A(y^2-y+1)+(y+1)(By+C)$$

$$2y=Ay^2-Ay+A+By^2+Cy+By+C, 2y=(A+B)y^2+y(-A+B+C)+A+C$$

$$\begin{cases}A+B=0\\-A+B+C=2\\A+C=0\end{cases}\Rightarrow B=-A\Rightarrow\begin{cases}-2A+C=2\\2A+C=0\end{cases}\Rightarrow 3C=2\Rightarrow C=\frac{2}{3}\Rightarrow A=-\frac{2}{3}\Rightarrow B=\frac{2}{3}$$

$$\Omega=\int_0^\infty\left(\frac{-\frac{2}{3}}{y+1}+\frac{\frac{2}{3}(y+1)}{y^2-y+1}\right)dy=-\frac{2}{3}\ln(y+1)\Big|_0^\infty+\frac{1}{3}\int_0^\infty\frac{2y-1+3}{y^2-y+1}dy=$$

$$=-\frac{2}{3}\ln(y+1)+\frac{1}{3}\ln(y^2-y+1)\Big|_0^\infty+\int_0^\infty\frac{1}{\left(y-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}\ln\left(\frac{y^2-y+1}{y^2+2y+1}\right)\Big|_0^\infty+\frac{1}{\frac{\sqrt{3}}{2}}\arctan\left(\frac{y-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^\infty=$$

$$=\frac{1}{3}(\ln 1-\ln 1)+\frac{2}{\sqrt{3}}\left(\arctan\infty-\arctan\left(-\frac{1}{\sqrt{3}}\right)\right)=$$

$$=\frac{1}{3}(0-0)+\frac{2}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}\cdot\frac{4\pi}{6}=\frac{8\pi}{6\sqrt{3}}=\frac{4\pi}{3\sqrt{3}}$$

Solution 2 by Daniel Sitaru – Romania

$$\int_0^\infty\frac{dx}{x^p+1}=\frac{\pi}{p\sin\left(\frac{\pi}{p}\right)}$$

$$\Omega=\int_0^\infty\frac{dx}{x^{\frac{2}{3}}+1}=\frac{\pi}{\frac{3}{2}\sin\left(\frac{\pi}{\frac{3}{2}}\right)}=\frac{2\pi}{3\sin\frac{2\pi}{3}}=\frac{2\pi}{3\sin\left(\pi-\frac{\pi}{3}\right)}=\frac{2\pi}{3\cdot\sin\frac{\pi}{3}}=\frac{2\pi}{\frac{3\sqrt{3}}{2}}=\frac{4\pi}{3\sqrt{3}}$$

2848.

$$\textbf{Find :}$$
$$\int_0^{\frac{\pi}{6}} \frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2} = \frac{\sin^2(x)}{\sin^2(x)+4\sin(x)\cos(x)+4\cos^2(x)} =$$
$$= \frac{\tan^2(x)}{(\tan(x)+2)^2}, \quad \begin{cases} \tan(x) \to t \\ dx = \dfrac{dt}{1+t^2} \end{cases}$$
$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2}dx = \int_0^{\frac{\sqrt{3}}{3}} \frac{t^2}{(t+2)^2}\cdot\frac{dt}{t^2+1}$$
$$\frac{t^2}{(t+2)^2} = \frac{A}{(t+2)^2}+\frac{B}{t+2}+\frac{Ct+D}{t^2+1}$$
$$t^2 = A(t^2+1)+B(t+2)(t^2+1)+(Ct+D)(t+2)^2$$
$$\begin{cases} B+C=0 \\ A+2B+4C+D=1 \\ B+4C+4D=0 \\ A+2B+4D=0 \end{cases} \to \begin{cases} A=\dfrac{4}{5}, \quad B=-\dfrac{4}{25} \\ C=\dfrac{4}{25}, \quad D=-\dfrac{3}{25} \end{cases}$$
$$\Omega = \Omega_1+\Omega_2+\Omega_3$$
$$\Omega_1 = \int_0^{\frac{\sqrt{3}}{3}} \frac{\frac{4}{5}}{(t+2)^2}dt = -\frac{4}{5}\cdot\frac{1}{t+2}\Big|_0^{\frac{\sqrt{3}}{3}} = \frac{2(2\sqrt{3}-1)}{55}$$
$$\Omega_2 = \int_0^{\frac{\sqrt{3}}{3}} \frac{-\frac{4}{25}}{t+2}dt = -\frac{4}{25}\cdot\ln(t+2)\Big|_0^{\frac{\sqrt{3}}{3}} = -\frac{4}{25}\ln\left(\frac{6+\sqrt{3}}{6}\right)$$
$$\Omega_3 = \int_0^{\frac{\sqrt{3}}{3}} \frac{\frac{4}{25}t-\frac{3}{25}}{t^2+1}dt = \frac{1}{25}\int_0^{\frac{\sqrt{3}}{3}} \frac{4t-3}{t^2+1}dt = \frac{2}{25}\int_0^{\frac{\sqrt{3}}{3}} \frac{d(t^2+1)}{t^2+1} =$$
$$\frac{2}{25}\cdot\ln(t^2+1)\Big|_0^{\frac{\sqrt{3}}{3}} - \frac{3}{25}\cdot\arctan(t)\Big|_0^{\frac{\sqrt{3}}{3}} = \frac{2}{25}\ln\left(\frac{4}{3}\right)-\frac{\pi}{50}$$

Then:

$$\Omega = \Omega_1+\Omega_2+\Omega_3 = \frac{2(2\sqrt{3}-1)}{55}-\frac{4}{25}\ln\left(\frac{6+\sqrt{3}}{6}\right)+\frac{2}{25}\ln\left(\frac{4}{3}\right)-\frac{\pi}{50} =$$
$$\frac{4\sqrt{3}-2}{55}+\frac{4}{25}\ln\left(\frac{4}{2\sqrt{3}+1}\right)-\frac{\pi}{50} =$$
$$\frac{4\sqrt{3}-2}{55}+\frac{8}{25}\ln(2)-\frac{4}{25}\ln(2\sqrt{3}+1)-\frac{\pi}{50}$$

2849. Find:

$$\Omega=\int_0^\infty \frac{dx}{x\sqrt{x}+x+\sqrt{x}+1}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\Omega=\int_0^\infty \frac{dx}{x\sqrt{x}+x+\sqrt{x}+1}=\int_0^\infty \frac{dx}{(\sqrt{x}+1)(x+1)}\overset{(x=y^2)}{=}$$

$$=\int_0^\infty \frac{2y\,dy}{(y+1)(y^2+1)}=\int_0^\infty\left(\frac{-1}{y+1}+\frac{y}{y^2+1}+\frac{1}{y^2+1}\right)dy=$$

$$=\left(-\ln(y+1)+\frac{1}{2}\ln(y^2+1)\right)\Big|_0^\infty+\arctan y|_0^\infty=$$

$$=\left(\ln\frac{\sqrt{y^2+1}}{y+1}\right)\Bigg|_0^\infty+\arctan\infty-\arctan 0=\ln 1-\ln 1+\frac{\pi}{2}-0=\frac{\pi}{2}$$

2850. Find a closed form:

$$\Omega=\int_0^\infty \frac{x\ln x}{\sinh x}dx$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\overset{\sinh x\to\frac{e^x-e^{-x}}{2}}{\Rightarrow}\int_0^\infty x\ln x\left(2\sum_{r=0}^\infty e^{-(2r+1)x}\right)dx=2\sum_{r=0}^\infty\int_0^\infty x\ln x\,e^{-(2r+1)x}\,dx$$

$$I(s)=\int_0^\infty x^s e^{-ax}\,dx=\frac{\Gamma(s+1)}{a^{s+1}}$$

$$I'(s)=\int_0^\infty x^s\ln x\,e^{-ax}\,dx=\frac{(a^{s+1}\Gamma'(s+1)-\Gamma(s+1)a^{s+1}\ln a)}{a^{2s+2}}$$

$$I'(1)=\frac{(a^2\Gamma'(2)-\Gamma(2)a^2\ln a)}{a^4}=\frac{(\Gamma'(2)-\Gamma(2)\ln a)}{a^2}$$

$$\Gamma'(2)=\psi(2)\Gamma(2)$$

$$I'(1)=\frac{\Gamma(2)(\psi(2)-\ln a)}{a^2}\ :(a=2r+1)$$

$$\int_0^{\infty} x\ln x\, e^{-(2r+1)x}\,dx = \frac{\left(1-\gamma-\ln\left((2r+1)\right)\right)}{(2r+1)^2}$$

$$\Omega = 2\sum_{r=0}^{\infty}\left(\frac{1-\gamma}{(2r+1)^2}-\frac{\ln\left((2r+1)\right)}{(2r+1)^2}\right) = 2\frac{\pi^2}{8}(1-\gamma)-2\sum_{r=0}^{\infty}\frac{\ln\left((2r+1)\right)}{(2r+1)^2}$$

$$= \frac{\pi^2(1-\gamma)}{4}-2\sum_{r=0}^{\infty}\frac{\ln\left((2r+1)\right)}{(2r+1)^2}$$

$$\textit{Dirichlet Lambda Function}: \ \lambda(s) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}$$

$$\lambda'(s) = -\sum_{n=0}^{\infty}\frac{\ln\left((2n+1)\right)}{(2n+1)^s}$$

$$\lambda(s) = (1-2^{-s})\zeta(s)$$

$$\lambda'(s) = 2^{-s}\ln 2\,\zeta(s)+(1-2^{-s})\zeta'(s) \overset{\lambda'(2)}{\Longrightarrow} 2^{-2}\ln 2\,\zeta(2)+(1-2^{-2})\zeta'(2)$$

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \Rightarrow \ln\zeta(s)$$

$$= s\ln 2+(s-1)\ln\pi+\ln\left(\sin\left(\frac{\pi s}{2}\right)\right)+\ln\Gamma(1-s)+\ln\zeta(1-s)$$

$$\ln\zeta(s) = s\ln 2+(s-1)\ln\pi+\ln\left(\sin\left(\frac{\pi s}{2}\right)\right)+\ln\Gamma(1-s)+\ln\zeta(1-s) \Rightarrow \frac{\zeta'(s)}{\zeta(s)}$$

$$= \ln 2+\ln\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\psi(1-s)-\frac{\zeta'(1-s)}{\zeta(1-s)}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \ln 2\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\psi(1-s)-\frac{\zeta'(1-s)}{\zeta(1-s)}$$

$$\textit{Evaluating limit at } s\longrightarrow 2: \textit{let } s = 2+\epsilon$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)} = \ln 2\pi+\frac{\pi}{2}\cot\left(\pi+\frac{\pi\epsilon}{2}\right)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)} \Rightarrow \frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}$$

$$= \ln 2\pi+\frac{\pi}{2}\left(\frac{2}{\pi\epsilon}-\frac{\pi\epsilon}{6}+O(\epsilon^3)\right)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)} = \ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\psi(-1-\epsilon) = \psi(-\epsilon)+\frac{1}{1+\epsilon}\ , \ \psi(-\epsilon) = \psi(1-\epsilon)+\frac{1}{\epsilon}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)} = \ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\left(\psi(1-\epsilon)+\frac{1}{\epsilon}+\frac{1}{1+\epsilon}\right)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)} = \ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\left(-\gamma+\frac{1}{\epsilon}+1-\epsilon+O(\epsilon^2)\right)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\Rightarrow \frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)} = \ln 2\pi-\frac{\pi^2\epsilon}{12}+\gamma-1-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2)}{\zeta(2)}=\ln 2\pi+\gamma-1-\frac{\zeta'(-1)}{\zeta(-1)}$$

$$=\ln 2\pi+\gamma-1-\frac{\frac{1}{12}-\ln A}{\left(-\frac{1}{12}\right)}\overset{A:\ Glaisher-Kinkelin\ constant}{\Longrightarrow}\ln 2\pi+\gamma-1-(-1+12\ln A)$$

$$\zeta'(2)=\zeta(2)(\ln 2\pi+\gamma-12\ln A)=\frac{\pi^2}{6}(\gamma+\ln 2\pi-12\ln A)$$

$$\lambda'(2)=2^{-2}\ln 2\,\zeta(2)+(1-2^{-2})\zeta'(2)=\frac{\pi^2\ln 2}{24}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{8}$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+2\left(-\sum_{r=0}^{\infty}\frac{\ln\left((2r+1)\right)}{(2r+1)^2}\right)=\frac{\pi^2(1-\gamma)}{4}+2\lambda'(2)$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+2\left(\frac{\pi^2\ln 2}{24}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{8}\right)$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+\frac{\pi^2\ln 2}{12}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{4}$$

$$\boxed{\Omega=\pi^2\left(\frac{3+4\ln 2+3\ln\pi-36\ln A}{12}\right)}$$

2851. ***Find*** **:**

$$\Delta=\int_0^1\int_0^1\frac{x^2+y^2+2}{\sqrt{x^2y^2+x^2+y^2+1}}dxdy$$

Proposed by Samuel Ajibade-Nigeria

Solution by Shirvan Tahirov-Azerbaijan

$$\Delta=\int_0^1\int_0^1\frac{x^2+y^2+2}{\sqrt{x^2y^2+x^2+y^2+1}}dxdy=\int_0^1\int_0^1\frac{x^2+y^2+1+1}{\sqrt{(x^2+1)(y^2+1)}}dxdy=$$

$$=\int_0^1\int_0^1\frac{(x^2+1)+(y^2+1)}{\sqrt{(x^2+1)(y^2+1)}}dxdy=$$

$$=\int_0^1\int_0^1\frac{(x^2+1)}{\sqrt{(x^2+1)(y^2+1)}}dxdy+\int_0^1\int_0^1\frac{(y^2+1)}{\sqrt{(x^2+1)(y^2+1)}}dxdy=$$

$$=\left(\int_0^1\left(\sqrt{(x^2+1)}\right)dx\times\int_0^1\frac{1}{\sqrt{(y^2+1)}}dy\right)+\left(\int_0^1\left(\sqrt{(y^2+1)}\right)dy\times\int_0^1\frac{1}{\sqrt{(x^2+1)}}dx\right)$$

$$=$$

$$=\left(\left(\frac{1}{2}\left|x\sqrt{x^2+1}+\ln\left(x+\sqrt{x^2+1}\right)\right|_0^1\right)\times\left|\ln\left(y+\sqrt{y^2+1}\right)\right|_0^1\right)+$$

$$+\left(\left(\frac{1}{2}\left|y\sqrt{y^2+1}+\ln\left(y+\sqrt{y^2+1}\right)\right|_0^1\right)\times\left|\ln\left(x+\sqrt{x^2+1}\right)\right|_0^1\right)=$$

$$=\left(\frac{1}{2}(\sqrt{2}+\ln(1+\sqrt{2}))\times(\ln(1+\sqrt{2})-\ln(1))\right)+\left(\frac{1}{2}(\sqrt{2}+\ln(1+\sqrt{2}))\times(\ln(1+\sqrt{2})-\ln(1))\right)=$$

$$=2\left(\frac{1}{2}(\sqrt{2}+\ln(1+\sqrt{2}))\times\ln(1+\sqrt{2})\right)$$

$$=(\sqrt{2}+1-1+(\ln(1+\sqrt{2})\times\ln(1+\sqrt{2})))=$$

$$=(\sigma+\ln(\sigma)-1)\times ln^2(\sigma)=\ln(\sigma^\sigma)+ln^2(\sigma)-\ln(\sigma)$$

$\therefore$ *where σ is the silver ratio*

2852. ***Find*** **:**

$$\Delta=\int_0^\infty\frac{e^{-\pi x\phi}\sinh\left(\frac{\pi x}{\phi}\right)}{\sinh(\pi x)}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Shirvan Tahirov-Azerbaijan

$$\Delta=\int_0^\infty\frac{e^{-\pi x\phi}\sinh\left(\frac{\pi x}{\phi}\right)}{\sinh(\pi x)}dx=\int_0^\infty\frac{\frac{1}{2}\left(e^{-\pi x\phi}e^{\frac{\pi x}{\phi}}-e^{-\pi x\phi}e^{-\frac{\pi x}{\phi}}\right)}{\frac{1}{2}\left(\frac{1-e^{-2\pi x}}{e^{-\pi x}}\right)}dx=$$

$$=\int_0^\infty\frac{e^{-2\pi x}-e^{-2\pi\phi x}}{1-e^{-2\pi x}}dx=\sum_{j\ge0}\int_0^\infty\left(e^{-2\pi(j+1)x}-e^{-2\pi(j+\phi)x}\right)dx=$$

$$=\sum_{j\ge0}\left(\int_0^\infty e^{-2\pi(j+1)x}dx-\int_0^\infty e^{-2\pi(j+\phi)x}dx\right)=$$

$$=\sum_{j\ge0}\left(\frac{1}{2\pi(j+1)}-\frac{1}{2\pi(j+\phi)}\right)=\frac{1}{2\pi}\sum_{j\ge0}\left(\frac{1}{j+1}-\frac{1}{j+\phi}\right)=\frac{H_{\frac{1}{\phi}}}{2\pi}$$

Note **:**

$$\therefore\sinh(x)=\frac{e^x-e^{-x}}{2},\qquad\therefore\int_0^\infty e^{-ax}\,dx=\frac{1}{a},\qquad\therefore\frac{1}{1-e^{-2\pi x}}=\sum_{j\ge0}e^{-2\pi jx}$$

2853. ***Find*** **:**

$$\Omega=\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^n\frac{(-1)^k.2^{n-k}}{k+1}.\binom{n}{k}$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^{n}\frac{(-1)^k.2^{n-k}}{k+1}.\binom{n}{k}=\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^{n}\frac{(-1)^k.2^n.2^{-k}}{k+1}.\binom{n}{k}=$$

$$=\lim_{n\to\infty}\frac{n+2}{2^n}.2^n\sum_{k=0}^{n}\frac{(-1)^k}{2^k(k+1)}.\binom{n}{k}=\lim_{n\to\infty}(n+2)\sum_{k=0}^{n}(-\frac{1}{2})^k\binom{n}{k}\int_0^1 x^k dx=$$

$$=\lim_{n\to\infty}(n+2)\int_0^1\sum_{k=0}^{n}(-\frac{x}{2})^k\binom{n}{k}dx=\lim_{n\to\infty}(n+2)\int_0^1\left(1-\frac{x}{2}\right)^n dx\overset{\begin{cases}t=1-\frac{x}{2}\\ dx=-2dt\end{cases}}{\cong}$$

$$=\lim_{n\to\infty}(n+2)\left(\int_{\frac{1}{2}}^{1}2t^n dt\right)=\lim_{n\to\infty}(n+2)\left(\frac{2}{n+1}.t^{n+1}\Big|_{\frac{1}{2}}^{1}\right)=$$

$$=\lim_{n\to\infty}(n+2)\left(\frac{2-2^{-n}}{n+1}\right)=\lim_{n\to\infty}(\frac{n+2}{n+1}.(2-2^{-n}))=\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$$

2854. *Evaluate*:

$$S=\sum_{k=0}^{\infty}\frac{1}{3^{k+1}}\sum_{n=0}^{k}\frac{1}{4^n}\binom{2n}{n}\binom{k}{n}\frac{1}{(2n+1)(4n+1)}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,dx=\frac{\pi}{2}\frac{\binom{2n}{n}}{4^n(2n+1)}$$

$$\frac{\binom{2n}{n}}{4^n(2n+1)}=\frac{2}{\pi}\int_0^1 x^{2n}\sqrt{1-x^2}\,dx$$

$$3S=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{3^k}\sum_{n=0}^{k}\binom{k}{n}\left(\int_0^1 x^{2n}\sqrt{1-x^2}\,dx\right)\left(\int_0^1 y^{4n}\,dy\right)$$

$$\frac{3\pi}{2}S=\sum_{k=0}^{\infty}\frac{1}{3^k}\iint_0^1\sqrt{1-x^2}\left(\underbrace{\sum_{n=0}^{k}\binom{k}{n}(x^2y^4)^n}_{(1+x^2y^4)^k}\right)dx\,dy$$

$$\frac{3\pi}{2}S=\iint_0^1\sqrt{1-x^2}\left(\sum_{k=0}^{\infty}\left(\frac{1+x^2y^4}{3}\right)^k\right)dx\,dy$$

$$\frac{3\pi}{2}S=\iint_0^1\sqrt{1-x^2}\left(\frac{3}{2-x^2y^4}\right)dx\,dy$$

$$\frac{\pi}{2}S=\iint_0^1\frac{\sqrt{1-x^2}}{2-x^2y^4}dx\,dy\overset{x\to\sin x}{\Longrightarrow}\int_0^1\left(\int_0^{\frac{\pi}{2}}\frac{(\cos x)^2\,dx}{2-y^4(\sin x)^2}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\left(\int_0^{\frac{\pi}{2}}\frac{dx}{2+(2-y^4)(\tan x)^2}\right)dy\overset{\substack{\tan x\to x\\ k=2-y^4}}{\Longrightarrow}\int_0^1\left(\int_0^{\infty}\frac{dx}{(1+x^2)(2+kx^2)}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{k}\left(\int_0^{\infty}\frac{dx}{(1+x^2)\left(\frac{2}{k}+x^2\right)}\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{k\left(\frac{2}{k}-1\right)}\left(\int_0^{\infty}\left(\frac{1}{(1+x^2)}-\frac{1}{\left(\frac{2}{k}+x^2\right)}\right)dx\right)dy$$

$$\frac{\pi}{2}S=\int_0^1\frac{1}{y^4}\left(\tan^{-1}\infty-\sqrt{\frac{k}{2}}\tan^{-1}\infty\right)dy,\qquad \sqrt{2}S=\int_0^1\frac{\left(\sqrt{2}-\sqrt{2-y^4}\right)}{y^4}dy$$

$$\overset{By\ Parts}{\Longrightarrow}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2}{3}\int_0^1\frac{dy}{\sqrt{2-y^4}}\overset{y^4\to 2y}{\Longrightarrow}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\int_0^{\frac{1}{2}}\frac{y^{-\frac{3}{4}}dy}{\sqrt{1-y}}$$

$$\frac{\pi}{2}S=\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\int_0^{\frac{1}{2}}y^{-\frac{3}{4}}(1-y)^{-\frac{1}{2}}dy$$

$(\textit{Incomplete beta function})$

$$\beta(k;x,y)=\int_0^k t^{x-1}(1-t)^{y-1}\,dt$$

$$\frac{\pi}{2}S=\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{5}{4}}}{3}\beta\left(\frac{1}{2};\frac{1}{4},\frac{1}{2}\right)$$

$$S=\frac{2}{\pi}\left(\frac{1-\sqrt{2}}{3}\right)+\frac{2^{-\frac{1}{4}}}{3\pi}\beta\left(\frac{1}{2};\frac{1}{4},\frac{1}{2}\right)$$

2855. ***Find :***

$$\Omega=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\frac{(-1)^{k_1+k_2+k_3+\dots+k_n}}{k_1k_2k_3\dots k_n(k_1+k_2+k_3+\dots+k_n)}$$

Proposed by Omkumar Rao-India

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\frac{(-1)^{k_1+k_2+k_3+\dots+k_n}}{k_1k_2k_3\dots k_n(k_1+k_2+k_3+\dots+k_n)}=$$

$$=\int_0^1\left(\sum_{k_n=1}^{\infty}\frac{(-1)^{k_1+k_2+k_3+\dots+k_n}x^{k_1+k_2+k_3+\dots+k_n-1}}{k_1k_2k_3\dots k_n}\right)dx=$$

$$=\int_0^1\frac{1}{x}\left(\sum_{k_n=1}^{\infty}\frac{(-1)^{k_1+k_2+k_3+\dots+k_n}x^{k_1+k_2+k_3+\dots+k_n}}{k_1k_2k_3\dots k_n}\right)dx$$

$$=\int_0^1\frac{1}{x}\left(\prod_{m=1}^{n}\left(\sum_{k_m=1}^{\infty}\frac{(-1)^{k_m}x^{k_m}}{k_m}\right)\right)dx=$$

$$\int_0^1\frac{1}{x}[-\ln(1+x)]^n dx=\int_0^1\frac{(-1)^n ln^n(1+x)}{x}dx\underbrace{\overbrace{=}^{y=\ln(1+x),x=e^y-1}}_{dx=e^ydy}(-1)^n\int_0^{\ln(2)}\frac{y^ne^y}{e^y-1}dy$$

$$\therefore\int_0^1\frac{(-1)^n ln^n(1+x)}{x}dx=(-1)^n\int_0^{\ln(2)}\frac{y^ne^y}{e^y-1}dy$$

$$n=1\Rightarrow\ (-1)^1\int_0^1\frac{ln^1(1+x)}{x}dx=-\int_0^1\frac{\ln(1+x)}{x}dx$$

$$\therefore\int_0^1\frac{\ln(1+x)}{x}dx=\int_0^1\sum_{j=1}^{\infty}\frac{(-1)^{j-1}x^j}{j}\frac{1}{x}dx=$$

$$\sum_{j=1}^{\infty}\int_0^1\left(\frac{(-1)^{j-1}x^{j-1}}{j}\right)dx=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\int_0^1 x^{j-1}\,dx=$$

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^2}=\frac{\pi^2}{12}.\ -\int_0^1\frac{\ln(1+x)}{x}dx=-\frac{\pi^2}{12}$$

2856. Find:

$$\Omega=\lim_{n\to\infty}\left[\frac{1}{n}\int_0^1\ln\left(1+e^{n\sqrt{x}}\right)dx\right]$$

Proposed by Vasile Mircea Popa-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 \ln\left(1+e^{n\sqrt{x}}\right)dx\right]=\lim_{n\to\infty}\left[\frac{1}{n}\int_0^1\left(n\sqrt{x}+\ln\left(1+e^{-n\sqrt{x}}\right)\right)dx\right]=$$

$$=\lim_{n\to\infty}\left[\frac{1}{n}\int_0^1 n\sqrt{x}dx+\frac{1}{n}\int_0^1\ln\left(1+e^{-n\sqrt{x}}\right)dx\right]=\int_0^1\sqrt{x}dx+\lim_{n\to\infty}\frac{1}{n}\int_0^1\ln\left(1+e^{-n\sqrt{x}}\right)dx$$

$$=\frac{2}{3}$$

$$\therefore 0\le e^{-n\sqrt{x}}\le 1$$

$$\therefore 0\le \ln\left(1+e^{-n\sqrt{x}}\right)\le \ln(2)$$

$$\therefore 0\le \lim_{n\to\infty}\frac{1}{n}\int_0^1\ln\left(1+e^{-n\sqrt{x}}\right)dx\le\lim_{n\to\infty}\frac{\ln(2)}{n}=0$$

2857. ***Prove that:***

$$\int_0^\infty\frac{x^2}{(x^2+4)^3}dx+\int_0^\infty\frac{y^2}{4(y^2+25)^3}dy+\int_0^\infty\frac{z^2}{9(z^2+100)^3}dz+\cdots$$

$$=\frac{\pi^3}{96}+\frac{3\pi}{32}-\frac{15\pi^2}{256}\coth\pi-\frac{7\pi^3}{256}(\operatorname{csch}\pi)^2-\frac{\pi^4}{128}\coth\pi\,(\operatorname{csch}\pi)^2$$

Proposed by Omkumar Rao-India

Solution by proposer

$$K=\int_0^\infty\frac{x^2}{(x^2+4)^3}dx+\int_0^\infty\frac{y^2}{4(y^2+25)^3}dy+\int_0^\infty\frac{z^2}{9(z^2+100)^3}dz+\cdots=$$

$$=\sum_{n=1}^\infty\frac{1}{n^2}\int_0^\infty\frac{x^2}{(x^2+(n^2+1)^2)^3}dx=\frac{\pi}{16}\sum_{n=1}^\infty\frac{1}{n^2(n^2+1)^3}$$

$$K=\frac{\pi}{16}\left[\sum_{n=1}^\infty\frac{1}{n^2}-\sum_{n=1}^\infty\frac{1}{n^2+1}-\sum_{n=1}^\infty\frac{1}{(n^2+1)^2}-\sum_{n=1}^\infty\frac{1}{(n^2+1)^3}\right]$$

$$K=\frac{\pi}{16}\left[\frac{\pi^2}{6}-\left(\frac{\pi\coth(\pi)-1}{2}\right)-\left(\frac{\pi\coth(\pi)+\pi^2(\operatorname{csch}(\pi))^2-2}{4}\right)\right.$$
$$\left.-\left(\frac{3\pi\coth(\pi)+3\pi^2(\operatorname{csch}(\pi))^2-8+2\pi^3(\operatorname{csch}(\pi))^2\coth(\pi)}{16}\right)\right]$$

$$\int_0^\infty \frac{x^2}{(x^2+4)^3}dx+\int_0^\infty \frac{y^2}{4(y^2+25)^3}dy+\int_0^\infty \frac{z^2}{9(z^2+100)^3}dz+\cdots$$
$$=\frac{\pi^3}{96}+\frac{3\pi}{32}-\frac{15\pi^2}{256}\coth\pi-\frac{7\pi^3}{256}(\operatorname{csch}\pi)^2-\frac{\pi^4}{128}\coth\pi\,(\operatorname{csch}\pi)^2$$

2858. Find:

$$\Omega=\prod_{n=0}^{\infty}\left(\frac{1024n^2+1024n+60}{1024n^2+1024n+31}\right)\left(\prod_{n=2}^{\infty}\left(\frac{n^{14}-2n^8+n^2-n^{12}+2n^6-1}{n^{14}}\right)\right)$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\Omega=\prod_{n=0}^{\infty}\frac{\left(n+\frac{1}{16}\right)\left(n+\frac{15}{16}\right)}{\left(n+\frac{1}{32}\right)\left(n+\frac{31}{32}\right)}\left(\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^3\left(1+\frac{1}{n^2}+\frac{1}{n^4}\right)^2\right)$$

$$\Omega=\left(\frac{\Gamma\left(\frac{1}{32}\right)\Gamma\left(\frac{31}{32}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{15}{16}\right)}\right)\left(\frac{1}{2}\right)^3\left(\prod_{n=2}^{\infty}\left(1-\frac{\left(e^{\frac{i\pi}{3}}\right)^2}{n^2}\right)\left(1-\frac{\left(e^{\frac{-i\pi}{3}}\right)^2}{n^2}\right)\right)^2$$

$$\Omega=\frac{1}{4}\cos\left(\frac{\pi}{32}\right)\left(\left(\frac{\sin\left(\pi e^{\frac{i\pi}{3}}\right)}{\pi e^{\frac{i\pi}{3}}}\right)\cdot\left(\frac{\sin\left(\pi e^{\frac{-i\pi}{3}}\right)}{\pi e^{\frac{-i\pi}{3}}}\right)\right)^2=\frac{1}{4}\cos\left(\frac{\pi}{32}\right)\frac{\left(\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\right)^4}{9\pi^4}$$

$$\prod_{n=0}^{\infty}\left(\frac{1024n^2+1024n+60}{1024n^2+1024n+31}\right)\left(\prod_{n=2}^{\infty}\left(\frac{n^{14}-2n^8+n^2-n^{12}+2n^6-1}{n^{14}}\right)\right)$$
$$=\frac{\left(\cosh\left(\frac{\sqrt{3}\pi}{2}\right)\right)^4\cos\left(\frac{\pi}{32}\right)}{36\pi^4}$$

2859.

Find :

$$\Omega=\int_0^\infty \frac{1}{x+1+\sqrt[3]{x^2}+\sqrt[3]{x}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Odeyemi Gideon-Nigeria

$$\sqrt[3]{x} \to x$$

$$\Omega = 3\int_0^\infty \frac{x^2}{x^3+x+x^2+1}dx = 3\int_0^\infty \frac{x^2}{(x+1)(x^2+1)}dx =$$

$$= \frac{3}{2}\left(\int_0^\infty \left(\frac{1}{x^2+1} - \frac{2x}{2x^2+2} - \frac{1}{1+x}\right)dx\right) =$$

$$= \frac{3}{2}\left(\arctan(x) - \frac{1}{2}\ln(x^2+1) - \ln(1+x)\right)\Bigg|_0^\infty =$$

$$= \frac{3}{2}\left(\arctan(x) - ln\frac{\sqrt{x^2+1}}{x+1}\right)\Bigg|_0^\infty = \frac{3}{2}\left(\frac{\pi}{2} - 0\right) = \frac{3\pi}{4}$$

2860. Find:

$$S = \sum_{n=1}^\infty \frac{(-1)^{n-1}H_n^{(2)}(n^2+1)}{n(n+1)(n+2)(n+3)}$$

$$where\ H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}\ represents\ the\ second-order\ harmonic\ numbers$$

Proposed by Omkumar Rao-India

Solution by Shirvan Tahirov-Azerbaijan

$$S = \sum_{n=1}^\infty (-1)^{n-1}H_n^{(2)}\left(\frac{1}{6n} - \frac{1}{n+1} + \frac{5}{2(n+2)} - \frac{5}{3(n+3)}\right) =$$

$$= \sum_{n=1}^\infty (-1)^{n-1}H_n^{(2)}\int_0^1 \left(\frac{1}{6}x^{n-1} - x^n + \frac{5}{2}x^{n+1} - \frac{5}{3}x^{n+2}\right)dx =$$

$$= \sum_{n=1}^\infty (-1)^{n-1}H_n^{(2)}x^{n-1}\int_0^1 \left(\frac{1}{6} - x + \frac{5}{2}x^2 - \frac{5}{3}x^3\right)dx =$$

$$= \int_0^1 \left(\frac{1}{6} - x + \frac{5}{2}x^2 - \frac{5}{3}x^3\right)\left(-\frac{Li_2(-x)}{x(1+x)}\right)dx =$$

$$= -\left(\underbrace{\int_0^1 \frac{Li_2(-x)}{6x}dx}_{-\frac{1}{8}\zeta(3)} - \underbrace{\int_0^1 \frac{16\,Li_2(-x)}{3(1+x)}dx}_{\frac{4}{3}\zeta(3)-\frac{4}{9}\pi^2\ln 2} + \underbrace{\int_0^1 \frac{25}{6}Li_2(-x)\,dx}_{\frac{25}{3}\ln 2-\frac{25}{72}\pi^2-\frac{25}{6}} - \underbrace{\int_0^1 \frac{5xLi_2(-x)}{3}dx}_{\frac{5}{24}-\frac{5}{72}\pi^2}\right)$$

$$\int_0^1 \frac{Li_2(-x)}{6x}dx = \frac{1}{6}\int_0^1 \sum_{n=1}^\infty \frac{(-1)^n x^{n-1}}{n^2} = \frac{1}{6}\sum_{n=1}^\infty \frac{(-1)^n}{n^3} = -\frac{1}{8}\zeta(3)$$

$$\int_0^1 \frac{16\,Li_2(-x)}{3(1+x)}dx \overset{By\ parts}{\cong} \frac{16}{3}\left(-\frac{\pi^2}{12}\ln 2+\int_0^1 \frac{(\ln(1+x))^2}{x}dx\right)=\frac{4}{3}\zeta(3)-\frac{4}{9}\pi^2\ln 2$$

$$\int_0^1 \frac{25}{6}Li_2(-x)\,dx \overset{By\ parts}{\cong} \frac{25}{6}\left(2\ln 2+\int_0^1 \ln(1+x)\,dx\right)=\frac{25}{3}\ln 2-\frac{25}{72}\pi^2-\frac{25}{6}$$

$$\int_0^1 \frac{5xLi_2(-x)}{3}dx \overset{By\ parts}{\cong} \frac{5}{3}\left(-\frac{\pi^2}{24}+\frac{1}{2}\int_0^1 x\ln(1+x)\,dx\right)=\frac{5}{24}-\frac{5}{72}\pi^2$$

$$=-\left(-\frac{1}{8}\zeta(3)-\left(\frac{4}{3}\zeta(3)-\frac{4}{9}\pi^2\ln 2\right)+\left(\frac{25}{3}\ln 2-\frac{25}{72}\pi^2-\frac{25}{6}\right)-\left(\frac{5}{24}-\frac{5}{72}\pi^2\right)\right)$$

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n^{(2)}(n^2+1)}{n(n+1)(n+2)(n+3)}=\frac{35}{24}\zeta(3)-\frac{4}{9}\pi^2\ln 2-\frac{25}{3}\ln 2+\frac{5}{18}\pi^2+\frac{35}{8}}$$

2861.

Find :

$$\Delta=\iiint_0^1 (x+2y+3z)e^{x+y+z}dxdydz$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Odeyemi Gideon-Nigeria

$$\iiint_0^1 (x+2y+3z)e^{x+y+z}dxdydz=$$

$$=\iiint_0^1 xe^{x+y+z}dxdydz+\iiint_0^1 2ye^{x+y+z}dxdydz+\iiint_0^1 3ze^{x+y+z}dxdydz=$$

$$=(1+2+3)\iiint_0^1 xe^{x+y+z}dxdydz\ (symmetry)=$$

$$=6\iiint_0^1 xe^{x+y+z}dxdydz=6\int_0^1 xe^x dx.\left(\int_0^1 e^y dy\right)^2=6(e-1)^2$$

2862. ***Evaluate***:

$$S=\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k} \quad ; a,b\in R^{+},0<a<b.$$

Proposed by Omkumar Rao-India

Solution by proposer

$$0<a<b: S=\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k}$$

$$0<x=\frac{a}{b}<1:\ =\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k}=\sum_{k=1}^{\infty}\frac{x^{2k}}{k}\left(\sum_{m=1}^{\infty}\frac{1}{m^{2k}}\right)$$

$$=\sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{m}\right)^{2k}\right)=-\sum_{m=1}^{\infty}\ln\left(1-\left(\frac{x}{m}\right)^{2}\right)=-\ln\left(\prod_{m=1}^{\infty}\left(1-\left(\frac{x}{m}\right)^{2}\right)\right)$$

$$\frac{\sin\pi z}{\pi z}=\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^{2}\right)\Rightarrow S=\ln\left(\frac{\pi a}{b\sin\left(\frac{\pi a}{b}\right)}\right)$$

$$\boxed{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k}=\ln\left(\frac{\pi a}{b\sin\left(\frac{\pi a}{b}\right)}\right)}$$

2863. ***Find*** :

$$\Delta=\sum_{n\in Z\setminus\{0\}}\frac{1}{n^2}\left(\frac{1}{(4n+1)^2}+\frac{1}{(4n-1)^2}\right)$$

Proposed by Odeyemi Gideon-Nigeria

Solution by Nawar Alasadi-Iraq

$$f(n)=\frac{1}{n^2(4n+1)^2}-\frac{1}{n^2(4n-1)^2}$$

$$f(n)=\frac{2}{n^2}+32\left(\frac{1}{4n+1}-\frac{1}{4n-1}\right)+16\left(\frac{1}{(4n+1)^2}+\frac{1}{(4n-1)^2}\right)$$

$$\sum_{n\in Z\setminus\{0\}}f(n)=2\sum_{n=1}^{\infty}f(n),\quad \sum_{n=1}^{\infty}\frac{2}{n^2}=\frac{\pi^2}{3}$$

$$32\sum_{n=1}^{\infty}\left(\frac{1}{4n+1}-\frac{1}{4n-1}\right)=32\left(1-\frac{\pi}{4}\right)=8\pi-32$$

$$16\sum_{n=1}^{\infty}\left(\frac{1}{(4n+1)^2}+\frac{1}{(4n-1)^2}\right)=16\left(\frac{\pi^2}{8}-1\right)=2\pi^2-16$$

$$2\sum_{n=1}^{\infty} f(n) = 2\left(\frac{\pi^2}{3} + 8\pi - 32 + 2\pi^2 - 16\right) = \frac{14\pi^2}{3} + 16\pi - 96$$

2864. ***Find*** :

$$\Delta = \int_0^1 x\ln\big(x\sin(x)\big)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Quadri Faruk Temitope-Nigeria

$$\therefore\ Recall\ \ that : \ln(AB) = \ln(A) + \ln(B)$$

$$\Delta = \int_0^1 x\ln\big(x\sin(x)\big)dx = \int_0^1 x\ln(x)dx + \int_0^1 x\ln\big(\sin(x)\big)dx =$$

$$\therefore\ Recall\ \ that : \ln(\sin(x)) = -\ln(2) - \sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}$$

$$\Delta = \underbrace{\int_0^1 x\ln(x)dx}_{I.B.P} - \ln(2)\int_0^1 xdx - \sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\int_0^1 x\cos(2nx)\,dx}_{I.B.P} =$$

$$= \frac{x^2}{2}\ln(2)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{x}dx - \frac{x^2}{2}\ln(2)\Big|_0^1 -$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\sin(2n)}{n} + \frac{\cos(2n)}{4n^2} - \frac{1}{4n^2}\right] = -\frac{1}{2}\int_0^1 x\,dx - \frac{\ln(2)}{2} -$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} = -\frac{1}{4} - \frac{\ln(2)}{2} -$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} =$$

$$-\frac{1}{4} - \frac{\ln(2)}{2} - \frac{1}{2}Cl_2(2) - \frac{1}{4}Cl_3(2) + \frac{1}{4}\zeta(3)$$

$$\Delta = \int_0^1 x\ln\big(x\sin(x)\big)dx$$

$$\Delta = -\frac{1}{4} - \frac{\ln(2)}{2} - \frac{1}{2}Cl_2(2) - \frac{1}{4}Cl_3(2) + \frac{1}{4}\zeta(3)$$

Note :

$$\therefore \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2} = Cl_2(\theta) \qquad \therefore \sum_{n=1}^{\infty}\frac{\cos(\theta n)}{n^3} = Cl_3(\theta)$$

$$Cl_n(\theta) - Clausen\ \ function$$

Solution 2 by Yang Silva Cartolin-Peru

$$We\ \ known\ that:$$

$$\sin(x)=\frac{e^{ix}-e^{-ix}}{2i}\to 2ie^{ix}\sin(x)=1-e^{2ix}$$

$$-2ixe^{ix}\sin(x)=x(1-e^{2ix})\to \log(2)+i\left(x-\frac{\pi}{2}\right)+\log\big(xsin(x)\big)=$$
$$\log(x)+\log(1-e^{2ix})$$

$$\Re\{\log\big(xsin(x)\big)\}=\log(x)+\Re\{\log(1-e^{2ix})\}-\log(2)\int_0^1 xdx$$

$$There\ fore: I=\int_0^1 xlog(x)\,dx\ +\Re\left\{\int_0^1 \mathrm{x}\log(1-e^{2ix})\right\}dx-\log(2)\int_0^1 xdx$$

$$I=\xi_1+\xi_2-\frac{1}{2}\log(2)--(1)$$

$$\xi_1=\int_0^1 xlog(x)\,dx\ \ I.B.P:\begin{cases}u=\log(x), & du=\dfrac{dx}{x}\\ dv=xdx, & v=\dfrac{x^2}{2}\end{cases}$$

$$\xi_1=\left[\log(x)\frac{x^2}{2}\right]_0^1-\frac{1}{2}\int_0^1 xdx=-\frac{1}{4}$$

$$\xi_2=\Re\left\{\int_0^1 \mathrm{x}\log(1-e^{2ix})\right\}=-\Re\left\{\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 xe^{2ixn}dx\right\}$$

$$\xi_2=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 xcos(2nx)dx=-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sin(2n)}{2n}-\frac{\cos(2n)-1}{4n^2}\right)$$

$$\xi_2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3},$$

$$\xi_2=-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

$$There\ fore:$$

$$I=-\frac{1}{4}-\frac{\ln(2)}{2}-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

Solution 3 by Omkumar Rao-India

$$I=\underbrace{\int_0^1 x\ln(x)\,dx}_{-\frac{1}{4}}+\underbrace{\int_0^1 x\ln(\sin x)\,dx}_{\ln(\sin x)=-\ln 2-\sum_{m=1}^{\infty}\frac{\cos(2mx)}{m}}$$

$$I=-\frac{1}{4}+\int_0^1 x\left(-\ln 2-\sum_{m=1}^{\infty}\frac{\cos(2mx)}{m}\right)dx=-\frac{1}{4}-\frac{1}{2}\ln 2-\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x\cos(2mx)\,dx$$

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\sin(2m)}{2m}-\frac{1}{4m^2}+\frac{\cos(2m)}{4m^2}\right)$$

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\underbrace{\sum_{m=1}^{\infty}\frac{\sin(2m)}{m^2}}_{Im\left(Li_2(e^{2i})\right)/Cl_2(2)}+\frac{1}{4}\underbrace{\sum_{m=1}^{\infty}\frac{1}{m^3}}_{\zeta(3)}-\frac{1}{4}\underbrace{\sum_{m=1}^{\infty}\frac{\cos(2m)}{m^3}}_{\Re\left(Li_3(e^{2i})\right)/Cl_3(2)}$$

$$\left(Cl_2(2)\right)=Clausen\ function\ of\ order\ 2$$
$$\left(Cl_3(2)\right)=Clausen\ function\ of\ order\ 3$$

$$\boxed{I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\left(Im\left(Li_2(e^{2i})\right)\right)+\frac{1}{4}\zeta(3)-\frac{1}{4}\Re\left(Li_3(e^{2i})\right)}$$

Or

$$\boxed{I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\left(Cl_2(2)\right)+\frac{1}{4}\zeta(3)-\frac{1}{4}\left(Cl_3(2)\right)}$$

2865. ***Find*** :

$$\Delta=\int_0^{\infty}\frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$I=\int_0^{\infty}\frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx=\int_0^{\infty}\frac{e^{-2x}\left(x+x^{\frac{2}{3}}\right)}{1+e^{-2x}}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-2(n+1)x}\left(x+x^{\frac{2}{3}}\right)dx$$

$$I=-\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}e^{-2nx}\left(x+x^{\frac{2}{3}}\right)dx\,.Let: S(k)=\sum_{n=1}^{\infty}(-1)^n\int_0^{\infty}e^{-2nx}x^k dx$$

$$We\ \ do: 2nx=t\to dx=\frac{dt}{2n}\to S(k)=\frac{1}{2^{k+1}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{k+1}}\int_0^{\infty}t^k e^{-t}dt$$

$$S(k)=-\frac{\Gamma(k+1)}{2^{k+1}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{k+1}}=-\frac{\Gamma(k+1)}{2^{k+1}}\eta(k+1)=-\frac{\Gamma(k+1)}{2^{k+1}}(1-2^{-k})\zeta(k+1)$$

Therefore:

$$I=-\left(S(1)+S\left(\frac{2}{3}\right)\right)=\frac{\Gamma(2)}{2^3}\zeta(2)+\frac{\Gamma\left(\frac{5}{3}\right)}{2^{\frac{5}{3}}}(1-2^{-\frac{2}{3}})\zeta\left(\frac{5}{3}\right)$$

$$I=\frac{\pi^2}{48}+\frac{\Gamma\left(\frac{5}{3}\right)}{\sqrt[3]{2^5}}\zeta\left(\frac{5}{3}\right)-\frac{\Gamma\left(\frac{5}{3}\right)}{\sqrt[3]{2^7}}\zeta\left(\frac{5}{3}\right),\qquad I=\frac{\pi^2}{48}+\Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right)\left(\frac{2\sqrt[3]{2}-\sqrt[3]{4}}{8}\right)$$

Solution 2 by Samuel Ajibade-Nigeria

$$\Delta=\int_0^\infty\frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx=\int_0^\infty\frac{x+x^{\frac{2}{3}}}{1+e^{2x}}dx=\int_0^\infty\frac{x+x^{\frac{2}{3}}}{1-(-e^{-2x})}e^{-2x}dx$$

$$\Delta=\int_0^\infty\sum_{n=0}^\infty(-1)^n e^{-2x(n+1)}\left(x+x^{\frac{2}{3}}\right)dx\quad\{geometric\ series:\ |e^{-2x}|<1\}$$

$$\Delta=\int_0^\infty\sum_{k=1}^\infty(-1)^{k-1}e^{-2kx}\left(x+x^{\frac{2}{3}}\right)dx\quad\{n+1=k\}$$

$$\Delta=\sum_{k=1}^\infty(-1)^{k-1}\int_0^\infty\left(x+x^{\frac{2}{3}}\right)e^{-2kx}dx\quad\{Fubini-Tonelli\ Theorem\}$$

$$\Delta=\sum_{k=1}^\infty(-1)^{k-1}\left(\frac{1}{(2k)^2}+\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{(2k)^{\frac{2}{3}+1}}\right)\quad\{Laplace\ \ Transforms\}$$

$$\Delta=\frac{1}{4}\sum_{k=1}^\infty(-1)^{k-1}\frac{1}{k^2}+\frac{2^{-\frac{2}{3}}}{3}\Gamma\left(\frac{2}{3}\right)\sum_{k=1}^\infty(-1)^{k-1}\frac{1}{k^{\frac{5}{3}}}$$

$$\Delta=\frac{1}{4}\eta(2)+\frac{2^{-\frac{2}{3}}}{3}\frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)}\eta\left(\frac{5}{3}\right),\qquad\Delta=\frac{\pi^2}{48}+\Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right)\left(\frac{2\sqrt[3]{2}-\sqrt[3]{4}}{8}\right)$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$\Delta=\int_0^\infty\frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx=\sum_{n=1}^\infty(-1)^{n-1}\int_0^\infty(x+\sqrt[3]{x^2})e^{-2nx}dx=$$

$$\sum_{n=1}^\infty(-1)^{n-1}\int_0^\infty\left[\frac{y}{2n}+\left(\frac{y}{2n}\right)^{\frac{2}{3}}\right]e^{-y}\frac{dy}{2n}=-\sum_{n=1}^\infty\frac{(-1)^n}{2n}\int_0^\infty\left[\frac{y}{2n}+\left(\frac{y}{2n}\right)^{\frac{2}{3}}\right]e^{-y}dy=$$

$$-\sum_{n=1}^\infty\frac{(-1)^n}{(2n)^2}\int_0^\infty ye^{-y}dy-\sum_{n=1}^\infty\frac{(-1)^n}{(2n)^{\frac{5}{3}}}\int_0^\infty y^{\frac{2}{3}}e^{-y}dy=$$

$$-\sum_{n=1}^\infty\frac{(-1)^n}{(2n)^2}\Gamma(2)-\sum_{n=1}^\infty\frac{(-1)^n}{(2n)^{\frac{5}{3}}}\Gamma\left(\frac{5}{3}\right)=-\frac{1}{4}\left(-\frac{\pi^2}{12}\right)+\frac{1}{2^{\frac{5}{3}}}\Gamma\left(\frac{5}{3}\right)\eta\left(\frac{5}{3}\right)=$$

$$\frac{\pi^2}{48}+\frac{1}{2^{\frac{5}{3}}}\left(1-2^{1-\frac{5}{3}}\right)\zeta\left(\frac{5}{3}\right)=\frac{\pi^2}{48}+\frac{\left(2^{\frac{2}{3}}-1\right)}{2^{\frac{7}{3}}}\zeta\left(\frac{5}{3}\right)$$

$$\Delta=\frac{\pi^2}{48}+\Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right)\left(\frac{2\sqrt[3]{2}-\sqrt[3]{4}}{8}\right)$$

Solution 4 by Omkumar Rao-India

$$\int_0^\infty \frac{x^{k-1}}{e^x+1}dx=\Gamma(k)\eta(k)$$

$$I=\underbrace{\int_0^\infty \frac{x}{1+e^{2x}}dx}_{2x\to x}+\underbrace{\int_0^\infty \frac{x^{\frac{2}{3}}}{1+e^{2x}}dx}_{2x\to x}=\frac{1}{4}\underbrace{\int_0^\infty \frac{x}{1+e^x}dx}_{\Gamma(2)\eta(2)}+\frac{1}{2^{\frac{5}{3}}}\underbrace{\int_0^\infty \frac{x^{\frac{2}{3}}}{1+e^x}dx}_{\Gamma\left(\frac{5}{3}\right)\eta\left(\frac{5}{3}\right)}$$

$$I=\frac{1}{4}\left(\frac{\pi^2}{12}\right)+\frac{1}{2^{\frac{5}{3}}}\left(\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\eta\left(\frac{5}{3}\right)\right)$$

$$\boxed{\int_0^\infty \frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx=\frac{\pi^2}{48}+\frac{\Gamma\left(\frac{2}{3}\right)\eta\left(\frac{5}{3}\right)}{3.2^{\frac{2}{3}}}}$$

2866. ***Find*** **:**

$$\Delta=\int_0^\infty \frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$I=\int_0^\infty \frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx=\int_0^\infty \frac{x^{\frac{1}{2}}e^{-2\pi x^{\frac{1}{2}}}\ln(x)}{1+e^{-2\pi x^{\frac{1}{2}}}}dx$$

$$We\ do: x=t^2\to\ dx=2tdt$$

$$I=4\int_0^\infty \frac{t^2e^{-2\pi t}\ln(t)}{1+e^{-2\pi t}}dt=4\sum_{n=1}^\infty(-1)^{n-1}\int_0^\infty t^2e^{-2\pi nt}\ln(t)\,dt$$

$$We\ do: x=2\pi nt\to\ dt=\frac{dx}{2\pi n}$$

$$I=\frac{1}{2\pi^3}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\int_0^\infty x^2e^{-x}\ln\left(\frac{x}{2\pi n}\right)dx=$$

$$\frac{1}{2\pi^3}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\left[\int_0^\infty x^2e^{-x}\ln(x)dx-(\ln(2\pi)+\ln(n))\int_0^\infty x^2e^{-x}dx\right]$$

$$We\ know\ that: \Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx\to\ \Gamma'(z)=\int_0^\infty x^{z-1}e^{-x}\ln(x)\,dx$$

$$I=\frac{1}{2\pi^3}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}[\Gamma'(3)\ -(\ln(2\pi)+\ln(n))\ \Gamma(3)\,]=$$

$$\frac{\Gamma'(3)}{2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}-\frac{ln(2\pi)}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}-\frac{1}{n^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\ln(n)=$$

$$\frac{1}{\pi^3}\left(\frac{3}{2}-\gamma\right)\eta(3)-\frac{ln(2\pi)}{\pi^3}\eta(3)-\frac{1}{\pi^3}\left(-\eta'(3)\right)=$$

$$\frac{3\zeta(3)}{4\pi^3}\left(\frac{3}{2}-\gamma\right)-\frac{3ln(2\pi)\zeta(3)}{4\pi^3}-\frac{1}{4\pi^3}\left(-3\zeta'(3)-\zeta(3)\ln(2)\right)=$$

$$\frac{3\zeta(3)}{4\pi^3}\left(\frac{3}{2}-\gamma-ln(2\pi)+\frac{\zeta'(3)}{\zeta(3)}+\frac{1}{3}\ln(2)\right)=\frac{3\zeta(3)}{4\pi^3}\left(\ln\left(\frac{2^{-\frac{2}{3}}e^{\frac{3}{2}}}{\pi}\right)+\frac{\zeta'(3)}{\zeta(3)}-\gamma\right)$$

Solution 2 by Omkumar Rao-India

$$\Delta=\int_0^{\infty}\frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx\overset{x\to x^2}{\cong}2\int_0^{\infty}\frac{x^2\ln(x)}{1+e^{2\pi x}}dx\overset{2\pi x\to x}{\cong}\frac{1}{4\pi^3}\underbrace{\int_0^{\infty}\frac{x^2\ln(x)}{1+e^x}dx}_{P'(3)}$$

$$-\frac{\ln(2\pi)}{4\pi^3}\int_0^{\infty}\frac{1}{1+e^x}dx$$

$$P(k)=\int_0^{\infty}\frac{x^{k-1}}{1+e^x}dx=\Gamma(k)\eta(k)$$

$$P'(k)=\left(\Gamma(k)(1-2^{1-k})\zeta(k)\right)'=$$

$$\underbrace{\Gamma'(k)}_{\Gamma(k)\psi(k)}(1-2^{1-k})\zeta(k)+\Gamma(k)\zeta(k)(2^{1-k}\ln 2)+\Gamma(k)(1-2^{1-k})\zeta'(k)$$

$$P'(3)=\underbrace{(\Gamma(3))}_{2}\underbrace{(\psi(3))}_{\frac{3}{2}-\gamma}\left(\frac{3}{4}\zeta(3)\right)+\underbrace{(\Gamma(3))}_{2}(\zeta(3))\left(\frac{1}{4}\ln 2\right)+\underbrace{(\Gamma(3))}_{2}\left(\frac{3}{4}\zeta'(3)\right)$$

$$=\frac{1}{4\pi^3}\left((3-2\gamma)\left(\frac{3}{4}\zeta(3)\right)+\zeta(3)\left(\frac{1}{2}\ln 2\right)+\left(\frac{3}{2}\zeta'(3)\right)\right)-\frac{\ln(2\pi)}{4\pi^3}(\Gamma(1)\eta(1))$$

$$\therefore\int_0^{\infty}\frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx$$

$$=\frac{1}{4\pi^3}\left((3-2\gamma)\left(\frac{3}{4}\zeta(3)\right)+\zeta(3)\left(\frac{1}{2}\ln 2\right)+\left(\frac{3}{2}\zeta'(3)\right)\right)-\frac{(\ln(2\pi))(\ln 2)}{4\pi^3}$$

Solution 3 by Lara Gangler-Hungary

$$\Delta=\int_0^{\infty}\frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx\ \therefore\ y=2\pi\sqrt{x}\to x=\frac{y^2}{4\pi^2}\to dx=\frac{y}{2\pi^2}dy$$

$$\text{I}=\int_0^{\infty}\frac{\frac{y}{2\pi}\ln\frac{y^2}{4\pi^2}}{1+e^y}\frac{y}{2\pi^2}dy=\frac{1}{2\pi^3}\int_0^{\infty}\frac{y^2[\ln(y)-\ln(2\pi)]}{1+e^y}dy=\frac{1}{2\pi^3}(I_1-I_2)$$

As $0 \leq e^{-y} \leq 1$, the integrand within both integrals can be extended to a power series:

$$I_1 = \int_0^\infty \frac{y^2 \ln(y)}{1+e^y} dy = \int_0^\infty \frac{y^2 \ln(y)\, e^{-y}}{1+e^{-y}} dy = \int_0^\infty y^2 \ln(y)\, e^{-y} \sum_{k=0}^\infty e^{-ky} (-1)^k dy =$$

$$= \int_0^\infty \sum_{k=0}^\infty y^2 \ln(y)\; e^{-(k+1)y} (-1)^k\, dy = \sum_{k=0}^\infty \int_0^\infty y^2 \ln(y)\; e^{-(k+1)y} (-1)^k dy$$

$Note\ 1^*$: **gamma function and its derivative**

$$\int_0^\infty y^{a-1}\; e^{-by} \ln(y)\, dy = \Gamma(a) \frac{\Psi(a) - \ln(b)}{b^a}$$

$$I_1 = \sum_{k=0}^\infty \int_0^\infty y^2 \ln(y)\; e^{-(k+1)y} (-1)^k dy = \sum_{k=0}^\infty \Gamma(3) \frac{\Psi(3) - \ln(k+1)}{(k+1)^3} (-1)^k =$$

$$\sum_{k=1}^\infty 2 \frac{\left(\frac{1}{2} + 1 + \Psi(1)\right) - \ln(k)}{k^3} (-1)^{k-1} = 2(I_{11} - I_{12})$$

$$I_{11} = \sum_{k=1}^\infty \int_0^\infty \frac{1 + \frac{1}{2} + \Psi(1)}{k^3} (-1)^{k-1} = \sum_{k=1}^\infty \frac{\frac{3}{2} - \gamma}{k^3} (-1)^{k-1} = \left(\frac{3}{2} - \gamma\right) \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^3}$$

$$= \left(\frac{3}{2} - \gamma\right) \eta(3) = \left(\frac{3}{2} - \gamma\right)\left(1 - \frac{1}{2^{1-3}}\right) \zeta(3) = \frac{9 - 6\gamma}{8} \zeta(3)$$

where γ is the Euler − Mascheroni constant: $\Psi(1) = -\gamma$

$$= \int_0^\infty e^{-y} \ln(y)\, dy$$ **and η is the Dirichlet eta function**: $\eta(a)$

$$= (1 - 2^{1-a}) \zeta(a)$$

$Note\ 2^*$: **derivative of the Dirichlet eta function**

$$\sum_{k=1}^\infty \frac{\ln(k)}{k^a} (-1)^{k-1} = -[2^{1-a} \ln(2)\, \zeta(a) + (1 - 2^{1-a}) \zeta'(a)]$$

$$\to I_{12} = \sum_{k=1}^\infty \frac{\ln(k)}{k^3} (-1)^{k-1} = -[2^{1-2} \ln(2)\, \zeta(3) + (1 - 2^{1-2}) \zeta'(a)]$$

$$= -\frac{\ln(2)}{4} \zeta(3) - \frac{3}{4} \zeta'(3)$$

$$I_1 = 2(I_{11} - I_{12}) = 2\left(\frac{9 - 6\gamma}{8} \zeta(3) + \frac{\ln(2)}{4} \zeta(3) + \frac{3}{4} \zeta'(3)\right)$$

$$= \frac{9 + 2\ln(2) - 6\gamma}{4} \zeta(3) + \frac{3}{2} \zeta'(3)$$

$$I_2=\ln(2\pi)\int_0^\infty \frac{y^2}{1+e^y}dy=\ln(2\pi)\int_0^\infty y^2e^{-y}\sum_{k=0}^\infty e^{-ky}(-1)^k dy=$$

$$=\ln(2\pi)\sum_{k=0}^\infty\int_0^\infty y^2 e^{-(k+1)y}(-1)^k dy=\ln(2\pi)\sum_{k=0}^\infty\Gamma(3)\frac{(-1)^k}{(k+1)^3}$$

$$=2\eta(3)\ln(2\pi)=\frac{3}{2}\zeta(3)\ln(2\pi)$$

$$I=\frac{1}{2\pi^3}(I_1-I_2)=\frac{1}{2\pi^3}\left(\frac{9+2\ln(2)-6\gamma}{4}\zeta(3)+\frac{3}{2}\zeta'(3)-\frac{3}{2}\zeta(3)\ln(2\pi)\right)$$

$$I=\frac{1}{8\pi^3}\left[(9-\ln(16\pi^6)-6\gamma)\zeta(3)+\frac{6}{4}\zeta'(3)\right]$$

Note 1^* Statement: $\int_0^\infty y^{a-1}\ e^{-by}\ln(y)\,dy=\Gamma(a)\frac{\Psi(a)-\ln(b)}{b^a}$

Proof: Differentiating the left side with respect to *a:*

$$\frac{\partial}{\partial a}\int_0^\infty y^{a-1}\ e^{-by}dy=\frac{\partial}{\partial(a-1)}\int_0^\infty y^{a-1}\ e^{-by}dy=\int_0^\infty y^{a-1}\ e^{-by}\ln(y)\,dy$$

Differentiating the right side with respect to *a:*

$$\frac{\partial\frac{\Gamma(a)}{b^a}}{\partial a}=\frac{\Gamma(a)\Psi(a)b^a-\Gamma(a)b^a\ln(b)}{b^{2a}}=\Gamma(a)\frac{\Psi(a)-\ln(b)}{b^a}$$

where Ψ is the digamma function, the logarithmic derivative of the gamma function:

$$\Psi(a)=\frac{d\ln\Gamma(a)}{d\,a}=\frac{\frac{d\Gamma(a)}{d\,a}}{\Gamma(a)}=\frac{\Gamma'(a)}{\Gamma(a)}$$

$$\to\int_0^\infty y^{a-1}\ e^{-by}\ln(y)\,dy=\Gamma(a)\frac{\Psi(a)-\ln(b)}{b^a},QED$$

Note 2^* Statement: $\sum_{k=1}^\infty\frac{\ln(k)}{k^a}(-1)^{k-1}=-[2^{1-a}\ln(2)\,\zeta(a)+(1-2^{1-a})\zeta'(a)]$

Proof:

$$\sum_{k=1}^\infty\frac{\ln(k)}{k^a}(-1)^{k-1}=\sum_{k=1}^\infty k^{-a}(-1)^{k-1}\ln(k)=-\sum_{k=1}^\infty(-1)^{k-1}k^{-a}[-\ln(k)]$$

$$=-\sum_{k=1}^\infty(-1)^{k-1}\frac{dk^{-a}}{da}=-\frac{d}{da}\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k^a}=-\frac{d\eta(a)}{da}$$

$$=-\frac{d}{da}[(1-2^{1-a})\zeta(a)]=-[2^{1-a}\ln(2)\,\zeta(a)+(1-2^{1-a})\zeta'(a)],$$

QED

Solution 4 by Quadri Faruk Temitope-Nigeria

$$\Delta=\int_0^\infty\frac{\sqrt{x}\ln(x)}{1+e^{2\pi\sqrt{x}}}dx=\int_0^\infty\frac{x\ln(x^2)}{1+e^{2\pi x}}.2xdx=4\int_0^\infty\frac{x^2\ln(x)}{1+e^{2\pi x}}dx=$$

$$Recall\ that:\frac{1}{1+e^{2\pi x}}=\sum_{n=1}^\infty(-1)^{n-1}e^{-2\pi x}$$

$$= 4\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^{\infty} x^2 e^{-2\pi nx}\ln(x)\,dx = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\int_0^1 x^{s-1}e^{-2\pi nx}dx =$$

$$I(s) = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\int_0^{\infty} x^{s-1}e^{-2\pi nx}dx = \begin{cases} 2\pi n = y \ , \ x = \dfrac{y}{2\pi n} \\ dx = \dfrac{dy}{2\pi n} \end{cases}$$

$$I(s) = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\int_0^{\infty}\left(\frac{y}{2\pi n}\right)^{s-1} e^{-y}\frac{dy}{2\pi n}$$

$$= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\left(\frac{1}{2\pi n}\right)^{s}\underbrace{\int_0^{\infty} y^{s-1}e^{-y}dy}_{\Gamma(s)} =$$

This :

$$4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\left(\frac{1}{2\pi n}\right)^{s}\Gamma(s) = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{d}{ds}\frac{\Gamma(s)}{(2\pi n)^s} = -4\frac{d}{ds}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}\frac{\Gamma(s)}{(2\pi)^s} =$$

$$4\frac{d}{ds}\frac{\Gamma(s)}{(2\pi)^s}\eta(s)$$

$$\therefore \eta(s) = (1-2^{1-s})\zeta(s)$$

$$I(s) = 4\frac{d}{ds}\frac{\Gamma(s)(1-2^{1-s})\zeta(s)}{(2\pi)^s}$$

$$I(s) = 4\frac{\Gamma(s)\eta(s)}{(2\pi)^s}\left[\psi(s) - \ln(2\pi) + \frac{\eta'(s)}{\eta(s)}\right]$$

$$s = 3$$

$$I = 4\frac{\Gamma(3)\eta(3)}{(2\pi)^3}\left[\psi(3) - \ln(2\pi) + \frac{\eta'(3)}{\eta(3)}\right]$$

$$I = \frac{3\zeta(3)}{4\pi^3}\left(\frac{3}{2} - \gamma - \ln(2\pi) + \frac{\zeta'(3)}{\zeta(3)} + \frac{1}{3}\ln(2)\right) = \frac{3\zeta(3)}{4\pi^3}\left(\ln\left(\frac{2^{-\frac{2}{3}}e^{\frac{3}{2}}}{\pi}\right) + \frac{\zeta'(3)}{\zeta(3)} - \gamma\right)$$

2867. ***If*** $0 < a \le b$ *then:*

$$\int_a^b\int_a^b \frac{dxdy}{\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^2} \le \frac{3}{4}(b-a)\left(\sqrt[3]{b}-\sqrt[3]{a}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x, y > 0 \text{ then } \left(\sqrt[3]{x}+\sqrt[3]{y}\right)^2 \overset{AM-GM}{\ge} 4\sqrt[3]{xy} \text{ or, } \frac{1}{\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^2} \le \frac{1}{4\sqrt[3]{xy}}$$

$$now,\int_a^b\int_a^b\frac{dxdy}{\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^2}\le\frac{1}{4}\int_a^b\int_a^b\frac{dxdy}{\sqrt[3]{xy}}=\frac{1}{4}\int_a^b\frac{dx}{\sqrt[3]{x}}\cdot\int_a^b\frac{dx}{\sqrt[3]{y}}=$$

$$=\frac{1}{4}\left(\frac{3}{2}\right)^2\left(b^{\frac{2}{3}}-a^{\frac{2}{3}}\right)^2=\frac{9}{16}\left(b^{\frac{2}{3}}-a^{\frac{2}{3}}\right)^2$$

We need to show:

$$\frac{9}{16}\left(b^{\frac{2}{3}}-a^{\frac{2}{3}}\right)^2\le\frac{3}{4}(b-a)\left(\sqrt[3]{b}-\sqrt[3]{a}\right)$$

$$or,3\left(b^{\frac{2}{3}}-a^{\frac{2}{3}}\right)^2\le4(b-a)\left(\sqrt[3]{b}-\sqrt[3]{a}\right)\ or,3(u^2-v^2)^2\overset{\substack{b=u^3,b=v^3\\ b>u\ or\ u^3>v^3}}{\le}4(u^3-v^3)(u-v)$$

$$or,3(u+v)^2(u-v)^2\le4(u-v)^2(u^2+uv+v^2)or,3(u+v)^2\le4(u^2+uv+v^2)$$

$$true\ as\ 4(u^2+uv+v^2)=3(u^2+v^2)+(u^2+v^2)+4uv\overset{AM\ge GM}{\ge}$$

$$\ge3(u^2+v^2)+2uv+4uv=3(u+v)^2$$

Equality holds for $a=b$.

2868.

Let $m,n\in N$ *with* $n>m\ge1$ *and let* $g:[0,1]\to[0,\infty)$ *be a non-negative continuous function on* $[0,1]$ *such that* $x^m(1-x^n)g(x)\le x^2$ *for all* $x\in[0,1]$. *Prove that*:

$$1\le\int_0^1\frac{dx}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\le\frac{\pi}{2}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$x^m(1-x^n)g(x)\le x^2\Rightarrow1-x^2+x^m(1-x^n)g(x)\le1$$

$$\frac{1}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\ge1\Rightarrow\int_0^1\frac{dx}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\ge1$$

$$x^m(1-x^n)g(x)\ge0\ \Rightarrow1-x^2+x^m(1-x^n)g(x)\ge1-x^2$$

$$\frac{1}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\le\frac{1}{\sqrt{1-x^2}}\Rightarrow\int_0^1\frac{dx}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\le$$

$$\le\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\frac{\pi}{2}$$

$$\boxed{1\le\int_0^1\frac{dx}{\sqrt{1-x^2+x^m(1-x^n)g(x)}}\le\frac{\pi}{2}}$$

2869. *Let $0 < a \le b$ and let $f:[a^k, b^k] \to R$ be a continuous function where $k \ge 2$ is a positive integer. Prove that:*

$$\frac{b^{2k-1+2m}-a^{2k-1+2m}}{2k-1+2m}+\int_a^b x^{2m}f^2(x^k)\,dx \ge \frac{2}{k}\int_{a^k}^{b^k} y^{\frac{2m}{k}}f(y)\,dy$$

for any given integer parameter $m \ge 0$.

Proposed by Omkumar Rao-India

Solution by proposer

$$\underbrace{\frac{b^{2k-1+2m}-a^{2k-1+2m}}{2k-1+2m}}_{\int_a^b x^{2k-2+2m}\,dx}+\int_a^b x^{2m}f^2(x^k)\,dx \ge \frac{2}{k}\int_{a^k}^{b^k} y^{\frac{2m}{k}}f(y)\,dy$$

$$\int_a^b \left(x^{2m}f^2(x^k)+x^{2k-2+2m}\right)dx \ge \underbrace{\frac{2}{k}\int_{a^k}^{b^k} y^{\frac{2m}{k}}f(y)\,dy}_{y=x^k}$$

$$\int_a^b \left(x^{2m}f^2(x^k)+x^{2k-2+2m}\right)dx \ge \int_a^b 2x^{k-1+2m}f(x^k)\,dx$$

$$\int_a^b x^{2m}\left(x^{k-1}-f(x^k)\right)^2 dx \ge 0 \text{ as } x^{2m} \ge 0 \text{ and } \left(x^{k-1}-f(x^k)\right)^2 \ge 0 \text{ for all } x \in [a,b]$$

2870. *Find :*

$$\Delta=\int_0^{\frac{\pi}{2}} x^2\ln\left(\frac{1+\cos(x)}{\sqrt{x}}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Nawar Alasadi-Iraq

$$\Delta=\int_0^{\frac{\pi}{2}} x^2\ln\left(\frac{1+\cos(x)}{\sqrt{x}}\right)dx = I_1 - I_2$$

$$I_2=\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\ln(x)dx=\frac{1}{2}\left[\frac{x^3}{3}\ln(x)-\frac{x^3}{9}\right]_0^{\frac{\pi}{2}}=\frac{\pi^3}{48}\ln\left(\frac{\pi}{2}\right)-\frac{\pi^3}{144}$$

$$I_1=\int_0^{\frac{\pi}{2}} x^2\ln(1+\cos(x))dx=\int_0^{\frac{\pi}{2}} x^2\ln\left(2\cos^2\left(\frac{x}{2}\right)\right)dx=$$

$$\ln(2)\int_0^{\frac{\pi}{2}} x^2dx+2\int_0^{\frac{\pi}{2}} x^2\ln\left(2\cos\left(\frac{x}{2}\right)\right)dx \overset{\left(t=\frac{x}{2}\right)}{\cong} \frac{\pi^3}{24}\ln(2)+16\int_0^{\frac{\pi}{4}} t^2\ln(\cos(t))\,dt$$

$$\therefore \ln(\cos(t))=-\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos(2nt)$$

$$\int_0^{\frac{\pi}{4}} t^2\ln(\cos(t))\,dt=\frac{\pi^3}{192}\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\underbrace{\int_0^{\frac{\pi}{4}} t^2\cos(2nt)\,dt}_{J_n}$$

$$J_n=\left[t^3\frac{\sin(2nt)}{2n}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}2t\frac{\sin(2nt)}{2n}dt=$$

$$\frac{\pi^2}{32n}\sin\left(\frac{n\pi}{2}\right)+\left[t\frac{\cos(2nt)}{2n^2}\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{\cos(2nt)}{2n^2}dt=$$

$$\left(\frac{\pi^2}{32n}-\frac{1}{4n^3}\right)\sin\left(\frac{n\pi}{2}\right)+\frac{\pi}{8n^2}\cos\left(\frac{n\pi}{2}\right)$$

$$I_1=-\frac{\pi^3}{24}\ln(2)+16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}J_n=-\frac{\pi^3}{24}\ln(2)+16(S_{even}+S_{old})$$

$$S_{even}(n=2k)=\sum_{k=1}^{\infty}-\frac{1}{2k}\left(\frac{\pi}{8(2k)^2}(-1)^k\right)=\frac{\pi}{64}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}=\frac{3\pi}{256}\zeta(3)$$

$$S_{old}(n=2k-1)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\left(\frac{\pi^2}{32(2k-1)}-\frac{1}{4(2k-1)^3}\right)=$$

$$\frac{\pi^2}{32}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^4}=\frac{\pi^2}{32}G-\frac{1}{4}\beta(4)$$

$$I_1=-\frac{\pi^3}{24}\ln(2)+16(S_{even}+S_{old})=$$

$$-\frac{\pi^3}{24}\ln(2)+16\left(\frac{3\pi}{256}\zeta(3)+\frac{\pi^2}{32}G-\frac{1}{4}\beta(4)\right)=-\frac{\pi^3}{24}\ln(2)+\frac{3\pi}{16}\zeta(3)+\frac{\pi^2}{2}G-4\beta(4)$$

$$\Delta=I_1-I_2=\frac{\pi^3}{24}\ln(2)+\frac{3\pi}{16}\zeta(3)+\frac{\pi^2}{2}G-4\beta(4)-\frac{\pi^3}{48}\ln\left(\frac{\pi}{2}\right)+\frac{\pi^3}{144}=$$

$$\frac{3\pi}{16}\zeta(3)+\frac{\pi^2}{2}G-4\beta(4)-\frac{\pi^3}{48}\ln(2\pi)+\frac{\pi^3}{144}$$

Solution 2 by Yang Silva Cartolin-Peru

$$I=\int_0^{\frac{\pi}{2}} x^2\log(1+\cos(x))dx-\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\log(x)dx$$

$$\textit{We know that}: 1+\cos(x)=2\cos^2\left(\frac{x}{2}\right)$$

$$I=\log(2)\int_0^{\frac{\pi}{2}} x^2dx+2\int_0^{\frac{\pi}{2}} x^2\log\left(\cos\left(\frac{x}{2}\right)\right)dx-\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\log(x)dx$$

$$I=\frac{\pi^3}{24}\log(2)+2i_1-\frac{1}{2}i_2$$

$$\blacksquare i_2=\int_0^{\frac{\pi}{2}} x^2\log(x)dx\overset{I.B.P}{\Longrightarrow} i_2=\left[\frac{x^3}{3}\log(x)\right]_0^{\frac{\pi}{2}}-\frac{1}{3}\int_0^{\frac{\pi}{2}} x^2dx=\frac{\pi^3}{24}\log\left(\frac{\pi}{2}\right)-\frac{\pi^3}{72}\therefore$$

$$\blacksquare i_1=\int_0^{\frac{\pi}{2}} x^2\log\left(\cos\left(\frac{x}{2}\right)\right)dx$$

$$\textit{We know that}: \cos\left(\frac{x}{2}\right)=\frac{e^{\frac{ix}{2}}+e^{-\frac{ix}{2}}}{2}\Rightarrow 2e^{\frac{ix}{2}}\cos\left(\frac{x}{2}\right)=1+e^{ix}$$

$$\log(2)+\log\left(e^{\frac{ix}{2}}\right)+\log\left(\cos\left(\frac{x}{2}\right)\right)=\log(1+e^{ix})$$

$$\log\left(\cos\left(\frac{x}{2}\right)\right)=\log(1+e^{ix})-\log(2)-\frac{ix}{2}$$

$$\Re\left\{\log\left(\cos\left(\frac{x}{2}\right)\right)\right\}=\Re\{\log(1+e^{ix})\}-\log(2)$$

$$\Rightarrow i_1=\Re\left\{\int_0^{\frac{\pi}{2}} x^2\log(1+e^{ix})\,dx\right\}-\log(2)\int_0^{\frac{\pi}{2}} x^2dx$$

$$i_1=\Re\left\{\int_0^{\frac{\pi}{2}} x^2\log(1+e^{ix})\,dx\right\}-\frac{\pi^3}{24}\log(2)$$

$$\textit{We know that}: \log(1+z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n\overset{z=e^{ix}}{\Longrightarrow}\log(1+e^{ix})=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{inx}$$

$$\Re\{\log(1+e^{ix})\}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos(nx)$$

$$\Rightarrow i_1=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{2}} x^2\cos(nx)\,dx-\frac{\pi^3}{24}\log(2)$$

$$i_1=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{\pi^2}{4n}\sin\left(\frac{\pi n}{2}\right)-\frac{2}{n^3}\sin\left(\frac{\pi n}{2}\right)+\frac{\pi}{n^2}\cos\left(\frac{\pi n}{2}\right)\right]-\frac{\pi^3}{24}\log(2)$$

$$i_1 = -\frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\sin\left(\frac{\pi n}{2}\right) - \pi\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right) - \frac{\pi^3}{24}\log(2)$$

Let:

$$\gg S_1 = -\frac{\pi^2}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\sin\left(\frac{\pi n}{2}\right)$$

$$\xRightarrow{n=2k+1} S_1 = \frac{\pi^2}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^4} \therefore$$

$$\gg S_2 = -\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right) \xRightarrow{n=2k} S_2 = \frac{\pi}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$\textit{We know that}: \blacksquare \textit{Dirichlet Beta Function}: \beta(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \beta(2) = G \wedge \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4} = \beta(4)$$

$$\blacksquare \textit{Dirichlet Beta Function}: \eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} = \eta(3) = \frac{3}{4}\zeta(3)$$

$$\Rightarrow i_1 = \frac{\pi^2}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^4} + \frac{\pi}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3} - \frac{\pi^3}{24}\log(2)$$

$$i_1 = \frac{\pi^2}{4}G - 2\beta(4) + \frac{3\pi}{32}\zeta(3) - \frac{\pi^3}{24}\log(2) \therefore$$

$$I = \frac{\pi^3}{24}log(2) + 2\left(\frac{\pi^2}{4}G - 2\beta(4) + \frac{3\pi}{32}\zeta(3) - \frac{\pi^3}{24}\log(2)\right) + \frac{1}{2}\left(\frac{\pi^3}{24}\log\left(\frac{\pi}{2}\right) - \frac{\pi^3}{72}\right)$$

$$I = -\frac{\pi^3}{48}log(2\pi) + \frac{\pi^2}{2}G - 4\beta(4) + \frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{144} \therefore$$

Solution 3 by Omkumar Rao-India

$$K = \underbrace{\ln 2\int_0^{\frac{\pi}{2}} x^2\,dx}_{\frac{\pi^3}{24}\ln 2} - \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\ln x\,dx}_{\frac{\pi^3}{48}\ln\left(\frac{\pi}{2}\right)-\frac{\pi^3}{144}} + \underbrace{2\int_0^{\frac{\pi}{2}} x^2\ln\left(\cos\left(\frac{x}{2}\right)\right)dx}_{M}$$

$$K = \frac{\pi^3}{24}\ln 2 - \frac{\pi^3}{48}\ln\left(\frac{\pi}{2}\right) + \frac{\pi^3}{144} + M$$

$$M=2\int_0^{\frac{\pi}{2}}x^2\ln\left(\cos\left(\frac{x}{2}\right)\right)dx\overset{x\to 2x}{\cong}16\int_0^{\frac{\pi}{4}}x^2\underbrace{\ln(\cos(x))}_{-\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\cos(2mx)}dx$$

$$=16\int_0^{\frac{\pi}{4}}x^2\left(-\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\cos(2mx)\right)dx$$

$$M=-\frac{\pi^3}{12}\ln 2+16\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\int_0^{\frac{\pi}{4}}x^2\cos(2mx)\,dx\right)$$

$$M=-\frac{\pi^3}{12}\ln 2+16\left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{\pi^2}{32m}\sin\left(\frac{m\pi}{2}\right)+\frac{\pi}{8m^2}\cos\left(\frac{m\pi}{2}\right)-\frac{1}{4m^3}\sin\left(\frac{m\pi}{2}\right)\right)\right)$$

$$M=-\frac{\pi^3}{12}\ln 2+16\left(\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi^2}{32m^2}\sin\left(\frac{m\pi}{2}\right)\right)+\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi}{8m^3}\cos\left(\frac{m\pi}{2}\right)\right)-\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}}{4m^4}\sin\left(\frac{m\pi}{2}\right)\right)\right)$$

$$\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi^2}{32m^2}\sin\left(\frac{m\pi}{2}\right)\right)=\frac{\pi^2}{32}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{(2p-1)^2}=\frac{\pi^2}{32}G$$

$$\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi}{8m^3}\cos\left(\frac{m\pi}{2}\right)\right)=\frac{\pi}{64}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^3}=\frac{3\pi}{256}\zeta(3)$$

$$\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}}{4m^4}\sin\left(\frac{m\pi}{2}\right)\right)=\frac{1}{4}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{(2p-1)^4}=\frac{1}{4}\beta(4)$$

$$\boxed{M=-\frac{\pi^3}{12}\ln 2+16\left(\frac{\pi^2}{32}G+\frac{3\pi}{256}\zeta(3)-\frac{1}{4}\beta(4)\right)}$$

$$K=\frac{\pi^3}{24}\ln 2-\frac{\pi^3}{48}\ln\left(\frac{\pi}{2}\right)+\frac{\pi^3}{144}-\frac{\pi^3}{12}\ln 2+16\left(\frac{\pi^2}{32}G+\frac{3\pi}{256}\zeta(3)-\frac{1}{4}\beta(4)\right)$$

$$K=\frac{\pi^3}{144}+\frac{\pi^2}{2}G+\frac{3\pi}{16}\zeta(3)-4\beta(4)-\frac{\pi^3}{48}\ln(2\pi)$$

2871. ***Find :***

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\psi^{(0)}\left(\frac{1}{2}+\frac{1}{2}x\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\psi^{(0)}\left(\frac{1}{2}+\frac{1}{2}x\right)dx=\int_0^{\frac{1}{2}}x^3\left(\psi\left(\frac{1}{2}+\frac{1}{2}x\right)-\psi\left(\frac{1}{2}-\frac{1}{2}x\right)\right)dx=$$

$$\pi\int_0^{\frac{1}{2}}x^3\tan\left(\frac{\pi x}{2}\right)dx\overset{t=\frac{\pi x}{2}}{\triangleq}\frac{16}{\pi^3}\int_0^{\frac{\pi}{4}}t^3\tan(t)\,dt\overset{IBP's}{\triangleq}$$

$$=\frac{16}{\pi^3}\left(-\left[t^3 ln(cos(t))\right]_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}t^2\ln(\cos(t))\,dt\right)=\frac{1}{8}\ln(2)+\frac{48}{\pi^3}\underbrace{\int_0^{\frac{\pi}{4}}t^2\ln(\cos(t))\,dt}_{A}$$

$$A=\int_0^{\frac{\pi}{4}}t^2\ln(\cos(t))\,dt=\int_0^{\frac{\pi}{4}}t^2\left(-\ln(2)-\sum_{k=1}^{\infty}\frac{(-1)^k\cos(2kt)}{k}\right)dt=$$

$$-\frac{\pi^3}{192}\ln(2)-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}}t^2\cos(2kt)\,dt=$$

$$-\frac{\pi^3}{192}\ln(2)-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(-\frac{1}{4}\frac{1}{k^3}\sin\left(\frac{\pi k}{2}\right)+\frac{\pi}{8}\frac{1}{k^2}\cos\left(\frac{\pi k}{2}\right)+\frac{\pi^2}{32}\frac{1}{k}\sin\left(\frac{\pi k}{2}\right)\right)$$

$$\therefore\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}\sin\left(\frac{\pi k}{2}\right)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=-\beta(4)$$

$$\therefore\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\cos\left(\frac{\pi k}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)^3}=-\frac{3}{32}\zeta(3)$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sin\left(\frac{\pi k}{2}\right)=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G$$

$$A=-\frac{\pi^3}{192}\ln(2)+\frac{1}{4}(-\beta(4))-\frac{\pi}{8}\left(-\frac{3}{32}\zeta(3)\right)-\frac{\pi^2}{32}(-G)=$$

$$-\frac{1}{4}\beta(4)+\frac{3\pi}{256}\zeta(3)+\frac{\pi^2}{32}G-\frac{\pi^3}{192}\ln(2)$$

$$\Omega=\frac{1}{8}\ln(2)+\frac{48}{\pi^3}\left(-\frac{1}{4}\beta(4)+\frac{3\pi}{256}\zeta(3)+\frac{\pi^2}{32}G-\frac{\pi^3}{192}\ln(2)\right)$$

$$\Omega=-\frac{12}{\pi^3}\beta(4)+\frac{9}{16\pi^2}\zeta(3)+\frac{3G}{2\pi}-\frac{1}{8}ln(2)$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\psi^{(0)}\left(\frac{1}{2}+\frac{x}{2}\right)dx\xRightarrow{We\ do:x\to-x}\Omega=-\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\psi^{(0)}\left(\frac{1}{2}-\frac{x}{2}\right)dx$$

$$Thus:2\Omega=\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\left[\psi^{(0)}\left(\frac{1}{2}+\frac{x}{2}\right)-\psi^{(0)}\left(\frac{1}{2}-\frac{x}{2}\right)\right]dx$$

We know that:

$$\psi^{(0)}(z)-\psi^{(0)}(1-z)=-\pi\cot(\pi z)$$

$$\Rightarrow 2\Omega=\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\cot\left(\pi\left(\frac{1+x}{2}\right)\right)dx=\pi\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\tan\left(\frac{\pi x}{2}\right)dx$$

$$\xRightarrow{Even\ Function}2\Omega=2\pi\int_0^{\frac{1}{2}}x^3\tan\left(\frac{\pi x}{2}\right)dx$$

$$\Omega=\pi\int_0^{\frac{1}{2}}x^3\tan\left(\frac{\pi x}{2}\right)dx\xRightarrow{We\ do:u=\frac{\pi x}{2}}\Omega=\frac{16}{\pi^3}\int_0^{\frac{\pi}{4}}u^3\tan(u)\,du$$

$$I.B.P:\begin{cases}a=u^3\to da=3u^2du\\ db=\tan(u)\,du\to b=-\log(cos(u))\end{cases}$$

$$\Rightarrow\Omega=\frac{16}{\pi^3}\left(\left[-u^3\log(cos(u))\right]_0^{\frac{\pi}{4}}+3\int_0^{\frac{\pi}{4}}u^2\log(cos(u))\,du\right)$$

We know that: *Fourier Series*

$$\log(cos(x))=-\log(2)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cos(2kx)$$

$$\Rightarrow\Omega=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\log(2)+3\int_0^{\frac{\pi}{4}}u^2\left(-\log(2)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\cos(2ku)\right)du\right)$$

$$\Omega=\frac{1}{8}\log(2)+\frac{48}{\pi^3}\left(-\log(2)\int_0^{\frac{\pi}{4}}u^2du-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}}u^2\cos(2ku)\,du\right)$$

$$\Omega=\frac{1}{8}\log(2)+\frac{48}{\pi^3}\left(-\frac{\pi^3}{192}\log(2)\right)$$

$$-\frac{48}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{\pi^2}{32k}\sin\left(\frac{\pi k}{2}\right)+\frac{\pi}{8k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^3}\sin\left(\frac{\pi k}{2}\right)\right)$$

$$\Omega=-\frac{1}{8}\log(2)$$

$$-\frac{48}{\pi^3}\left(\frac{\pi^2}{32}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sin\left(\frac{\pi k}{2}\right)+\frac{\pi}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\cos\left(\frac{\pi k}{2}\right)\right.$$

$$\left.-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}\sin\left(\frac{\pi k}{2}\right)\right)$$

$$\textit{Let}: \blacksquare S_1=\frac{\pi^2}{32}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sin\left(\frac{\pi k}{2}\right)-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}\sin\left(\frac{\pi k}{2}\right)$$

$$\xRightarrow{k=2n+1}S_1=\frac{\pi^2}{32}\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2n+1)^2}(-1)^n-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{(2n+1)^4}(-1)^n$$

$$S_1=-\frac{\pi^2}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}\therefore$$

$$\blacksquare S_2=\frac{\pi}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\cos\left(\frac{\pi k}{2}\right)\xRightarrow{k=2n}S_2=\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{(2n)^3}(-1)^n$$

$$S_2=-\frac{\pi}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\therefore$$

$$\Rightarrow\Omega=-\frac{1}{8}\log(2)+\frac{3}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{3}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}-\frac{12}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}$$

$$\textit{We know that}: \blacksquare \textit{Dirichlet Beta Function}: \beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\beta(2)=G\wedge\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=\beta(4)$$

$$\blacksquare \textit{Dirichlet Beta Function}: \eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\eta(3)=\frac{3}{4}\zeta(3)$$

$$Therefore: \Omega=-\frac{1}{8}\log(2)+\frac{3G}{2\pi}+\frac{9\zeta(3)}{16\pi^2}-\frac{12\beta(4)}{\pi^3}$$

$$\Omega=-\frac{1}{8}\log(2)+\frac{8}{9\pi^3}\left(\frac{27\pi^2}{16}G+\frac{81\pi}{128}\zeta(3)-\frac{27}{2}\beta(4)\right)\therefore$$

Solution 3 by Omkumar Rao-India

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}x^3\psi^{(0)}\left(\frac{1}{2}+\frac{1}{2}x\right)dx\overset{\frac{1}{2}+\frac{1}{2}x\to x}{\cong}2\int_{\frac{1}{4}}^{\frac{3}{4}}(2x-1)^3\psi(x)\,dx\overset{x\to 1-x}{\cong}-2\int_{\frac{1}{4}}^{\frac{3}{4}}(2x-1)^3\psi(1-x)\,dx$$

$$2I=\int_{\frac{1}{4}}^{\frac{3}{4}}2(2x-1)^3[\psi(x)-\psi(1-x)]\,dx=-2\pi\int_{\frac{1}{4}}^{\frac{3}{4}}(2x-1)^3\cot(\pi x)\,dx\overset{2x-1\to 2x}{\cong}16\pi\int_{-\frac{1}{4}}^{\frac{1}{4}}x^3\tan(\pi x)\,dx$$

$$I=16\pi\int_0^{\frac{1}{4}}x^3\tan(\pi x)\,dx\overset{\pi x\to x}{\cong}\frac{16}{\pi^3}\int_0^{\frac{\pi}{4}}x^3\tan(x)\,dx\overset{By\ parts}{\cong}\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2\right.$$

$$\left.+3\int_0^{\frac{\pi}{4}}x^2\underbrace{\ln(\cos(x))}_{-\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\cos(2mx)}{m}}dx\right)$$

$$I=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2+3\int_0^{\frac{\pi}{4}}x^2\left(-\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\cos(2mx)}{m}\right)dx\right)$$

$$=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2+3\left(-\frac{\pi^3}{192}\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\int_0^{\frac{\pi}{4}}x^2\cos(2mx)\,dx\right)\right)$$

$$I=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2+3\left(-\frac{\pi^3}{192}\ln 2+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{\pi^2}{32m}\sin\left(\frac{m\pi}{2}\right)+\frac{\pi}{8m^2}\cos\left(\frac{m\pi}{2}\right)-\frac{1}{4m^3}\sin\left(\frac{m\pi}{2}\right)\right)\right)\right)$$

$$I=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2+3\left(-\frac{\pi^3}{192}\ln 2+\underbrace{\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi^2}{32m^2}\sin\left(\frac{m\pi}{2}\right)\right)}_{\frac{\pi^2}{32}G}+\underbrace{\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}\pi}{8m^3}\cos\left(\frac{m\pi}{2}\right)\right)}_{\frac{3\pi}{256}\zeta(3)}\right.\right.$$

$$\left.\left.-\underbrace{\sum_{m=1}^{\infty}\left(\frac{(-1)^{m-1}}{4m^4}\sin\left(\frac{m\pi}{2}\right)\right)}_{\frac{1}{4}\beta(4)}\right)\right)$$

$$I=\frac{16}{\pi^3}\left(\frac{\pi^3}{128}\ln 2+3\left(-\frac{\pi^3}{192}\ln 2+\frac{\pi^2}{32}G+\frac{3\pi}{256}\zeta(3)-\frac{1}{4}\beta(4)\right)\right)$$

$$\boxed{\int_{-\frac{1}{2}}^{\frac{1}{2}} x^3\psi^{(0)}\left(\frac{1}{2}+\frac{1}{2}x\right)dx=\frac{3}{2\pi}G+\frac{9}{16\pi^2}\zeta(3)-\frac{1}{8}\ln 2-\frac{12}{\pi^3}\beta(4)}$$

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Find :

$$\Omega=\int_0^1 x^2\left(arctan^3(x)+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Nawar Alasadi-Iraq

$$Let:\ \Omega=\int_0^1 x^2\left(arctan^3(x)+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx=$$

$$\int_0^1 x^2\,arctan^3(x)dx+\int_0^1 x^2\ln\left(\frac{-\ln(x)}{1-x^3}\right)dx=I_1+I_2$$

$$For:\int_0^1 x^2\ln(-\ln(x))\,dx\ \ let\ \ x=e^{-t}\to\ \ dx=-e^{-t}dt$$

$$\int_0^\infty e^{-3t}\ln(t)\,dt\overset{3t=v}{=}\frac{1}{3}\int_0^\infty e^{-v}(\ln(v)-\ln(3))dv=-\frac{\ln(3)+\gamma}{3}$$

$$For:\int_0^1 x^2\ln(1-x^3)\,dx\ \ let\ \ v=x^3\to\ \ dv=3x^2dx$$

$$\frac{1}{3}\int_0^1\ln(1-v)\,dv=\frac{1}{3}[-(1-v)\ln(1-v)-v]\Big|_0^1=-\frac{1}{3}$$

$$I_2=-\frac{\ln(3)+\gamma}{3}+\frac{1}{3}=\frac{1-\gamma-\ln(3)}{3}$$

$$I_1=\int_0^1 x^2\,arctan^3(x)dx\ \ \left(I.B.P\ (u=arctan^3(x),dv=x^2dx)\right)$$

$$I_1=\left[\frac{x^3}{3}arctan^3(x)\right]_0^1-\int_0^1\frac{x^3}{1+x^2}arctan^2(x)dx=$$

$$\frac{\pi^3}{192}-\int_0^1\left(x-\frac{x}{1+x^2}\right)arctan^2(x)dx=\frac{\pi^3}{192}-J_1+J_2$$

$$J_1=\int_0^1 x\,arctan^2(x)dx\ \ \left(I.B.P\ (u=arctan^2(x),dv=xdx)\right)$$

$$J_1=\left[\frac{x^2}{2}arctan^2(x)\right]_0^1-\int_0^1\frac{x^2}{1+x^2}\arctan(x)\,dx=$$

$$\frac{\pi^2}{32}-\int_0^1\left(1-\frac{1}{1+x^2}\right)\arctan(x)\,dx=\frac{\pi^2}{32}-\left(\frac{\pi}{4}-\frac{1}{2}\ln(2)\right)+\frac{\pi^2}{32}=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\ln(2)$$

$$J_2=\int_0^1\frac{x}{1+x^2}arctan^2(x)dx\ \{x=\tan(u)\}\to\int_0^{\frac{\pi}{4}}u^2\tan(u)\,du=$$

$$[-u^2\ln(\cos(u))]_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}u\ln(\cos(u))\,du=\frac{\pi^2}{32}\ln(2)+2\int_0^{\frac{\pi}{4}}u\ln(\cos(u))\,du$$

$$Using\ \ \ln(\cos(u))\ =\ -\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos(2nu)$$

$$2\int_0^{\frac{\pi}{4}}u\ln(\cos(u))\,du=-\frac{\pi^2}{16}\ln(2)+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{\pi}{4}}ucos(2nu)\,du=$$

$$-\frac{\pi^2}{16}\ln(2)+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\pi}{8}\sin\left(\frac{\pi n}{2}\right)+\frac{\cos\left(\frac{\pi n}{2}\right)-1}{4n^2}\right)=$$

$$-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{n-1}\left(\cos\left(\frac{\pi n}{2}\right)-1\right)}{n^3}=$$

$$-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}G+\frac{1}{2}\left(-\frac{7}{8}\zeta(3)-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k-1}{k^3}\right)=$$

$$-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}G+\frac{1}{2}\left(-\frac{7}{8}\zeta(3)+\frac{7}{32}\zeta(3)\right)=-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

$$J_2=\frac{\pi^2}{32}\ln(2)-\frac{\pi^2}{16}\ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

$$J_2=-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

$$I_1=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}\ln(2)-\frac{\pi^2}{32}\ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

$$I_2=\frac{1-\gamma-\ln(3)}{3}$$

$$\blacklozenge=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\left(\frac{\pi^2+16}{32}\right)\ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)+\frac{1-\gamma-\ln(3)}{3}$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Omega=\int_0^1x^2\left(arctan^3(x)+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx$$

$$\Omega=\int_0^1x^2\arctan^3(x)\,dx+\int_0^1x^2\log(-log(x))\,dx-\int_0^1x^2\log(1-x^3)\,dx$$

$$I=\xi_1+\xi_2-\xi_3$$

$$\blacksquare\ \xi_1=\int_0^1 x^2\arctan^3(x)\,dx$$

$$We\ do: t=\arctan(x)\to x=\tan(t)\to dx=\sec^2(t)dt$$

$$x=1\to t=\frac{\pi}{4}\wedge x=0\to t=0\Rightarrow \xi_1=\int_0^{\frac{\pi}{4}} t^3\tan^2(t)\sec^2(t)\,dt$$

$$I.B.P:\begin{cases} u=t^3\to du=3t^2dt \\ dv=\tan^2(t)\sec^2(t)\,dt\to v=\frac{1}{3}\tan^3(t)\end{cases}$$

$$\Rightarrow \xi_1=\left[\frac{t^3}{3}\tan^3(t)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} t^2\tan^3(t)\,dt=\frac{\pi^3}{192}-i_1$$

$$\gg i_1=\int_0^{\frac{\pi}{4}} t^2\tan^3(t)\,dt\xRightarrow{\tan^2(x)=\sec^2(x)-1} i_1=\int_0^{\frac{\pi}{4}} t^2\tan(t)\sec^2(t)\,dt-\int_0^{\frac{\pi}{4}} t^2\tan(t)dt$$

$$i_1=a-b\ggg a=\int_0^{\frac{\pi}{4}} t^2\tan(t)\sec^2(t)\,dt$$

$$I.B.P:\begin{cases} u=t^2\to du=2tdt \\ dv=\tan(t)\sec^2(t)\,dt\to v=\frac{1}{2}\tan^2(t)\end{cases}$$

$$\Rightarrow a=\left[\frac{t^2}{2}\tan^2(t)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} t\tan^2(t)\,dt=\frac{\pi^2}{32}-\int_0^{\frac{\pi}{4}} tsec^2(t)dt+\int_0^{\frac{\pi}{4}} tdt$$

$$a=\frac{\pi^2}{16}-\int_0^{\frac{\pi}{4}} tsec^2(t)dt$$

$$I.B.P:\begin{cases} u=t\to du=dt \\ dv=\sec^2(t)\,dt\to v=\tan(t)\end{cases}\Rightarrow a=\frac{\pi^2}{16}-[t\tan(t)]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\tan(t)\,dt$$

$$a=\frac{\pi^2}{16}-\frac{\pi}{4}+[-\log(cos(t))]_0^{\frac{\pi}{4}}=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log(2)\ \therefore\ggg b=\int_0^{\frac{\pi}{4}} t^2\tan(t)\,dt$$

$$I.B.P:\begin{cases} u=t^2\to du=2tdt \\ dv=\tan(t)\to v=-\log(cos(t))\end{cases}$$

$$\Rightarrow b=\left[-t^2\log(cos(t))\right]_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}tlog(\cos(t))dt=\frac{\pi^2}{32}\log(2)+2\int_0^{\frac{\pi}{4}}tlog(\cos(t))dt$$

$$Fourier\ Series:\ \log(cos(x))=-\log(2)-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\cos(2kx)$$

$$\Rightarrow b=\frac{\pi^2}{32}\log(2)-2\log(2)\int_0^{\frac{\pi}{4}}tdt-2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{4}}t\cos(2kt)\,dt$$

$$b=-\frac{\pi^2}{32}\log(2)-2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left[\frac{\pi}{8k}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{4k^2}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{4k^2}\right]$$

$$b=-\frac{\pi^2}{32}\log(2)-\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sin\left(\frac{\pi k}{2}\right)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\cos\left(\frac{\pi k}{2}\right)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$We\ do$:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sin\left(\frac{\pi k}{2}\right)\xRightarrow{k=2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\cos\left(\frac{\pi k}{2}\right)\xRightarrow{k=2n}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$\Rightarrow b=-\frac{\pi^2}{32}\log(2)+\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{7}{16}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$$

$$Dirichlet\ Beta\ Function:\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s},\qquad\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\beta(2)=G$$

$$Dirichlet\ Beta\ Function:\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\eta(3)=\frac{3}{4}\zeta(3)\Rightarrow b=-\frac{\pi^2}{32}\log(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)\ \therefore$$

$Therefore$:

$$i_1=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log(2)-\left(-\frac{\pi^2}{32}\log(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)\right)$$

$$i_1=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{32}\log(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\ \therefore$$

$This$:

$$\xi_1=\frac{\pi^3}{192}-\left(\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log(2)+\frac{\pi^2}{32}\log(2)-\frac{\pi}{4}G+\frac{21}{64}\zeta(3)\right)$$

$$\xi_1=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}\log(2)-\frac{\pi^2}{32}\log(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)\therefore$$

$$\blacksquare\xi_2=\int_0^1 x^2\log(-log(x))\,dx$$

$$We\ do: t=-\log(x)\to x=e^{-t}\to dx=-e^{-t}dt$$

$$x=1\to t=0\wedge x=0\to t=\infty\Rightarrow\xi_2=\int_0^\infty e^{-3t}\log(t)\,dt$$

$$We\ do: x=3t\to dt=\frac{dx}{3}$$

$$\Rightarrow\xi_2=\frac{1}{3}\int_0^\infty e^{-x}\log\left(\frac{x}{3}\right)dx=\frac{1}{3}\left(\int_0^\infty e^{-x}\log(x)\,dx-\log(3)\int_0^\infty e^{-x}dx\right)$$

$$\xi_2=\frac{1}{3}\left(\int_0^\infty e^{-x}\log(x)\,dx-\log(3)\right)$$

$$Gamma\ Function: \Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx\to\Gamma'(z)=\int_0^\infty x^{z-1}\log(x)e^{-x}dx$$

$$\xi_2=\frac{1}{3}(\Gamma'(1)-\log(3))=-\frac{\gamma}{3}-\frac{1}{3}\log(3)\therefore$$

$$\blacksquare\xi_3=\int_0^1 x^2\log(1-x^3)\,dx=-\sum_{n=1}^\infty\frac{1}{n}\int_0^1 x^{3n+2}dx=-\frac{1}{3}\sum_{n=1}^\infty\frac{1}{n(n+1)}=-\frac{1}{3}\therefore$$

Thus:

$$I=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}\log(2)-\frac{\pi^2}{32}\log(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\gamma}{3}-\frac{1}{3}\log(3)+\frac{1}{3}$$

Solution 3 by Lara Gangler-Hungary

$$\mathrm{I}=\int_0^1 x^2\left((\tan^{-1}x)^3+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx$$

$$\mathrm{I}=\int_0^1 x^2(\tan^{-1}x)^3dx+\int_0^1 x^2\ln\left(\frac{\ln(x)}{x^3-1}\right)dx=\mathrm{I}_1+\mathrm{I}_2$$

Calculating I_1 Integration by parts – 1:

$$\left(\frac{x^3}{3}(\tan^{-1}x)^3\right)'=x^2(\tan^{-1}x)^3+\frac{x^3}{3}3(\tan^{-1}x)^2\frac{1}{1+x^2}$$

$$\frac{x^3}{3}(\tan^{-1}x)^3\Big|_0^1=\mathrm{I}_1+\int_0^1 x^3(\tan^{-1}x)^2\frac{dx}{1+x^2}$$

$$\frac{\pi^3}{192} = I_1 + \int_0^1 x^3(\tan^{-1} x)^2 \frac{dx}{1+x^2} = I_1 + \int_0^1 (\tan^{-1} x)^2 \frac{(x^3+x-x)dx}{1+x^2}$$

$$= I_1 + \int_0^1 \left(x(\tan^{-1} x)^2 - \frac{x(\tan^{-1} x)^2}{1+x^2} \right) dx = I_1 + J_{11} - J_{12}$$

Calculating J_{11} Integration by parts – 2:

$$\left(\frac{x^2}{2}(\tan^{-1} x)^2\right)' = x(\tan^{-1} x)^2 + \frac{x^2}{2} 2\tan^{-1} x \frac{1}{1+x^2}$$

$$\frac{x^2}{2}(\tan^{-1} x)^2\Big|_0^1 = J_{11} + \int_0^1 x^2 \tan^{-1} x \frac{dx}{1+x^2} = J_{11} + \int_0^1 x^2 \tan^{-1} x \frac{(x^2+1-1)}{1+x^2} dx =$$

$$= J_{11} + \int_0^1 \left(\tan^{-1} x - \frac{\tan^{-1} x}{1+x^2}\right) dx$$

$$= J_{11} + \left[\left(x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right) - \frac{1}{2}(\tan^{-1} x)^2\right]\Big|_0^1$$

$$\frac{\pi^2}{32} = J_{11} + \frac{\pi}{4} - \frac{1}{2}\ln(2) - \frac{\pi^2}{32} \to J_{11} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

Calculating J_{12} Integration by parts – 3:

$$\left(\frac{1}{2}\ln(1+x^2)(\tan^{-1} x)^2\right)' = \frac{x(\tan^{-1} x)^2}{1+x^2} + \frac{1}{2}\ln(1+x^2)\, 2\frac{\tan^{-1} x}{1+x^2}$$

$$\left(\frac{1}{2}\ln(1+x^2)(\tan^{-1} x)^2\right)\Big|_0^1 = J_{12} + \int_0^1 \ln(1+x^2)\frac{\tan^{-1} x}{1+x^2} dx = J_{12} + K_{12}$$

$$\frac{\pi^2}{32}\ln(2) = J_{12} + K_{12}$$

Calculating K_{12}

$$K_{12} = \int_0^1 \ln(1+x^2)\frac{\tan^{-1} x}{1+x^2} dx$$

$$x = \tan y \to dx = \sec^2 y\, dy \ \&\ 1+x^2 = \sec^2 y$$

$$K_{12} = \int_0^{\pi/4} \ln(\sec^2 y)\frac{y}{\sec^2 y}\sec^2 y\, dy = -\int_0^{\frac{\pi}{4}} 2y\, ln(\cos y)\, dy = -2\, L_{12}$$

Calculating L_{12} $L_{12} = \int_0^{\frac{\pi}{4}} y\, ln(\cos y)\, dy = \int_0^{\pi/4} y\left(-\ln\, 2 + \sum_{k=1}^{\infty}\frac{\cos 2ky}{k}(-1)^{k+1}\right) dy = =$
$-\frac{1}{2}\left(\frac{\pi}{4}\right)^2 \ln 2 + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\int_0^{\frac{\pi}{4}} y\cos 2ky\, dy = = -\frac{\pi^2}{32}\ln(2) +$
$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\frac{2ky\sin 2ky + \cos 2ky}{(2k)^2}\Big|_0^{\frac{\pi}{4}} == -\frac{\pi^2}{32}\ln(2) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^3}\left(\frac{k\pi}{2}\sin\frac{k\pi}{2} + \cos\frac{k\pi}{2} - 1\right) = =$
$-\frac{\pi^2}{32}\ln(2) - \frac{1}{4}\times\frac{3}{4}\zeta(3) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^3}\left(\frac{k\pi}{2}\sin\frac{k\pi}{2} + \cos\frac{k\pi}{2}\right) = = -\frac{\pi^2}{32}\ln(2) - \frac{3\zeta(3)}{16} +$
$\sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{4(2k)^3}(k\pi\sin k\pi + \cos k\pi) + + \sum_{k=1}^{\infty}\frac{(-1)^{2k}}{4(2k-1)^3}\left(\frac{(2k-1)\pi}{2}\sin\frac{(2k-1)\pi}{2} +\right.$
$\left.\cos\frac{(2k-1)\pi}{2}\right) = -\frac{\pi^2}{32}\ln(2) - \frac{3\zeta(3)}{16} + \sum_{k=1}^{\infty}\left(\frac{(-1)^{2k+1}}{4(2k)^3}\cos k\pi + \frac{(-1)^{2k}}{4(2k-1)^3}\frac{(2k-1)\pi}{2}\sin\frac{(2k-1)\pi}{2}\right) =$

$$=-\frac{\pi^2}{32}\ln(2)-\frac{3\zeta(3)}{16}+\sum_{k=1}^{\infty}\left(\frac{-1}{32k^3}(-1)^k+\frac{\pi}{8(2k-1)^2}(-1)^{k-1}\right)==-\frac{\pi^2}{32}\ln(2)-\frac{3\zeta(3)}{16}+\frac{1}{32}\frac{3\zeta(3)}{4}+\frac{\pi G}{8}=\frac{\pi G}{8}-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{128}$$

$$\to K_{12}=\int_0^1\ln(1+x^2)\frac{\tan^{-1}x}{1+x^2}dx=\frac{\pi^2}{16}\ln(2)+\frac{21\zeta(3)}{64}-\frac{\pi G}{4}$$

$$\frac{\pi^2}{32}\ln(2)=J_{12}+K_{12}$$

$$\to J_{12}=\int_0^1\frac{x(\tan^{-1}x)^2}{1+x^2}dx=\frac{\pi G}{4}-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{64}$$

$$\frac{\pi^3}{192}=I_1+J_{11}-J_{12}$$

$$I_1=\frac{\pi^3}{192}+\frac{\pi G}{4}-\frac{\pi^2}{32}\ln(2)-\frac{21\zeta(3)}{64}-\left(\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\ln(2)\right)$$

$$\to I_1=\int_0^1 x^2(\tan^{-1}x)^3=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{\pi^2}{32}\ln(2)+\frac{\pi G}{4}-\frac{1}{2}\ln(2)-\frac{21\zeta(3)}{64}$$

Calculating I_2: $I_2=\int_0^1 x^2\ln\left(\frac{\ln(x)}{x^3-1}\right)dx=\int_0^1 x^2\ln\left(\frac{-\ln(x)}{1-x^3}\right)dx=I_{21}-I_{22}$

Calculating I_{21}: $u=-\ln(x)\to x=e^{-u}\to dx=-e^{-u}\,du$

$$I_{21}=\int_0^1 x^2\ln(-\ln(x))dx=\int_0^{\infty}e^{-2u}\ln(u)e^{-u}\,du=\int_0^{\infty}e^{-3u}\ln(u)du$$

$$w=3u\to du=\frac{dw}{3}$$

$$I_{21}=\int_0^{\infty}e^{-3u}\ln(u)du=\frac{1}{3}\int_0^{\infty}e^{-w}\ln\left(\frac{w}{3}\right)dw=\frac{1}{3}\int_0^{\infty}e^{-w}(\ln(w)-\ln(3))dw$$

$$=-\frac{\gamma+\ln(3)}{3}$$

Calculating I_{22}

$$I_{22}=\int_0^1 x^2\ln(1-x^3)dx=\int_0^1-x^2\sum_{k=1}^{\infty}\frac{x^{3k}}{k}dx=\sum_{k=1}^{\infty}-\frac{1}{k}\int_0^1 x^{3k+2}dx=\sum_{k=1}^{\infty}-\frac{1}{k}\frac{1}{3k+3}$$

$$=-\frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=-\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=-\frac{1}{3}$$

$$I_2=I_{21}-I_{22}=-\frac{\gamma+\ln(3)}{3}+\frac{1}{3}=\frac{1-\gamma-\ln(3)}{3}$$

$$I=I_1+I_2=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{\pi^2}{32}\ln(2)+\frac{\pi G}{4}-\frac{1}{2}\ln(2)+\frac{\pi G}{4}-\frac{21\zeta(3)}{64}+\frac{1-\gamma-\ln(3)}{3}$$

$$\to I=\int_0^1 x^2\left((\tan^{-1}x)^3+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx=$$

$$I=\frac{1}{3}+\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{\pi^2}{32}\ln(2)+\frac{\pi G}{4}-\frac{1}{2}\ln(2)-\frac{1}{3}\ln(3)-\frac{21\zeta(3)}{64}-\frac{\gamma}{3}$$

Solution 4 by Samuel Ajibade-Nigeria

$$I=\int_0^1 x^2\left(\arctan^3(x)+\ln\left(\frac{\ln(x)}{x^3-1}\right)\right)dx$$

$$I=\int_0^1 x^2 arctan^3(x)dx+\int_0^1 x^2 ln\left(\frac{ln(x)}{x^3-1}\right)dx$$

$$:I=J+K\quad\cdots\ equation\ 1, J=\int_0^1 x^2arctan^3(x)dx$$

$$Using\ IBP: u=arctan^3(x), dv=x^2,\quad du=\frac{3arctan^2(x)}{1+x^2}dx, v=\frac{x^3}{3}$$

$$J=\frac{x^3}{3}arctan^3(x)\Big|_0^1-\int_0^1\frac{x^3}{1+x^2}arctan^2(x)dx$$

$$J=\frac{\pi^3}{192}-\int_0^1\left(x-\frac{x}{1+x^2}\right)arctan^2(x)dx$$

$$J=\frac{\pi^3}{192}-\int_0^1 x\,arctan^2(x)dx+\int_0^1\frac{x\,arctan^2(x)}{1+x^2}dx$$

$$Let\ J=\frac{\pi^3}{192}-J_1+J_2\quad\cdots\ equation\ 2,\quad J_1=\int_0^1 x\,arctan^2(x)dx$$

$$IBP: u=arctan^2(x), du=\frac{2\,arctan(x)}{1+x^2}, v=\frac{x^2}{2}$$

$$J_1=\frac{x^2}{2}arctan^2(x)\Big|_0^1-\int_0^1\frac{x^2\,arctan(x)}{1+x^2}dx$$

$$J_1=\frac{\pi^2}{32}-\int_0^1 arctan(x)dx+\int_0^1\frac{arctan(x)}{1+x^2}dx$$

$$J_1=\frac{\pi^2}{32}-\left(x\,arctan(x)-\frac{1}{2}ln(1+x^2)\right)\Big|_0^1+\frac{1}{2}arctan^2(x)\Big|_0^1$$

$$J_1=\frac{\pi^2}{32}-\frac{\pi}{4}+\frac{1}{2}ln(2)+\frac{\pi^2}{32}$$

$$J_1=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}ln(2)$$

$$J_2=\int_0^1\frac{x\,arctan^2(x)}{1+x^2}dx$$

$$Let\ t=arctan(x), dt=\frac{dx}{1+x^2}, t\in\left[0,\frac{\pi}{4}\right]$$

$$J_2=\int_0^{\frac{\pi}{4}}t^2\ \tan(t)\,dt$$

$$By\ IBP: u=t^2, du=2tdt, v=-ln(\cos(t))$$

$$J_2=-t^2ln(\cos(t))\Big|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}t\,ln(\cos(t))\,dt$$

By Fourier expansion of $ln(cos(t))$

$$J_2=\frac{\pi^2}{32}ln(2)-2\int_0^{\frac{\pi}{4}}t\left(ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}cos(2nt)\right)dt$$

$$J_2=\frac{\pi^2}{32}ln(2)-2ln(2)\int_0^{\frac{\pi}{4}}t\,dt\ -2\int_0^{\frac{\pi}{4}}t\sum_{n=1}^{\infty}\frac{(-1)^n}{n}cos(2nt)\,dt$$

Fubini – Tonelli Theorem

$$J_2=\frac{\pi^2}{32}ln(2)-ln(2)\frac{\pi^2}{16}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}t\,cos(2nt)\,dt$$

$$By\ IBP: u=t, v=\frac{sin(2nt)}{2n}\quad\left\{\begin{array}{l}NB: sin\left(\frac{\pi}{2}n\right)=\begin{cases}0 & n\ is\ even\\ \pm1 & n\ is\ odd\end{cases}\\ \therefore\ sin\left(\frac{\pi}{2}(2k+1)\right)=(-1)^k\\ cos\left(\frac{\pi}{2}n\right)=\begin{cases}0 & n\ is\ odd\\ \pm1 & n\ is\ even\end{cases}\\ \therefore\ cos\left(\frac{\pi}{2}(2k)\right)=(-1)^k\end{array}\right.$$

$$J_2=-\frac{\pi^2}{32}ln(2)-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(t\frac{sin(2nt)}{2n}+\frac{cos(2nt)}{4n^2}\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$J_2=-\frac{\pi^2}{32}ln(2)-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\pi\,sin\left(\frac{\pi}{2}n\right)}{8n}+\frac{cos\left(\frac{\pi}{2}n\right)-1}{4n^2}\right)$$

$$J_2=-\frac{\pi^2}{32}ln(2)-\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}sin\left(\frac{\pi}{2}n\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}cos\left(\frac{\pi}{2}n\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

For the first series, let $n=2k+1$ (the odd integer sub)

For the second series, let $n=2m$ (the even integer sub): $n\geq 2$

$$J_2=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}-\frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{8m^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

$$J_2=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G+\frac{1}{16}\eta(3)-\frac{1}{2}\eta(3)$$

$$J_2=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{7}{16}\eta(3)$$

$$J_2=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

Combining J_1 & J_2 to get J from equation 2

$$J=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}ln(2)-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)$$

Now solving for K

$$K=\int_0^1 x^2 ln\left(\frac{ln(x)}{x^3-1}\right)dx=\int_0^1 x^2 ln\left(\frac{-ln(x)}{1-x^3}\right)dx$$

$$K=\int_0^1 x^2 ln(-ln(x))\,dx-\int_0^1 x^2 ln(1-x^3)\,dx$$

Let $K=K_1-K_2$ $\cdots$ *equation 3*

$$K_1=\int_0^1 x^2 ln(-ln(x))\,dx$$

$$x=e^{-t}, dx=-e^{-t}dt, t\in(0,\infty)$$

$$K_1=\int_0^\infty ln(t)\,e^{-3t}dt$$

Evaluating via the derivative of the gamma function

$$K_1=\frac{\partial}{\partial a}\bigg|_{a=0}\int_0^\infty t^a\,e^{-3t}dt=\frac{1}{3}\frac{\partial}{\partial a}\frac{\Gamma(a+1)}{3^a}\bigg|_{a=0}$$

$$K_1=\frac{1}{3}\frac{\Gamma(a+1)}{3^a}(\psi(a+1)-ln(3))|_{a=0}$$

$$K_1=\frac{1}{3}(\psi(1)-ln(3))$$

$$K_1=-\frac{\gamma}{3}-\frac{ln(3)}{3},\qquad K_2=\int_0^1 x^2 ln(1-x^3)\,dx$$

$$t=1-x^3, dt=-3x^2dx$$

$$K_2=\frac{1}{3}\int_0^1 ln(t)dt=\frac{t}{3}(ln(t)-1)\bigg|_0^1,\qquad K_2=-\frac{1}{3}$$

$$\therefore$$

$$K=\frac{1}{3}-\frac{\gamma}{3}-\frac{ln(3)}{3}$$

Finally: From equation 1, $I=J+K$

$$I=\frac{\pi^3}{192}-\frac{\pi^2}{16}+\frac{\pi}{4}-\frac{1}{2}ln(2)-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)+\frac{1}{3}-\frac{\gamma}{3}-\frac{ln(3)}{3}$$

$$\boxed{I=\frac{1}{192}(\pi^3-12\pi^2-6\pi^2ln(2)+48\pi-96ln(2)+48\pi G-63\zeta(3)-64\gamma-64ln(3)+64)\blacksquare}$$

2873. ***Prove that:***

$$\int_0^\infty\frac{x^2}{(x^2+4)^2}dx+\int_0^\infty\frac{y^2}{4(y^2+25)^2}dy+\int_0^\infty\frac{z^2}{9(z^2+100)^2}dz+\cdots=$$

$$=\frac{\pi^2}{24}\left(\pi-3\coth\pi+\frac{3}{\pi}\right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Omkumar Rao-India

$$K_n = \frac{1}{n^2}\int_0^\infty \frac{x^2}{(x^2+k^2)^2}dx \qquad : (k = n^2+1)$$

$$K_n \overset{x\to k\tan x}{\Rightarrow} \frac{1}{kn^2}\int_0^{\frac{\pi}{2}} (\sin x)^2\, dx = \frac{\pi}{4n^2(n^2+1)}$$

$$S = \sum_{n=1}^{\infty} \frac{\pi}{4n^2(n^2+1)} = \frac{\pi}{4}\left(\frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2+1}\right)$$

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2-n^2\pi^2} \overset{x\to i\pi}{\Rightarrow} \coth(\pi) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{\pi\coth(\pi)-1}{2}$$

$$S = \frac{\pi}{4}\left(\frac{\pi^2}{6} - \frac{\pi\coth(\pi)-1}{2}\right) = \frac{\pi^2}{24}\left(\pi - 3\coth\pi + \frac{3}{\pi}\right)$$

2874. Find:

$$I = \int_0^{+\infty}\frac{dx}{\sqrt{x^7+1}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Nawar Alasadi-Iraq

$$I = \int_0^{+\infty}\frac{dx}{\sqrt{x^7+1}}$$

$$x^7 = t \Rightarrow x = t^{\frac{1}{7}} \Rightarrow dx = \frac{1}{7}t^{-\frac{6}{7}}dt$$

$$I = \frac{1}{7}\int_0^{+\infty}\frac{t^{-\frac{6}{7}}}{\sqrt{t+1}}dt$$

$$= \frac{1}{7}\int_0^{+\infty}\frac{t^{\frac{1}{7}-1}}{(1+t)^{\frac{1}{7}+\frac{5}{14}}}dt$$

$$\mathrm{B}(x,y) = \int_0^{+\infty}\frac{u^{x-1}}{(1+u)^{x+y}}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\begin{aligned} I &= \frac{1}{7}\mathrm{B}\left(\frac{1}{7},\frac{5}{14}\right) \\ &= \frac{1}{7}\frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{5}{14}\right)}{\Gamma\left(\frac{1}{7}+\frac{5}{14}\right)} \\ &= \frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{5}{14}\right)}{7\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{5}{14}\right)}{7\sqrt{\pi}} \end{aligned}$$

2875. ***Find***:

$$\int_0^{\frac{\pi}{2}} \frac{(\sin x)^2(\sin x-\cos x)^{2026}}{(\sin x+\cos x)^{2028}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Omkumar Rao-India

$$\int_0^{\frac{\pi}{2}} \frac{(\sin x)^2(\sin x-\cos x)^{2026}}{(\sin x+\cos x)^{2028}}dx \overset{x\to\frac{\pi}{2}-x}{\Longrightarrow} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^2(\sin x-\cos x)^{2026}}{(\sin x+\cos x)^{2028}}dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)^{2026}\cdot\frac{1}{(\sin x+\cos x)^2}dx =$$

$$\overset{\frac{\sin x-\cos x}{\sin x+\cos x}\to y}{\Longrightarrow} \int_{-1}^{1}\frac{y^{2026}}{2}dy = \frac{1}{4054}$$

Observation:

$$dy = \left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)' dx$$

$$= \frac{(cosx+sinx)(\sin x+\cos x)-(\sin x-\cos x)(cosx-sinx)}{(\sin x+\cos x)^2}dx =$$

$$= \frac{(\sin x+\cos x)^2+(\sin x-\cos x)^2}{(\sin x+\cos x)^2}dx = \frac{2}{(\sin x+\cos x)^2}dx$$

2876. Find a closed form:

$$K=\int_0^\infty \frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}dx$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^\infty \frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}dx \overset{x\to x^2}{\cong} 8\int_0^\infty \frac{(\ln x)^2 x^2}{(x^4+1)(x^2+1)}dx$$

$$=8\left(\int_0^1 \frac{(\ln x)^2 x^2}{(x^4+1)(x^2+1)}dx+\underbrace{\int_1^\infty \frac{(\ln x)^2 x^2}{(x^4+1)(x^2+1)}dx}_{x\to\frac{1}{x}}\right)$$

$$K=16\int_0^1 \frac{(\ln x)^2 x^2}{(x^4+1)(x^2+1)}dx=8\int_0^1 \frac{x^2+1}{x^4+1}(\ln x)^2\,dx-8\int_0^1 \frac{(\ln x)^2}{x^2+1}dx$$

$$\int_0^1 \frac{x^2+1}{x^4+1}(\ln x)^2\,dx=\underbrace{\int_0^1 \frac{(\ln x)^2 x^2}{x^4+1}dx}_{x\to\frac{1}{x}}+\int_0^1 \frac{(\ln x)^2}{x^4+1}dx=\int_0^\infty \frac{(\ln x)^2}{x^4+1}dx=\frac{\partial^2\big(P(m,n)\big)}{\partial m^2}$$

$$P(m,n)=\int_0^\infty \frac{x^m}{x^n+1}=\frac{\pi}{n\sin\left(\frac{\pi(m+1)}{n}\right)}$$

$$K=8\left(P''(0,4)-\frac{\pi^3}{16}\right)=8\left(\frac{3\sqrt{2}\pi^3}{64}-\frac{\pi^3}{16}\right)=\frac{\pi^3}{8}(3\sqrt{2}-4)$$

$$\boxed{K=\int_0^\infty \frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}dx=\frac{\pi^3}{8}(3\sqrt{2}-4)}$$

2877. ***Find*** **:**

$$\Delta=\int_0^1 \frac{ln^2(1+x)ln^2(1-x)}{1+x}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$\#Note: a^2b^2 = \frac{1}{12}[(a+b)^4+(a-b)^4-2a^4-2b^4]$$

$$I=\frac{1}{12}\left[\int_0^1 \frac{log^4(1-x^2)}{1+x}dx+\int_0^1 \frac{\log^4\left(\frac{1-x}{1+x}\right)}{1+x}dx-2\int_0^1 \frac{\log^4(1-x)}{1+x}dx\right.$$
$$\left.-2\int_0^1 \frac{\log^4(1+x)}{1+x}\right]$$

$$I=\frac{1}{12}(i_1+i_2-2i_3-2i_4)$$

$$*\, i_4=\int_0^1 \frac{\log^4(1+x)}{1+x}=\left[\frac{1}{5}log^5(1+x)\right]_0^1=\frac{1}{5}log^5(2)\ \therefore$$

$$*\, i_3=\int_0^1 \frac{log^4(1-x)}{1+x}dx \xRightarrow{t=1-x\to dt=-dx} A_3=\int_0^1 \frac{log^4(t)}{2-t}dt=\frac{1}{2}\int_0^1 \frac{log^4(t)}{1-\frac{t}{2}}dt$$

$$i_3=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}\int_0^1 t^{n-1}\,log^4(t)dt$$

$$\#Note: \int_0^1 x^p\ln^q(x)dx=\frac{(-1)^q q!}{(p+1)^{q+1}}, i_3=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{24}{n^5}\right)=24\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^5}=24Li_5\left(\frac{1}{2}\right)\ \therefore$$

$$*\, i_2=\int_0^1 \frac{\log^4\left(\frac{1-x}{1+x}\right)}{1+x}dx,\ let: y=\frac{1-x}{1+x}\to dx=-\frac{2dy}{(1+y)^2}$$

$$i_2=\int_0^1 \frac{\log^4(y)}{1+y}dy=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 y^{n-1}\log^4(y)\,dy=24\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5}=24\eta(5)$$
$$=\frac{45}{2}\zeta(5)\ \therefore$$

$$*\, i_1=\int_0^1 \frac{log^4(1-x^2)}{1+x}dx=\int_0^1 \frac{(1-x)log^4(1-x^2)}{1-x^2}dx,\ let: x=t^{\frac{1}{2}}\to dx=\frac{1}{2}t^{-\frac{1}{2}}dt$$

$$i_1=\frac{1}{2}\int_0^1\left(\frac{1-t^{\frac{1}{2}}}{t^{\frac{1}{2}}}\right)\frac{log^4(1-t)}{1-t}dt,\ I.B.P:\begin{cases} u=\dfrac{1-\sqrt{t}}{\sqrt{t}}\to du=-\dfrac{1}{2t^{\frac{3}{2}}} \\ dv=\dfrac{log^4(1-t)}{1-t}dt\to v=-\dfrac{1}{5}log^5(1-t)\end{cases}$$

$$i_1=\frac{1}{2}\left(\left[-\frac{1}{5}\log^5(1-t)\left(\frac{1-\sqrt{t}}{\sqrt{t}}\right)\right]_0^1-\frac{1}{10}\int_0^1 t^{-\frac{3}{2}}\log^5(1-t)\,dt\right)$$

$$i_1=-\frac{1}{20}\int_0^1 t^{-\frac{3}{2}}\log^5(1-t)\,dt$$

$$\#Note: B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\to\frac{\partial}{\partial y}B(x,y)$$

$$=\int_0^1 t^{x-1}(1-t)^{y-1}\log(1-t)\,dt$$

$$i_1=-\frac{1}{20}\lim_{(x,y)\to\left(-\frac{1}{2},1\right)}\frac{\partial}{\partial y^5}\left(\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right)=\frac{\sqrt{\pi}}{10}\lim_{y\to 1}\frac{\partial}{\partial y^5}\left(\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\right)$$

$$Where: \frac{\partial}{\partial y^5}\left(\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\right)=\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(-\psi^{(0)}\left(y-\frac{1}{2}\right)^5+5\psi^{(0)}(y)\psi^{(0)}\left(y-\frac{1}{2}\right)\right)$$

$$-10\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(0)}(y)^2-\psi^{(1)}\left(y-\frac{1}{2}\right)+\psi^{(1)}(y)\right)\psi^{(0)}\left(y-\frac{1}{2}\right)^3$$

$$+10\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(0)}(y)^3-3\left(\psi^{(1)}\left(y-\frac{1}{2}\right)-\psi^{(1)}(y)\right)\psi^{(0)}(y)\right)\psi^{(0)}\left(y-\frac{1}{2}\right)^2$$

$$-\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(2)}\left(y-\frac{1}{2}\right)\mp\psi^{(2)}(y)\right)\psi^{(0)}\left(y-\frac{1}{2}\right)^2$$

$$-5\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(0)}(y)^4-6\left(\psi^{(1)}\left(y-\frac{1}{2}\right)-\psi^{(1)}(y)\right)\psi^{(0)}(y)^2\right)\psi^{(0)}\left(y-\frac{1}{2}\right)$$

$$+20\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(2)}\left(y-\frac{1}{2}\right)-\psi^{(2)}(y)\right)\psi^{(0)}(y)\psi^{(0)}\left(y-\frac{1}{2}\right)$$

$$-5\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(3\psi^{(1)}\left(y-\frac{1}{2}\right)^2+3\psi^{(1)}(y)^2-6\psi^{(1)}\left(y-\frac{1}{2}\right)\psi^{(1)}(y)\right)\psi^{(0)}\left(y-\frac{1}{2}\right)$$

$$+5\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(3)}\left(y-\frac{1}{2}\right)-\psi^{(3)}(y)\right)\psi^{(0)}\left(y-\frac{1}{2}\right)$$

$$+\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(\psi^{(0)}(y)^5-10\psi^{(0)}(y)^3\left(\psi^{(1)}\left(y-\frac{1}{2}\right)-\psi^{(1)}(y)\right)\right)$$

$$+\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(10\psi^{(1)}\left(y-\frac{1}{2}\right)\psi^{(2)}\left(y-\frac{1}{2}\right)-10\psi^{(1)}(y)\psi^{(2)}\left(y-\frac{1}{2}\right)\right)$$

$$-\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(10\psi^{(0)}(y)^2\left(\psi^{(2)}\left(y-\frac{1}{2}\right)-\psi^{(2)}(y)\right)+10\psi^{(1)}\left(y-\frac{1}{2}\right)\psi^{(2)}(y)\right)$$

$$+\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\left(10\psi^{(1)}(y)\psi^{(2)}(y)-\psi^{(4)}\left(y-\frac{1}{2}\right)+\psi^{(4)}(y)\right)$$

$$+5\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\psi^{(0)}(y)\left(3\psi^{(1)}\left(y-\frac{1}{2}\right)^2-6\psi^{(1)}(y)\psi^{(1)}\left(y-\frac{1}{2}\right)+3\psi^{(1)}(y)^2\right)$$

$$-\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\psi^{(0)}(y)\left(\psi^{(3)}\left(y-\frac{1}{2}\right)-\psi^{(3)}(y)\right)$$

$$\lim_{y\to 1}\frac{\partial}{\partial y^5}\left(\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)}\right)=\frac{-10\pi^2(12\zeta(3)+\log^3(4))+3(720\zeta(5)+120\zeta(3)\log^2(4)+\log^5(4))-9\pi^4\log(4))}{3\sqrt{\pi}}$$

$$i_1=-\frac{\pi^2}{3}(12\zeta(3)+\log^3(4))+\frac{1}{10}\left(720\zeta(5)+120\zeta(3)\log^2(4)+\log^5(4)\right)-\frac{3\pi^4}{10}\log(4)$$

Therefore:

$$I=\frac{1}{12}\left(-\frac{\pi^2}{3}(12\zeta(3)+\log^3(4))+\frac{1}{10}\left(720\zeta(5)+120\zeta(3)\log^2(4)+\log^5(4)\right)-\frac{3\pi^4}{10}\log(4)\right)+\frac{1}{12}\left(\frac{45}{2}\zeta(5)\right)-\frac{1}{6}\left(24Li_5\left(\frac{1}{2}\right)\right)-\frac{1}{6}\left(\frac{1}{5}log^5(2)\right)$$

$$I=-4Li_5\left(\frac{1}{2}\right)+\frac{63}{8}\zeta(5)+4\zeta(3)\log^2(2)-\frac{\pi^2}{3}\zeta(3)+\frac{7}{30}\log^5(2)-\frac{2\pi^2}{9}\log^3(2)-\frac{9}{2}\log(2)\zeta(4)$$

Solution 2 by Nawar Alasadi-Iraq

$$Let: a=\ln(1+x)\ \ and\ \ \ln(1-x)$$

$$\begin{cases}(a+b)^4=a^4+4a^3b+6a^2b^2+4b^3a+b^4\\(a-b)^4=a^4-4a^3b+6a^2b^2-4b^3a+b^4\\(a+b)^4+(a-b)^4=2a^4+12a^2b^2+2b^4\end{cases}$$

$$\to\ a^2b^2=\frac{1}{12}[(a+b)^4+(a-b)^4-2a^4-2b^4]$$

$$ln^2(1+x)ln^2(1-x)=$$

$$\frac{1}{12}\left[ln^4(1-x^2)+ln^4\left(\frac{1+x}{1-x}\right)-2ln^4(1+x)-2ln^4(1-x)\right]$$

$$Let: I=\int_0^1\frac{ln^2(1+x)ln^2(1-x)}{1+x}dx$$

$$I=\frac{1}{12}I_1+\frac{1}{12}I_2-\frac{1}{12}I_3-\frac{1}{6}I_4$$

$$Where:\begin{cases} I_1=\int_0^1\frac{ln^4(1-x^2)}{1+x}dx\\ I_2=\int_0^1\frac{ln^4\left(\frac{1+x}{1-x}\right)}{1+x}dx\\ I_3=\int_0^1\frac{ln^4(1-x)}{1+x}dx\\ I_4=\int_0^1\frac{ln^4(1+x)}{1+x}dx\end{cases}$$

Evaluating I_4 : *Let* $u=\ln(1+x)\to\quad du=\frac{1}{1+x}dx$

$$I_4=\int_0^{\ln(2)}u^4du=\frac{ln^5(2)}{5}$$

Evaluating I_3 : *Let* $u=1-x\to\quad dx=-du\,,1+x=2-u$

$$I_3=\int_0^1\frac{ln^4(u)}{2-u}du=\frac{1}{2}\int_0^1\frac{ln^4(u)}{1-\frac{u}{2}}du=$$

$$\frac{1}{2}\int_0^1 ln^4(u)\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^u du=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\int_0^1 u^n ln^4(u)du=$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\frac{24}{(n+1)^5}=24\sum_{m=1}^{\infty}\frac{1}{2^m m^5}=24Li_5\left(\frac{1}{2}\right)$$

Evaluating I_2 : *Let* $y=\frac{1-x}{1+x}\to\quad dx=\frac{-2}{(1+y)^2}dy,1+x=\frac{2}{1+y}$

$$I_2=\int_0^1\frac{ln^4(y)}{\frac{2}{1+y}}\frac{2}{(1+y)^2}dy=\int_0^1\frac{ln^4(y)}{1+y}dy=$$

$$-\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^n ln^4(y)\,dy=\sum_{n=0}^{\infty}(-1)^n\frac{24}{(1+n)^5}=$$

$$24\sum_{n=1}^{\infty}\frac{(-1)^{m-1}}{m^5}=24\left(1-2^{1-5}\right)\zeta(5)=\frac{45}{2}\zeta(5)$$

Evaluating I_1 : *Let* $t=1-x^2\to\quad dx=-\frac{1}{2\sqrt{1-t}}dt,1+x=\sqrt{1-t}$

$$I_1=\int_0^1\frac{ln^4(t)}{1+\sqrt{1-t}}\frac{dt}{2\sqrt{1-t}}=\frac{1}{2}\int_0^1\frac{1-\sqrt{1-t}}{t\sqrt{1-t}}ln^4(t)dt=$$

$$\frac{1}{2}\int_0^1\left(\frac{1}{\sqrt{1-t}}-1\right)\frac{ln^4(t)}{t}dt$$

$$Define\ \ F(a)=\int_0^1\left((1-t)^{-\frac{1}{2}}-1\right)t^{a-1}dt$$

$$F(a)=\frac{\Gamma(a)\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)}-\frac{1}{a}=\frac{G(a)-1}{a}$$

$$Where\ :\ \ G(a)=\frac{\Gamma(1+a)\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)}=\frac{2^{2a}\Gamma(1+a)^2}{\Gamma(1+2a)}$$

$$In\ \ G(a)=2aln(2)+2ln\Gamma(1+a)-ln\Gamma(1+2a)=$$

$$2\,(\ln 2)a-\zeta(2)a^5+2\zeta(2)a^3-\frac{7}{2}\zeta(4)a^4+6\zeta(5)a^5+o(a^6)$$

$$I_1=\frac{1}{2}F^{(4)}(0)=\frac{1}{10}G^{(5)}(0)=12\left[a^5\right]G(a)$$

$$Let\ \ S(a)=c_1a+c_2a^2+c_3a^3+c_4a^4+c_5a^5$$

$$c_1=2ln(2),c_2=-\zeta(2)\,,c_3=2\zeta(3),\qquad c_4=-\frac{7}{2}\zeta(4),c_5=6\,\zeta(5)$$

$$\left[a^5\right]e^{S(a)}=c_5+c_1c_4+c_2c_3+\frac{1}{2}{c_1}^2c_3+\frac{1}{2}{c_2}^2c_1+\frac{1}{6}{c_1}^3c_2+\frac{1}{120}{c_1}^5=$$

$$6\,\zeta(5)-7\zeta(4)\ln(2)-2\zeta(2)\zeta(3)+4\zeta(3)ln^2(2)+$$

$$\frac{5}{2}\zeta(4)\ln(2)-\frac{4}{3}\zeta(2)ln^3(2)+\frac{4}{15}ln^5(2)$$

$$I_1=72\zeta(5)-54\zeta(4)\ln(2)-24\zeta(2)\zeta(3)+$$

$$48\zeta(3)ln^2(2)-16\zeta(2)ln^3(2)+\frac{16}{5}ln^5(2)$$

$$Finally\,,\qquad substituting\ \ all\ \ parts\ \ I:$$

$$I=\frac{1}{12}I_1+\frac{1}{12}I_2-\frac{1}{12}I_3-\frac{1}{6}I_4$$

$$I=\left(6+\frac{15}{8}\right)\zeta(5)-4Li_5\left(\frac{1}{2}\right)-\frac{9}{2}\zeta(4)\ln(2)-2\zeta(2)\zeta(3)+$$

$$4\zeta(3)ln^2(2)-\frac{4}{3}\zeta(2)ln^3(2)+\left(\frac{4}{15}-\frac{1}{30}\right)ln^5(2)$$

$$Using\ :\ \zeta(2)=\frac{\pi^2}{6}\ \ and\ \ \zeta(4)=\frac{\pi^4}{90}$$

$$I=\frac{63}{8}\zeta(5)-4Li_5\left(\frac{1}{2}\right)-\frac{\pi^4}{20}\ln(2)-\frac{\pi^2}{3}\zeta(3)+$$

$$4\zeta(3)ln^2(2)-\frac{2\pi^2}{9}ln^3(2)+\frac{7}{30}ln^5(2)$$

2878. Find:

$$I\quad=\int_0^1\frac{\ln^2(x)\ln^4(1-x)}{x(1-x)}\,dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Nawar Alasadi-Iraq

$$I = \int_0^1 \frac{\ln^2(x)\ln^4(1-x)}{x}\,dx + \int_0^1 \frac{\ln^4(x)\ln^2(1-x)}{x}\,dx = I_1 + I_2$$

$$G(a,b) = \int_0^1 x^{a-1}(1-x)^b\,dx = \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)} \Rightarrow \ln\big(aG(a,b)\big) \equiv S(a,b)$$

$$S(a,b) = \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)}{k}\big(a^k + b^k - (a+b)^k\big) \Rightarrow G(a,b) = \frac{1}{a}\exp\big(S(a,b)\big)$$

$$I_1 = \frac{\partial^2}{\partial a^2}\frac{\partial^4}{\partial b^4}G(a,b)|_{a=b=0} = 48[a^3b^4]e^S, \quad I_2 = \frac{\partial^4}{\partial a^4}\frac{\partial^2}{\partial b^2}G(a,b)|_{a=b=0} = 48[a^5b^2]e^S$$

$$S(a,b) = -\zeta(2)ab + \zeta(3)ab(a+b) - \frac{\zeta(4)}{2}ab(2a^2+3ab+2b^2) + \cdots$$

$$e^S = [a^3b^4]\left(S + \frac{S^2}{2} + \frac{S^3}{6}\right) = 5\zeta(7) - 2\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) + \frac{1}{2}\zeta(2)^2\zeta(3)$$

$$[a^5b^2]e^S = [a^5b^2]\left(S + \frac{S^2}{2}\right) = 3\zeta(7) - \zeta(2)\zeta(5) - \zeta(3)\zeta(4)$$

$$\begin{aligned} I &= 48\big([a^3b^4]e^S + [a^5b^2]e^S\big) \\ &= 48\left(8\zeta(7) - 3\zeta(2)\zeta(5) - \frac{7}{2}\zeta(3)\zeta(4) + \frac{1}{2}\zeta(2)^2\zeta(3)\right) \\ &= 384\zeta(7) - 144\zeta(2)\zeta(5) - 168\zeta(3)\zeta(4) + 24\zeta(2)^2\zeta(3) \end{aligned}$$

Substitute $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}$:

$$\begin{aligned} I &= 384\zeta(7) - 144\left(\frac{\pi^2}{6}\right)\zeta(5) - 168\left(\frac{\pi^4}{90}\right)\zeta(3) + 24\left(\frac{\pi^4}{36}\right)\zeta(3) \\ &= 384\zeta(7) - 24\pi^2\zeta(5) - \frac{6}{5}\pi^4\zeta(3) \end{aligned}$$

Solution 2 by Yang Silva Cartolin-Peru

$$I = \int_0^1 \frac{\log^2(x)\log^4(1-x)}{x}\,dx + \int_0^1 \frac{\log^2(x)\log^4(1-x)}{1-x}\,dx$$

$$I = \xi_1 + \xi_2$$

$We\ will\ use\ the\ following\ generalization$:

$$\log^{\lambda}(1-z)=(-1)^{\lambda}\lambda!\sum_{n=\lambda}^{\infty}\frac{\begin{bmatrix}n\\\lambda\end{bmatrix}}{n!}z^n=(-1)^{\lambda}\lambda!\sum_{n=1}^{\infty}\frac{H_n^{[\lambda-1]}}{n}z^n$$

Where $\begin{bmatrix}n\\\lambda\end{bmatrix}$ *are the Stirling numbers of the first kind and* H_n *is the harmonic number*

$$\overset{\lambda=4}{\Longrightarrow}\log^4(1-z)=4\sum_{n=1}^{\infty}\frac{H_{n-1}^3-3H_{n-1}H_{n-1}^{(2)}+2H_{n-1}^{(3)}}{n}z^n$$

$$\log^4(1-z)=4\sum_{n=1}^{\infty}\frac{\left(H_n-\frac{1}{n}\right)^3-3\left(H_n-\frac{1}{n}\right)\left(H_n^{(2)}-\frac{1}{n^2}\right)+2\left(H^{(3)}-\frac{1}{n^3}\right)}{n}z^n$$

$$\log^4(1-z)=4\sum_{n=1}^{\infty}\frac{H_n^3-\frac{3H_n^2}{n}+\frac{3H_n}{n^2}-\frac{1}{n^3}}{n}z^n-12\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}-\frac{H_n}{n^2}-\frac{H_n^{(2)}}{n}+\frac{1}{n^3}}{n}z^n+8\sum_{n=1}^{\infty}\frac{H^{(3)}-\frac{1}{n^3}}{n}z^n$$

$\therefore$

Therefore:

$$\blacksquare\xi_1=\int_0^1\frac{\log^2(x)\log^4(1-x)}{x}dx$$

$$\xi_1=4\sum_{n=1}^{\infty}\frac{H_n^3-\frac{3H_n^2}{n}+\frac{3H_n}{n^2}-\frac{1}{n^3}}{n}\int_0^1x^{n-1}\log^2(x)\,dx$$

$$-12\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}-\frac{H_n}{n^2}-\frac{H_n^{(2)}}{n}+\frac{1}{n^3}}{n}\int_0^1x^{n-1}\log^2(x)\,dx+8\sum_{n=1}^{\infty}\frac{H^{(3)}-\frac{1}{n^3}}{n}\int_0^1x^{n-1}\log^2(x)\,dx$$

$$\textit{We know that}:\int_0^1\mathrm{x}^{\mathrm{p}}\log^q(x)=\frac{(-1)^qq!}{(p+1)^{q+1}}$$

$$\xi_1=8\sum_{n=1}^{\infty}\frac{H_n^3-\frac{3H_n^2}{n}+\frac{3H_n}{n^2}-\frac{1}{n^3}}{n^4}-24\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}-\frac{H_n}{n^2}-\frac{H_n^{(2)}}{n}+\frac{1}{n^3}}{n^4}+16\sum_{n=1}^{\infty}\frac{H^{(3)}-\frac{1}{n^3}}{n^4}$$

$$\xi_1=8\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-24\sum_{n=1}^{\infty}\frac{H_n^2}{n^5}+48\sum_{n=1}^{\infty}\frac{H_n}{n^6}-48\sum_{n=1}^{\infty}\frac{1}{n^7}-24\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$$

$$+24\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}+16\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}$$

Where:

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}=\frac{1}{240}\left(-34\pi^4\zeta(3)+80\pi^2\zeta(5)+3465\zeta(7)\right)$$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^5}=6\zeta(7)-\frac{\pi^2}{36}\left(\pi^2\zeta(3)+6\zeta(5)\right)\wedge\sum_{n=1}^{\infty}\frac{H_n}{n^6}=-\frac{\pi^4}{90}\zeta(3)-\frac{\pi^2}{6}\zeta(5)+4\zeta(7)$$

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}=\frac{1}{240}\left(2\pi^4\zeta(3)+80\pi^2\zeta(5)-765\zeta(7)\right)$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}=\frac{\pi^4}{45}\zeta(3)+\frac{5\pi^2}{6}\zeta(5)-10\zeta(7)\wedge\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}=18\zeta(7)-\frac{5\pi^2}{3}\zeta(5)$$

Therefore:

$$\xi_1=\frac{1}{30}\left(-34\pi^4\zeta(3)+80\pi^2\zeta(5)+3465\zeta(7)\right)-24\left(6\zeta(7)-\frac{\pi^2}{36}\left(\pi^2\zeta(3)+6\zeta(5)\right)\right)$$

$$+48\left(-\frac{\pi^4}{90}\zeta(3)-\frac{\pi^2}{6}\zeta(5)+4\zeta(7)\right)-48\zeta(7)-\frac{1}{10}\left(2\pi^4\zeta(3)+80\pi^2\zeta(5)-765\zeta(7)\right)$$

$$+24\left(\frac{\pi^4}{45}\zeta(3)+\frac{5\pi^2}{6}\zeta(5)-10\zeta(7)\right)+16\left(18\zeta(7)-\frac{5\pi^2}{3}\zeta(5)\right)$$

$$\xi_1=-\frac{2\pi^4}{3}\zeta(3)-16\pi^2\zeta(5)+240\zeta(7)\therefore$$

$$\blacksquare\,\xi_2=\int_0^1\frac{\log^2(x)\log^4(1-x)}{1-x}dx\xRightarrow{We\ do:1-x\to x}\xi_2=\int_0^1\frac{\log^2(1-x)\log^4(x)}{x}dx$$

Using the generalization mentioned above:

$$\overset{\lambda=2}{\Longrightarrow}\log^2(1-z)=2\sum_{n=1}^{\infty}\frac{H_{n-1}}{n}z^n=2\sum_{n=1}^{\infty}\frac{H_n-\frac{1}{n}}{n}z^n$$

$$\textit{Therefore}:\xi_2=2\sum_{n=1}^{\infty}\frac{H_n-\frac{1}{n}}{n}\int_0^1x^{n-1}\log^4(x)\,dx=48\left(\sum_{n=1}^{\infty}\frac{H_n}{n^6}-\sum_{n=1}^{\infty}\frac{1}{n^7}\right)$$

$$\xi_2=48\left(-\frac{\pi^4}{90}\zeta(3)-\frac{\pi^2}{6}\zeta(5)+3\zeta(7)\right)$$

$$\xi_2=-\frac{8\pi^4}{15}\zeta(3)-8\pi^2\zeta(5)+144\zeta(7)\therefore$$

Thus:

$$I=\left(-\frac{2\pi^4}{3}\zeta(3)-16\pi^2\zeta(5)+240\zeta(7)\right)+\left(-\frac{8\pi^4}{15}\zeta(3)-8\pi^2\zeta(5)+144\zeta(7)\right)$$

$$\boxed{I=-24\pi^2\zeta(5)+384\zeta(7)-\frac{6\pi^4}{5}\zeta(3)\therefore}$$

2879. ***Find :***

$$\Omega=\int_0^{\infty}\frac{x^2\sqrt{x}\ln(x)}{x^4+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Omkumar Rao-India

$$I(p)=\int_0^{\infty}\frac{x^p}{x^4+1}dx=\frac{\pi}{4\sin\left(\frac{(p+1)\pi}{4}\right)},\qquad I'(p)=\int_0^{\infty}\frac{x^p\ln x}{x^4+1}dx$$

$$I'(p)=-\frac{\pi^2}{16}\cos\left(\frac{(p+1)\pi}{4}\right)\left(\csc\left(\frac{(p+1)\pi}{4}\right)\right)^2$$

$$\Omega=I'\left(\frac{5}{2}\right)=-\frac{\pi^2}{16}\cos\left(\frac{7\pi}{8}\right)\left(\csc\left(\frac{7\pi}{8}\right)\right)^2=-\frac{\pi^2}{16}\frac{\left(-\frac{\sqrt{2+\sqrt{2}}}{2}\right)}{\left(\frac{2-\sqrt{2}}{4}\right)}=\frac{\pi^2}{16}(2+\sqrt{2})\sqrt{2+\sqrt{2}}$$

2880. Find:

$$I=\int_0^{\infty}\frac{a^x-b^x}{c^x-d^x}\,dx$$

Proposed by Zinadin Kebaili - Algeria

Solution by Samuel Adekunle Ajibade - Nigeria

$$I=\int_0^{\infty}\frac{a^x-b^x}{c^x-d^x}\,dx,\qquad I=\int_0^{\infty}\frac{\left(\frac{a}{c}\right)^x-\left(\frac{b}{c}\right)^x}{1-\left(\frac{d}{c}\right)^x}\,dx=\int_0^{\infty}\frac{\alpha^x-\beta^x}{1-\varphi^x}dx$$

$$\alpha,\beta,\varphi=\frac{a}{c},\frac{b}{c},\frac{d}{c}\text{ respectively}$$

$$I=\int_0^{\infty}(\alpha^x-\beta^x)\sum_{n=0}^{\infty}\varphi^{nx}\,dx\quad(\text{Geometric series: }|\varphi^x|<1\ \forall c>d$$

$$I=\sum_{n=0}^{\infty}\int_0^{\infty}(\alpha^x-\beta^x)\,\varphi^{nx}\,dx$$

(Fubini − Tonelli Theorem justifed by absolute convergence)

$$I=\sum_{n=0}^{\infty}\int_0^{\infty}\left(e^{\ln(\alpha)x}-e^{\ln(\beta)x}\right)e^{n\ln(\varphi)x}dx$$

$$\mathrm{I} = \sum_{n=0}^{\infty} -\frac{1}{n\ln(\varphi)+\ln(\alpha)} + \frac{1}{n\ln(\varphi)+\ln(\beta)} \quad \text{(Laplace Transforms)}$$

$$\mathrm{I} = \frac{1}{\ln(\varphi)}\sum_{n=0}^{\infty}\frac{1}{n+\frac{\ln(\beta)}{\ln(\varphi)}} - \frac{1}{n+\frac{\ln(\alpha)}{\ln(\varphi)}}$$

$$\mathrm{I} = \frac{1}{\ln(\varphi)}\sum_{n=0}^{\infty} -\left(\frac{1}{n+1} - \frac{1}{n+\frac{\ln(\beta)}{\ln(\varphi)}}\right) + \left(\frac{1}{n+1} - \frac{1}{n+\frac{\ln(\alpha)}{\ln(\varphi)}}\right)$$

Recall:

$$\psi(z)+\gamma = \sum_{n=0}^{\infty}\frac{1}{n+1} - \frac{1}{n+z}$$

γ **is the Euler – Mascheroni constant,** $\psi(z)$ **is the digamma function**

$$\mathrm{I} = \frac{1}{\ln(\varphi)}\left(-\psi\left(\frac{\ln(\beta)}{\ln(\varphi)}\right) - \gamma + \gamma + \psi\left(\frac{\ln(\alpha)}{\ln(\varphi)}\right)\right)$$

$$\mathrm{I} = \frac{1}{\ln(\varphi)}\left(\psi\left(\frac{\ln(\alpha)}{\ln(\varphi)}\right) - \psi\left(\frac{\ln(\beta)}{\ln(\varphi)}\right)\right)$$

$$\boxed{\mathrm{I} = \frac{1}{\ln(\varphi)}\left(\psi(\log_{\varphi}(\alpha)) - \psi(\log_{\varphi}(\beta))\right)}$$

Where:

$$\alpha, \beta, \varphi = \frac{a}{c}, \frac{b}{c}, \frac{d}{c} \text{ respectively}$$

Valid:

$$\forall c > a, b, d$$

2881. Find:

$$I = \int_0^1 \frac{\ln^2(x)}{(x-1)x^a}\,dx$$

Proposed by Zinadin Kebaili - Algeria

Solution by Samuel Adekunle Ajibade - Nigeria

$$I = \int_0^1 \frac{\ln^2(x)}{(x-1)x^a}\,dx = -\frac{\partial^2}{\partial b^2}\int_0^1 \frac{x^{b-a}}{1-x}\,dx\,\Bigg|_{b=0} \quad (Leibniz's\ rule)$$

$$I=-\frac{\partial^2}{\partial b^2}\int_0^1\sum_{n=0}^{\infty}x^{n+b-a}\,dx\Bigg|_{b=0}\quad(\textit{Geometric series}: |x|<1$$

$$I=-\sum_{n=0}^{\infty}\frac{\partial^2}{\partial b^2}\int_0^1 x^{n+b-a}\,dx\Bigg|_{b=0}\quad(\textit{Fubini}-\textit{Tonelli Theorem})$$

$$I=-\sum_{n=0}^{\infty}\frac{\partial^2}{\partial b^2}\frac{x^{n+b-a+1}}{n+b-a+1}\Bigg|_0^1\Bigg|_{b=0}=-\sum_{n=0}^{\infty}\frac{\partial^2}{\partial b^2}\frac{1}{n+b-a+1}\Bigg|_{b=0}$$

$$I=-\sum_{n=0}^{\infty}\frac{2}{(n+1-a)^3}=\psi^{(2)}(1-a)$$

$$I(a)=\psi^{(2)}(1-a)$$
$$\forall a<1$$

Where: $\psi^{(2)}(z)$ is the Tetragamma function

Example: when a=$\frac{1}{2}$

$$I\left(\frac{1}{2}\right)=\psi^{(2)}\left(\frac{1}{2}\right)=-14\zeta(3)$$

2882. Find:

$$I=\int_0^{\frac{\pi}{2}}\ln(a^2+b^2\tan^2(x))dx$$

Proposed by Zinadin Kebaili - Algeria

Solution by Samuel Adekunle Ajibade - Nigeria

$$I=\int_0^{\frac{\pi}{2}}\ln(a^2+b^2\tan^2(x))dx=\int_0^{\frac{\pi}{2}}\ln\left(a^2\left(1+\left(\frac{b}{a}\right)^2\tan^2(x)\right)\right)dx$$

$$I=\int_0^{\frac{\pi}{2}}\ln(a^2)dx+\int_0^{\frac{\pi}{2}}\ln\left(1+\left(\frac{b}{a}\right)^2\tan^2(x)\right)dx$$

$$I=\pi\ln|a|+J\quad\cdots\textit{equation }1$$

$$J=\int_0^{\frac{\pi}{2}}\ln(1+c\tan^2(x))dx\quad\cdots\left\{\left(\frac{b}{a}\right)^2=c\right\}$$

$$J=\int_0^c\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial c}ln(1+ctan^2(x))dx\,dc\ =\int_0^c\int_0^{\frac{\pi}{2}}\frac{tan^2(x)}{1+ctan^2(x)}dx\,dc$$

$$J=\int_0^c\frac{1}{c}\int_0^{\frac{\pi}{2}}\frac{1+ctan^2(x)-1}{1+ctan^2(x)}dx\,dc$$

$$J=\int_0^c\frac{1}{c}\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{1+ctan^2(x)}\right)dx\,dc\quad\cdots\ (2)$$

$$\textit{Solving the inner integral, say } K$$

$$K=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{1+ctan^2(x)}\right)dx\ =\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}\frac{1}{1+ctan^2(x)}dx$$

$$let\ t=tan(x),dx=\frac{dt}{1+t^2},t\in[0,\infty)$$

$$K=\frac{\pi}{2}-\int_0^{\infty}\frac{1}{(1+ct^2)(1+t^2)}dx$$

$$K=\frac{\pi}{2}+\frac{1}{1-c}\int_0^{\infty}\left(\frac{c}{1+ct^2}-\frac{1}{1+t^2}\right)dx$$

$$K=\frac{\pi}{2}+\frac{1}{1-c}\left(\sqrt{c}\,(\tan^{-1}(ct)-\tan^{-1}(t)\right)\Big|_0^{\infty}$$

$$K=\frac{\pi}{2}+\frac{\pi}{2}\frac{\sqrt{c}-1}{1-c}\ =\frac{\pi}{2}\left(\frac{\sqrt{c}-c}{1-c}\right)$$

$$\textit{Solving for } J \textit{ from equation } 2$$

$$J=\frac{\pi}{2}\int_0^c\frac{\sqrt{c}-c}{c(1-c)}dc\ =\frac{\pi}{2}\int_0^c\frac{1-\sqrt{c}}{\sqrt{c}(1-\sqrt{c})(1+\sqrt{c})}dc$$

$$J=\frac{\pi}{2}\int_0^c\frac{1}{\sqrt{c}(1+\sqrt{c})}dc$$

$$let\ u=1+\sqrt{c}\,,du=\frac{1}{2\sqrt{c}}dc,u\in[1,1+\sqrt{c}]$$

$$J=\pi\int_1^{1+\sqrt{c}}\frac{1}{u}du\ =\pi ln(1+\sqrt{c})\cdots\left\{\left(\frac{b}{a}\right)^2=c\right\}$$

$$J=\ \pi ln\left(1+\frac{|b|}{|a|}\right)=\pi ln(|a|+|b|)-\pi ln|a|$$

$$\textit{Expressing from } I \textit{ equation } 1$$

$$I = \pi ln(|a| + |b|)$$
$$\forall a, b \in \mathbb{R}$$

2883. Find:

$$\Omega = \lim_{x \to 0} \left(\frac{1 - \prod_{k=1}^{n} (\cos kx)^{\frac{1}{k^3}}}{x^2} \right)^{\frac{1}{x^2}}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$A(x) = \prod_{k=1}^{n} (\cos kx)^{\frac{1}{k^3}} \Rightarrow \ln A(x) = \sum_{k=1}^{n} \frac{\ln(\cos kx)}{k^3}$$

$$\Rightarrow \cos kx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

$$\Rightarrow \ln(1 + x) = x - \frac{x^2}{2} + O(x^3)$$

$$x \longrightarrow -\frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\ln(\cos kx) = \left(-\frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} \right) - \frac{1}{2} \left(-\frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} \right)^2 + O(x^6) \Rightarrow \ln(\cos kx)$$
$$= -\frac{k^2x^2}{2} - \frac{k^4x^4}{12} + O(x^6)$$

$$\ln A(x) = -\frac{x^2}{2} \sum_{k=1}^{n} \frac{1}{k} - \frac{x^4}{12} \sum_{k=1}^{n} k + O(x^6) = -\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + O(x^6) \Rightarrow A(x)$$
$$= e^{\left(-\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + O(x^6) \right)}$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2} + O(x^3)$$

$$A(x) = 1 + \left(-\frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 \right) + \frac{1}{2} \left(-\frac{x^2 H_n}{2} \right)^2 + O(x^6)$$
$$= 1 - \frac{x^2 H_n}{2} - \frac{n(n+1)}{24} x^4 + \frac{1}{8} (H_n)^2 x^4 + O(x^6)$$

$$1-A(x)=\frac{x^2H_n}{2}-x^4\left(\frac{1}{8}(H_n)^2-\frac{n(n+1)}{24}\right)+O(x^6)\Rightarrow\frac{1-A(x)}{x^2}$$

$$=\frac{H_n}{2}-x^2\left(\frac{1}{8}(H_n)^2-\frac{n(n+1)}{24}\right)+O(x^4)\quad:\left(\frac{1-A(x)}{x^2}=B(x)\right)$$

$$\Rightarrow\ln\Omega=\lim_{x\to0}\left(\frac{\ln B(x)}{x^2}\right)\Rightarrow B(x)=\frac{H_n}{2}\left(1-x^2\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)+O(x^4)\right)$$

$$\Rightarrow\ln B(x)=\ln\left(\frac{H_n}{2}\right)+\ln\left(1-x^2\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)+O(x^4)\right)$$

$$\ln B(x)=\ln\left(\frac{H_n}{2}\right)+\left(-x^2\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)+O(x^4)\right)$$

$$\Rightarrow\lim_{x\to0}\left(\frac{\ln B(x)}{x^2}\right)=\lim_{x\to0}\frac{\ln\left(\frac{H_n}{2}\right)}{x^2}-\lim_{x\to0}\frac{x^2\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)}{x^2}+O(x^2)$$

$$\ln\Omega=\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)\Rightarrow\Omega=e^{\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)}$$

$$\boxed{\Omega=\lim_{x\to0}\left(\frac{1-\prod_{k=1}^{n}(\cos kx)^{\frac{1}{k^3}}}{x^2}\right)^{\frac{1}{x^2}}=e^{\left(\frac{H_n}{4}-\frac{n(n+1)}{12H_n}\right)}}$$

2884. **Find:**

$$\lim_{n\to\infty}\left[\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)\int_{-1}^{\infty}\frac{(\cos x)^{2n}}{k^x}dx\right]$$

Proposed by Omkumar Rao-India

Solution by proposer

$$P_n=\prod_{k=1}^{n}\frac{2k}{2k-1},Q_n=\prod_{k=1}^{n}\frac{2k}{2k+1}\Rightarrow P_n=\frac{2^{2n}(n!)^2}{(2n)!}$$

$$\Rightarrow Stirling\ Approximation: n!=\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\Rightarrow$$

$$P_n=\frac{2^{2n}\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^2}{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}}\Rightarrow P_n=\sqrt{\pi n}$$

$$I = \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{k^x} dx \Rightarrow Since\ |\cos x| < 1\,, (\cos x)^{2n} \to 0\ everywhere\ except\ \cos x$$

$$= \pm 1, i.e.\ \ x = m\pi. The\ relevant\ points\ are\ x = 0, \pi, 2\pi, 3\pi.$$

$$As\ n \to \infty, all\ the\ area\ of\ the\ integral\ collapses\ onto\ tiny\ spikes\ at\ each\ m\pi.$$

$$x \to m\pi + x \Rightarrow \lim_{n\to\infty} \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{k^x} dx = \lim_{n\to\infty} \int_{-\delta}^{+\delta} \frac{(\cos x)^{2n}}{k^n} dx \Rightarrow$$

$$I = \sum_{m=0}^{\infty} \int_{-\delta}^{+\delta} \frac{1}{k^{m\pi+x}} \left(1 - \frac{x^2}{2}\right)^{2n} dx \Rightarrow \lim_{n\to\infty} \left(1 - \frac{x^2}{2}\right)^{2n} = e^{-nx^2} \Rightarrow$$

$$as\ x \to 0, n \to \infty$$

$$I = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \int_{-\delta}^{+\delta} \frac{e^{-nx^2}}{k^x} dx \Rightarrow I = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \int_{-\infty}^{+\infty} \frac{e^{-nx^2}}{k^0} dx = \sum_{m=0}^{\infty} \frac{1}{k^{m\pi}} \sqrt{\frac{\pi}{n}}$$

$$\lim_{n\to\infty} P_n Q_n = \sqrt{\pi n}. \frac{k^\pi}{k^\pi - 1}. \sqrt{\frac{\pi}{n}} = \frac{\pi k^\pi}{k^\pi - 1}$$

2885. Find :

$$\Omega = \lim_{n\to\infty} \left(n \int_0^1 \frac{x^n}{1 + x^n} dx \right)$$

Proposed by Vasile Mircea Popa-Romania

Solution by Omkumar Rao-India

$$n \int_0^1 \frac{x^n}{1 + x^n} dx \overset{x^n \to y}{\Longrightarrow} n \int_0^1 \frac{y \cdot \frac{1}{n} \cdot y^{\frac{1}{n}-1}}{1 + y} dy = \int_0^1 \frac{x^{\frac{1}{n}}}{1 + x} dx$$

$$\Omega = \lim_{n\to\infty} \left(\int_0^1 \frac{x^{\frac{1}{n}}}{1 + x} dx \right) \overset{Dominated\ Convergence\ Theorem}{\Longrightarrow} \int_0^1 \left(\lim_{n\to\infty} \frac{x^{\frac{1}{n}}}{1 + x} \right) dx =$$

$$= \int_0^1 \left(\frac{x^0}{1 + x} \right) dx = \int_0^1 \frac{1}{1 + x} dx = \ln 2$$

2886. Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\cos(x))}{x^2+(\ln(\cos(x)))^2}dx = \frac{\pi}{2}(1-\log_2(e))$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\cos(x))}{x^2+(\ln(\cos(x)))^2}dx = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\ln(\cos(x))}}{1+\left(\frac{x}{\ln(\cos(x))}\right)^2}dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\ln(\cos(x))}+\frac{x\tan(x)}{(\ln(\cos(x)))^2}-\frac{x\tan(x)}{(\ln(\cos(x)))^2}}{1+\left(\frac{x}{\ln(\cos(x))}\right)^2}dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\ln(\cos(x))}+\frac{x\tan(x)}{(\ln(\cos(x)))^2}}{1+\left(\frac{x}{\ln(\cos(x))}\right)^2}dx - \int_0^{\frac{\pi}{2}} \frac{\frac{x\tan(x)}{(\ln(\cos(x)))^2}}{1+\left(\frac{x}{\ln(\cos(x))}\right)^2}dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\left(\frac{x}{\ln(\cos(x))}\right)}{1+\left(\frac{x}{\ln(\cos(x))}\right)^2}dx - \underbrace{\int_0^{\frac{\pi}{2}} \frac{x\tan(x)}{x^2+(\ln(\cos(x)))^2}dx}_{-\ln(\cos(x))\to x} =$$

$$= \left[\tan^{-1}\left(\frac{x}{\ln(\cos(x))}\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\infty} \frac{\cos^{-1}(e^{-x})}{x^2+\cos^{-1}(e^{-x})}dx =$$

$$= \frac{\pi}{2} - \int_0^{\infty}\left(\underbrace{\int_0^{\infty} e^{-yx}\sin\left(y(\cos^{-1}(e^{-x}))\right)dy}_{e^{-x}\to x}\right)dx =$$

$$= \frac{\pi}{2} - \int_0^{\infty}\left(\underbrace{\int_0^{1} x^{y-1}\sin\left(y(\cos^{-1}(x))\right)dy}_{Now\ by\ parts}\right)dx =$$

$$= \frac{\pi}{2} - \int_0^{\infty}\left(\underbrace{\int_0^{1} x^{y}\cos\left(y(\cos^{-1}(x))\right)dx}_{x\to\cos x}\right)dy =$$

$$=\frac{\pi}{2}-\int_0^\infty\left(\int_0^{\frac{\pi}{2}}(\cos(x))^y\cos(yx)\,dx\right)dy=\frac{\pi}{2}-\int_0^\infty\Re\left(\int_0^{\frac{\pi}{2}}(\cos(x))^y e^{iyx}\,dx\right)dy$$

$$=\frac{\pi}{2}-\int_0^\infty\Re\left(\int_0^{\frac{\pi}{2}}((\cos(x))^2+i\sin(x)\cos(x))^y\,dx\right)dy$$

$$=\frac{\pi}{2}-\int_0^\infty\Re\left(\frac{1}{2^y}\int_0^{\frac{\pi}{2}}(1+e^{2ix})^y\,dx\right)dy$$

$$=\frac{\pi}{2}-\int_0^\infty\frac{1}{2^y}\Re\left(\int_0^{\frac{\pi}{2}}\left(\sum_{j=0}^{y}e^{i2jx}\right)dx\right)dy=\frac{\pi}{2}-\int_0^\infty\frac{1}{2^y}\Re\left(\sum_{j=0}^{y}\left(\int_0^{\frac{\pi}{2}}e^{i2jx}\right)dx\right)dy=$$

$$=\frac{\pi}{2}-\int_0^\infty\frac{\pi}{2^{y+1}}dy=\frac{\pi}{2}-\frac{\pi}{2\ln(2)}=\frac{\pi}{2}(1-\log_2(e))$$

$$\boxed{\int_0^{\frac{\pi}{2}}\frac{\ln(\cos(x))}{x^2+(\ln(\cos(x)))^2}dx=\frac{\pi}{2}(1-\log_2(e))}$$

2887. **Find a closed form:**

$$S=\sum_{n=1}^\infty\frac{H_n}{n^3(n+1)^3}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\frac{1}{n^3(n+1)^3}=\frac{1}{n^3}-\frac{1}{(n+1)^3}-\frac{3}{n^2}-\frac{3}{(n+1)^2}+\frac{6}{n}-\frac{6}{n+1}$$

$$\sum_{n=1}^\infty\frac{H_n}{(n+1)^3}=\sum_{k=2}^\infty\frac{H_k-\frac{1}{k}}{k^3}=\left(\sum_{k=1}^\infty\frac{H_k}{k^3}-1\right)-\left(\sum_{k=1}^\infty\frac{1}{k^4}-1\right)=\sum_{k=1}^\infty\frac{H_k}{k^3}-\zeta(4)$$

$$\Rightarrow\sum_{n=1}^\infty\frac{H_n}{(n)^3}-\sum_{n=1}^\infty\frac{H_n}{(n+1)^3}=\zeta(4)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}=\sum_{k=2}^{\infty}\frac{H_k-\frac{1}{k^2}}{k^2}=\left(\sum_{n=1}^{\infty}\frac{H_n}{(n)^2}-1\right)-(\zeta(3)-1)=$$

$$=\sum_{n=1}^{\infty}\frac{H_n}{(n)^2}-\zeta(3)=\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n}-\sum_{n=1}^{\infty}\frac{H_n}{n+1}=\zeta(2)$$

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3(n+1)^3}=\zeta(4)-6\sum_{n=1}^{\infty}\frac{H_n}{(n)^2}+3\zeta(3)+6\zeta(2)=$$

$$=\zeta(4)+3\zeta(3)+6\zeta(2)-6\big(2\zeta(3)\big)=\zeta(4)+6\zeta(2)-9\zeta(3)=\frac{\pi^4}{90}+6\left(\frac{\pi^2}{6}\right)-9\zeta(3)$$

$$\boxed{\sum_{n=1}^{\infty}\frac{H_n}{n^3(n+1)^3}=\frac{\pi^4}{90}+\pi^2-9\zeta(3)}$$

2888. ***Find***:

$$\Omega=\int_0^{\infty}\frac{x^3\tan^{-1}x}{x^8+x^4+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Omkumar Rao-India

$$\int_0^{\infty}\frac{x^3\tan^{-1}x}{x^8+x^4+1}dx\overset{x\to\frac{1}{x}}{\Longrightarrow}\int_0^{\infty}\frac{x^3\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{x}\right)\right)}{x^8+x^4+1}dx$$

$$2\Omega=\frac{\pi}{2}\int_0^{\infty}\frac{x^3}{x^8+x^4+1}dx\overset{x^4\to x}{\Longrightarrow}\frac{\pi}{8}\int_0^{\infty}\frac{1}{x^2+x+1}dx$$

$$\Omega=\frac{\pi}{16}\int_0^{\infty}\frac{1}{x^2+x+1}dx=\frac{\pi}{16}\left(\frac{2\pi}{3\sqrt{3}}\right)$$

$$\Omega=\frac{\pi^2\sqrt{3}}{72}$$

2889. Find:

$$I = \iiint_0^1 \sqrt{x + 2y + 3z}\, dx\, dy\, dz$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Samuel Ajibade-Nigeria

Basic rule: $\int (ax+b)^n\, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$

$$I = \iiint_0^1 \sqrt{x+2y+3z}\, dx\, dy\, dz = \frac{2}{3}\iint_0^1 \left[(1+2y+3z)^{\frac{3}{2}} - (2y+3z)^{\frac{3}{2}}\right] dy\, dz$$

$$I = \frac{2}{3}\cdot\frac{1}{5}\int_0^1 \left[(3+3z)^{\frac{5}{2}} - (1+3z)^{\frac{5}{2}} - (2+3z)^{\frac{5}{2}} - (3z)^{\frac{5}{2}}\right] dz$$

$$I = \frac{2}{15}\cdot\frac{2}{21}\left(6^{\frac{7}{2}} - 3^{\frac{7}{2}} - 4^{\frac{7}{2}} + 1 - 5^{\frac{7}{2}} + 2^{\frac{7}{2}} + 3^{\frac{7}{2}}\right)$$

$$\boxed{I = \frac{4}{315}\left(216\sqrt{6} - 125\sqrt{5} + 8\sqrt{2} - 127\right)}$$

2890. ***Evaluate***:

$$S = \frac{1}{1^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \frac{1}{11^4} - \frac{1}{13^4} + \frac{1}{15^4} + \cdots = \sum_{k=0}^{\infty}\frac{\varepsilon_k}{(2k+1)^4}$$

$$where:\ \varepsilon_k = \begin{cases} +1, & k \equiv 0,3 (mod\ 4) \\ -1, & k \equiv 1,2 (mod\ 4) \end{cases}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\frac{1}{p^4} = -\frac{1}{6}\int_0^1 x^{p-1}(\ln x)^3\, dx \overset{(p=2k+1)}{\Longrightarrow}$$

$$S = \sum_{k=0}^{\infty}\varepsilon_k\left(-\frac{1}{6}\int_0^1 x^{2k}(\ln x)^3\, dx\right) = -\frac{1}{6}\int_0^1 (\ln x)^3\left(\sum_{k=0}^{\infty}\varepsilon_k\, x^{2k}\right) dx$$

$$\sum_{k=0}^{\infty}\varepsilon_k x^{2k}=(1-x^2-x^4+x^6)+(x^8-x^{10}-x^{12}+x^{14})+\cdots \overset{Let\ x^2=m}{\Rightarrow}$$

$$(1-m-m^2+m^3)(1+m^4+m^8+\cdots)=\frac{(1-m)^2(1+m)}{1-m^4}=\frac{1-x^2}{1+x^4}$$

$$S=-\frac{1}{6}\left(\int_0^1(\ln x)^3.\frac{1-x^2}{1+x^4}dx\right)\Rightarrow -6S=\underbrace{\int_0^1\frac{(\ln x)^3}{1+x^4}dx}_{A}-\underbrace{\int_0^1\frac{x^2(\ln x)^3}{1+x^4}dx}_{B}$$

$$B=\int_0^1\frac{x^2(\ln x)^3}{1+x^4}dx\overset{x\to\frac{1}{x}}{\Rightarrow}-\int_1^\infty\frac{(\ln x)^3}{1+x^4}dx\Rightarrow$$

$$-6S=\int_0^1\frac{(\ln x)^3}{1+x^4}dx+\int_1^\infty\frac{(\ln x)^3}{1+x^4}dx=\int_0^\infty\frac{(\ln x)^3}{1+x^4}dx$$

$$J(b)=\int_0^\infty\frac{x^b}{1+x^4}dx\overset{-1<b<3}{\Rightarrow}\frac{\pi}{4\sin\left(\frac{\pi(b+1)}{4}\right)},S=-\frac{1}{6}J'''(0)$$

$$J'^{(b)}=-\frac{\pi^2}{16}\csc\left(\frac{\pi(b+1)}{4}\right)\cot\left(\frac{\pi(b+1)}{4}\right)\Rightarrow$$

$$J''(b)=\frac{\pi^3}{16}\left[\left(\cot\left(\frac{\pi(b+1)}{4}\right)\right)^2\csc\left(\frac{\pi(b+1)}{4}\right)+\left(\csc\left(\frac{\pi(b+1)}{4}\right)\right)^3\right]$$

$$J'''^{(b)}=\frac{\pi^4}{256}\csc\left(\frac{\pi(b+1)}{4}\right)\cot\left(\frac{\pi(b+1)}{4}\right)\left[1-6\left(\csc\left(\frac{\pi(b+1)}{4}\right)\right)^2\right]\Rightarrow$$

$$J'''^{(0)}=-\frac{11\sqrt{2}\pi^4}{256}\Rightarrow S=-\frac{1}{6}J'''^{(0)}=\frac{11\sqrt{2}\pi^4}{1536}$$

$$\boxed{S=\frac{1}{1^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}-\frac{1}{11^4}-\frac{1}{13^4}+\frac{1}{15^4}+\cdots=\sum_{k=0}^{\infty}\frac{\varepsilon_k}{(2k+1)^4}=\frac{11\sqrt{2}\pi^4}{1536}}$$

2891. ***Prove that:***

$$\int_0^\infty\frac{x^2}{(x^2+4)^2}dx+\int_0^\infty\frac{y^2}{4(y^2+25)^2}dy+\int_0^\infty\frac{z^2}{9(z^2+100)^2}dz+\cdots=$$

$$=\frac{\pi^2}{24}\left(\pi-3\coth\pi+\frac{3}{\pi}\right)$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Omkumar Rao-India

$$K_n=\frac{1}{n^2}\int_0^{\infty}\frac{x^2}{(x^2+k^2)^2}dx \qquad :(k=n^2+1)$$

$$K_n \overset{x\to k\tan x}{\Longrightarrow} \frac{1}{kn^2}\int_0^{\frac{\pi}{2}}(\sin x)^2\,dx=\frac{\pi}{4n^2(n^2+1)}$$

$$S=\sum_{n=1}^{\infty}\frac{\pi}{4n^2(n^2+1)}=\frac{\pi}{4}\left(\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right)$$

$$\cot x=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2\pi^2}\overset{x\to i\pi}{\Longrightarrow}\coth(\pi)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1}=\frac{\pi\coth(\pi)-1}{2}$$

$$S=\frac{\pi}{4}\left(\frac{\pi^2}{6}-\frac{\pi\coth(\pi)-1}{2}\right)=\frac{\pi^2}{24}\left(\pi-3\coth\pi+\frac{3}{\pi}\right)$$

2892. ***Find*** :

$$\int_0^{\arctan(2)}\frac{3\tan(x)-1}{4\cos^2(x)+\sin^2(x)}dx$$

Proposed by Henry Yang-Bangladseh

Solution by Chew Seong Cheong-Malaysia

$$\int_0^{\arctan(2)}\frac{3\tan(x)-1}{4\cos^2(x)+\sin^2(x)}dx\cdot\frac{\sec^2(x)}{\sec^2(x)}=$$

$$=\int_0^{\arctan(2)}\frac{(3\tan(x)-1)\sec^2(x)}{4+\tan^2(x)}dx=\int_0^2\frac{3t-1}{4+t^2}dt=$$

$$=\int_0^2\frac{3t}{4+t^2}dt-\int_0^2\frac{1}{4+t^2}dt=\frac{3}{2}\ln(4+t^2)-\frac{1}{2}\arctan\left(\frac{t}{2}\right)\Big|_0^2=$$

$$=\frac{3}{2}(\ln(8)-\ln(4))-\frac{1}{2}\left(\frac{\pi}{4}-0\right)=\frac{3}{2}\ln(2)-\frac{\pi}{8}$$

2893. **Find:**

$$\int_0^1\frac{x^{3i}-x^{-\frac{i}{3}}}{\ln(x)}dx$$

Proposed by Henry Yang-Bangladesh

Solution by Chew Seong Cheong-Malaysia

$$\textit{Using the following Frullani's integral:}$$

$$\therefore \int_0^1 \frac{x^a - x^b}{\ln(x)} dx = \ln\left(\frac{a+1}{b+1}\right)$$

$$\textit{valid for } a, b \in \mathbb{C} \textit{ with } \Re(a), \Re(b) > -1 \textit{ we have:}$$

$$\int_0^1 \frac{x^{3i} - x^{-\frac{i}{3}}}{\ln(x)} dx = \ln\left(\frac{3i+1}{-\frac{i}{3}+1}\right) = \ln\left(\frac{3(1+3i)}{3-i}\right) =$$

$$= \ln\left(\frac{3(1+3i)(3+i)}{(3-i)(3+i)}\right) = \ln\left(\frac{3(10i)}{10}\right) = \ln(3) + \ln(i) =$$

$$= \ln(3) + \ln\left(e^{i\frac{\pi}{2}}\right) = \ln(3) + i\frac{\pi}{2}$$

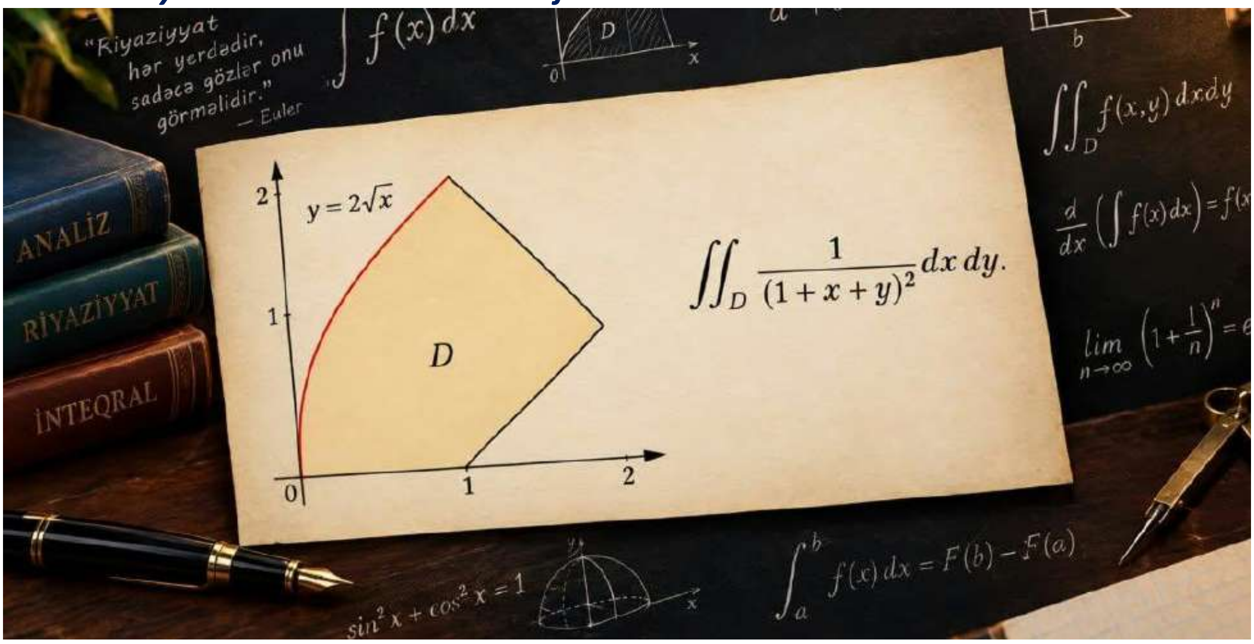
2894. Let be: $D_1 = \{(x,y) | 0 \le x \le 1;\ \ 0 \le y \le 2\sqrt{x}\}$

$$D_2 = \{(x,y) | 1 \le x \le 2;\ \ x - 1 \le y \le 3 - x\},\ D = D_1 \cap D_2. \text{ Find:}$$

$$\Omega = \iint_D \frac{1}{(x+y+1)^2} dy dx$$

Proposed by Nicole De Rosa-Italy

Solution by Shirvan Tahirov-Azerbaijan



$$\therefore \textit{The vertical line extends up ward from } y = 0 \textit{ to } y = 2\sqrt{x}$$
$$\textit{that is, it is bounded by them:}$$

$$\therefore \textit{The vertical line extends up ward from } y = x - 1 \textit{ to}$$
$$y = 3 - x \textit{ that is, it is bounded by them:}$$

Let us calculate the given integral using Fubini's theorem

$$\therefore \int_m^n \int_k^l f(x,y)dydx = \int_m^n \left(\int_k^l f(x,y)\,dy\right)dx$$

$$\Omega_1 = \int_0^1 \int_0^{2\sqrt{x}} \frac{1}{(x+y+1)^2}dydx = \int_0^1 \left(\int_0^{2\sqrt{x}} \frac{1}{(x+y+1)^2}dy\right)dx =$$

$$= \int_0^1 \left(\left[-\frac{1}{x+y+1}\right]_0^{2\sqrt{x}}\right)dx = \int_0^1 \left(\frac{1}{1+x} - \frac{1}{\left(1+\sqrt{x}\right)^2}\right)dx =$$

$$= (\ln(2) - 2\ln(2) - 1) - (\ln(1) - 2\ln(1)) = 1 - \ln(2)$$

$$\Omega_2 = \int_1^2 \int_{x-1}^{3-x} \frac{1}{(x+y+1)^2}dydx = \int_1^2 \left(\int_{x-1}^{3-x} \frac{1}{(x+y+1)^2}dy\right)dx =$$

$$= \int_1^2 \left(\left[-\frac{1}{x+y+1}\right]_{x-1}^{3-x}\right)dx = \int_1^2 \left(-\frac{1}{4} + \frac{1}{2x}\right)dx =$$

$$= -\frac{1}{2} + \frac{\ln(2)}{2} - \left(-\frac{1}{4} + \frac{\ln(1)}{2}\right) = -\frac{1}{4} + \frac{\ln(2)}{2}$$

$$\Omega = \Omega_1 + \Omega_2 = (1 - \ln(2)) + \left(-\frac{1}{4} + \frac{\ln(2)}{2}\right) = \frac{3}{4} - \frac{1}{2}\ln(2)$$

2895. Prove that:

$$\int_0^\infty e^{-(\ln x)^{2a}}\,dx = \frac{1}{a}\sum_{n=0}^\infty \frac{\Gamma\left(\frac{n}{a} + \frac{1}{2a}\right)}{\Gamma(2n+1)}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^\infty e^{-(\ln x)^{2a}}\,dx \overset{\ln x \to x}{\Rightarrow} \int_{-\infty}^\infty e^{-(x)^{2a}} e^x\,dx = \int_0^\infty e^{-(x)^{2a}}(e^x + e^{-x})\,dx =$$

$$= 2\int_0^\infty e^{-(x)^{2a}}\cosh(x)\,dx = 2\sum_{n=0}^\infty \frac{1}{\Gamma(2n+1)} \int_0^\infty x^{2n} e^{-(x)^{2a}}\,dx$$

$$\int_0^\infty x^{2n} e^{-(x)^{2a}}\,dx \overset{x^{2a} \to x}{\Rightarrow} \frac{1}{2a}\int_0^\infty x^{\left(\frac{n}{a} + \frac{1}{2a}\right)-1} e^{-x}\,dx = \frac{1}{2a}\Gamma\left(\frac{n}{a} + \frac{1}{2a}\right)$$

$$\boxed{\int_0^\infty e^{-(\ln x)^{2a}}\,dx = \frac{1}{a}\sum_{n=0}^\infty \frac{\Gamma\left(\frac{n}{a} + \frac{1}{2a}\right)}{\Gamma(2n+1)}}$$

2896. Prove that:

$$\sum_{k=1}^\infty \frac{p^k \sin kx}{k} = \tan^{-1}\left(\frac{p\sin(x)}{1 - p\cos(x)}\right) \qquad [0 < x < 2\pi, p^2 \le 1\ ]$$

$$\sum_{k=1}^{\infty}\frac{p^k\cos kx}{k}=-\frac{1}{2}\ln(p^2-2p\cos x+1)\qquad [0<x<2\pi\,, p^2\le 1\ ]$$

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\sin(2k-1)x}{2k-1}=\frac{1}{2}\tan^{-1}\left(\frac{2p\sin(x)}{1-p^2}\right)\qquad [0<x<2\pi\,, p^2\le 1\ ]$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\sum_{k=1}^{\infty}\frac{p^k\sin kx}{k}=\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)\qquad [0<x<2\pi\,, p^2\le 1\ ]$$

$$\sum_{k=1}^{\infty}\frac{p^k\sin kx}{k}=Im\left\{\sum_{k=1}^{\infty}\frac{(pe^{ix})^k}{k}\right\}=Im\{-\ln(1-pe^{ix})\}$$

$$1-pe^{ix}=\underbrace{(1-p\cos x)}_{Real\ part}+i\ \underbrace{(-p\sin x)}_{Imaginary\ part}$$

$$arg(1-pe^{ix})=\tan^{-1}\left(\frac{-p\sin(x)}{1-p\cos(x)}\right)=-\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)$$

$$Im\{-\ln(1-pe^{ix})\}=-arg(1-pe^{ix})=-\left[-\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)\right]$$

$$=\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)$$

$$\sum_{k=1}^{\infty}\frac{p^k\cos kx}{k}=-\frac{1}{2}\ln(p^2-2p\cos x+1)\qquad [0<x<2\pi\,, p^2\le 1\ ]$$

$$\sum_{k=1}^{\infty}\frac{p^k\cos kx}{k}=\Re\left\{\sum_{k=1}^{\infty}\frac{(pe^{ix})^k}{k}\right\}=\Re\{-\ln(1-pe^{ix})\}=-\ln\left|1-pe^{ix}\right|$$

$$=-\ln\left(\sqrt{1-2p\cos x+p^2}\right)=-\frac{1}{2}\ln(p^2-2p\cos x+1)$$

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\sin(2k-1)x}{2k-1}=\frac{1}{2}\tan^{-1}\left(\frac{2p\sin(x)}{1-p^2}\right)\qquad [0<x<2\pi\,, p^2\le 1\ ]$$

$$\boxed{\sum_{k=1}^{\infty}\frac{y^{2k-1}}{2k-1}=\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)}$$

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\sin(2k-1)x}{2k-1}=Im\left\{\frac{1}{2}\ln\left(\frac{1+pe^{ix}}{1-pe^{ix}}\right)\right\}=\frac{1}{2}Im\{\ln(1+pe^{ix})-\ln(1-pe^{ix})\}$$

$$Im\{-\ln(1-pe^{ix})\}=-arg(1-pe^{ix})=$$

$$=-\left[-\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)\right]=\tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)$$

$$Im\{\ln(1+pe^{ix})\} = arg(1+pe^{ix}) = \tan^{-1}\left(\frac{p\sin(x)}{1+p\cos(x)}\right) =$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{p\sin(x)}{1+p\cos(x)}\right) + \tan^{-1}\left(\frac{p\sin(x)}{1-p\cos(x)}\right)\right] =$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\frac{p\sin(x)}{1+p\cos(x)} + \frac{p\sin(x)}{1-p\cos(x)}}{1-\left(\frac{p\sin(x)}{1+p\cos(x)}\cdot\frac{p\sin(x)}{1-p\cos(x)}\right)}\right) =$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\frac{2p\sin x}{1-p^2(\cos x)^2}}{\frac{1-p^2}{1-p^2(\cos x)^2}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2p\sin(x)}{1-p^2}\right)$$

2897. ***Evaluate*:**

$$S = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k} \qquad ; a,b \in R^+, 0 < a < b.$$

Proposed by Omkumar Rao-India

Solution by proposer

$$0 < a < b: S = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k}$$

$$0 < x = \frac{a}{b} < 1: \ = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k} = \sum_{k=1}^{\infty}\frac{x^{2k}}{k}\left(\sum_{m=1}^{\infty}\frac{1}{m^{2k}}\right) =$$

$$= \sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{m}\right)^{2k}\right) = -\sum_{m=1}^{\infty}\ln\left(1-\left(\frac{x}{m}\right)^2\right) = -\ln\left(\prod_{m=1}^{\infty}\left(1-\left(\frac{x}{m}\right)^2\right)\right)$$

$$\frac{\sin\pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n}\right)^2\right) \Rightarrow S = \ln\left(\frac{\pi a}{b\sin\left(\frac{\pi a}{b}\right)}\right)$$

$$\boxed{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\left(\frac{a}{b}\right)^{2k} = \ln\left(\frac{\pi a}{b\sin\left(\frac{\pi a}{b}\right)}\right)}$$

2898. ***For any positive integer $k \geq 1$, find*:**

$$S(k) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn(m+n+k)}$$

Solution by Omkumar Rao-India

Solution by proposer

$$S(k)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn(m+n+k)}\quad (k\geq 1)$$

$$S(k)=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x^{m+n+k-1}\,dx=\int_0^1 x^{k-1}(\ln(1-x))^2\,dx$$

$$S(k)=\frac{d^2}{ds^2}\left[\int_0^1 x^{k-1}(\ln(1-x))^2\,dx\right]=\frac{d^2}{ds^2}\left(\beta(k,s)\right)=\frac{d^2}{ds^2}\left(\frac{\Gamma(k)\Gamma(s)}{\Gamma(k+s)}\right)$$

$$S(k)=\left[\frac{\Gamma(k)\Gamma(s)}{\Gamma(k+s)}\left\{\psi^{(1)}(s)-\psi^{(1)}(k+s)+\left(\psi(s)-\psi(k+s)\right)^2\right\}\right]$$

$$S(k)\overset{s=1}{\cong}\frac{\Gamma(k)}{\Gamma(k+1)}\left\{\psi^{(1)}(1)-\psi^{(1)}(k+1)+\left(\psi(1)-\psi(k+1)\right)^2\right\}=\frac{H_k^{(2)}+H_k^2}{k}$$

$$\boxed{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn(m+n+k)}=\frac{H_k^{(2)}+H_k^2}{k}}$$

2899. Find a closed form:

$$K=\int_0^{\infty}\frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}\,dx$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\int_0^{\infty}\frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}\,dx\overset{x\to x^2}{\cong}8\int_0^{\infty}\frac{(\ln x)^2x^2}{(x^4+1)(x^2+1)}\,dx=$$

$$=8\left(\int_0^1\frac{(\ln x)^2x^2}{(x^4+1)(x^2+1)}\,dx+\underbrace{\int_1^{\infty}\frac{(\ln x)^2x^2}{(x^4+1)(x^2+1)}\,dx}_{x\to\frac{1}{x}}\right)$$

$$K=16\int_0^1\frac{(\ln x)^2x^2}{(x^4+1)(x^2+1)}\,dx=8\int_0^1\frac{x^2+1}{x^4+1}(\ln x)^2\,dx-8\int_0^1\frac{(\ln x)^2}{x^2+1}\,dx$$

$$\int_0^1\frac{x^2+1}{x^4+1}(\ln x)^2\,dx=\underbrace{\int_0^1\frac{(\ln x)^2x^2}{x^4+1}\,dx}_{x\to\frac{1}{x}}+\int_0^1\frac{(\ln x)^2}{x^4+1}\,dx=\int_0^{\infty}\frac{(\ln x)^2}{x^4+1}\,dx=\frac{\partial^2(P(m,n))}{\partial m^2}$$

$$P(m,n)=\int_0^\infty\frac{x^m}{x^n+1}=\frac{\pi}{n\sin\left(\frac{\pi(m+1)}{n}\right)}$$

$$K=8\left(P''(0,4)-\frac{\pi^3}{16}\right)=8\left(\frac{3\sqrt{2}\pi^3}{64}-\frac{\pi^3}{16}\right)=\frac{\pi^3}{8}(3\sqrt{2}-4)$$

$$\boxed{K=\int_0^\infty\frac{(\ln x)^2\sqrt{x}}{(x^2+1)(x+1)}dx=\frac{\pi^3}{8}(3\sqrt{2}-4)}$$

2900. Find:

$$\Omega=\sum_{k=1}^\infty\frac{3k^3+2k+1}{(3k+1)^2k^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Nawar Alasadi-Iraq

$$\psi(1+z)=-\gamma+\sum_{k=1}^\infty\left(\frac{1}{k}-\frac{1}{k+z}\right)$$

$$\psi_1(1+z)=\sum_{k=1}^\infty\frac{1}{(k+z)^2}$$

$$\psi\left(\frac{1}{3}\right)=-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{3}{2}\ln3$$

$$\begin{aligned}\Omega&=\sum_{k=1}^\infty\frac{3k^3+2k+1}{(3k+1)^2k^3}\\
&=\sum_{k=1}^\infty\left(\frac{1}{k^3}-\frac{4}{k^2}+\frac{15}{k}-\frac{45}{3k+1}-\frac{6}{(3k+1)^2}\right)\\
&=\sum_{k=1}^\infty\frac{1}{k^3}-4\sum_{k=1}^\infty\frac{1}{k^2}+15\sum_{k=1}^\infty\left(\frac{1}{k}-\frac{1}{k+1/3}\right)-\frac{2}{3}\sum_{k=1}^\infty\frac{1}{(k+1/3)^2}\\
&=\zeta(3)-4\zeta(2)+15\left(\psi\left(\frac{4}{3}\right)+\gamma\right)-\frac{2}{3}\psi_1\left(\frac{4}{3}\right)\\
&=\zeta(3)-4\left(\frac{\pi^2}{6}\right)+15\left(\psi\left(\frac{1}{3}\right)+3+\gamma\right)-\frac{2}{3}\left(\psi_1\left(\frac{1}{3}\right)-9\right)\\
&=\zeta(3)-\frac{2\pi^2}{3}+15\left(-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{3}{2}\ln3+3+\gamma\right)-\frac{2}{3}\left(\psi_1\left(\frac{1}{3}\right)-9\right)\\
&=\zeta(3)-\frac{2\pi^2}{3}+45-\frac{5\sqrt{3}\pi}{2}-\frac{45}{2}\ln3-\frac{2}{3}\psi_1\left(\frac{1}{3}\right)+6\\
&=51-\frac{5\sqrt{3}\pi}{2}-\frac{45}{2}\ln3-\frac{2\pi^2}{3}-\frac{2}{3}\psi_1\left(\frac{1}{3}\right)+\zeta(3)\end{aligned}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru