

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\sum_{\text{cyc}} \left(\sqrt{\frac{\sin B + \sin C}{\sin A + \sin B}} \cdot \frac{\sqrt{\sin B + \sin C} + \sqrt{\sin A + \sin C}}{\sqrt{(\sin A + \sin B)(\sin B + \sin C) + \sin A + \sin C}} \right) \geq \sqrt{3\sqrt{3}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \forall A', B', C', x', y', z' > 0, \\ & \frac{x'}{y' + z'}(B' + C') + \frac{y'}{z' + x'}(C' + A') + \frac{z'}{x' + y'}(A' + B') \stackrel{\text{Walter Janous}}{\underset{\textcircled{1}}{\geq}} \sqrt{3 \sum_{\text{cyc}} A' B'} \\ & \text{Now, } \sum_{\text{cyc}} \left(\sqrt{\frac{\sin B + \sin C}{\sin A + \sin B}} \cdot \frac{\sqrt{\sin B + \sin C} + \sqrt{\sin A + \sin C}}{\sqrt{(\sin A + \sin B)(\sin B + \sin C) + \sin A + \sin C}} \right) \\ & = \sum_{\text{cyc}} \left(\frac{x}{z} \cdot \frac{x + y}{zx + y^2} \right) (x = \sqrt{\sin B + \sin C}, y = \sqrt{\sin C + \sin A}, z = \sqrt{\sin A + \sin B}) \\ & = \sum_{\text{cyc}} \left(\frac{x^2 y}{y^2 z + z^2 x} \cdot \frac{x(x + y)}{x^2 y} \right) = \sum_{\text{cyc}} \left(\frac{x^2 y}{y^2 z + z^2 x} \cdot \left(\frac{1}{x} + \frac{1}{y} \right) \right) \\ & = \frac{x'}{y' + z'}(B' + C') + \frac{y'}{z' + x'}(C' + A') + \frac{z'}{x' + y'}(A' + B') \\ & \left(x' = x^2 y, y' = y^2 z, z' = z^2 x, A' = \frac{1}{z}, B' = \frac{1}{x}, C' = \frac{1}{y} \right) \stackrel{\text{via } \textcircled{1}}{\geq} \sqrt{3 \sum_{\text{cyc}} \left(\frac{1}{xy} \right)} \stackrel{\text{Bergstrom}}{\geq} \\ & \sqrt{\frac{27}{\sum_{\text{cyc}} xy}} \geq \sqrt{\frac{27}{\sum_{\text{cyc}} x^2}} = \sqrt{\frac{27}{2 \sum_{\text{cyc}} \sin A}} = \sqrt{\frac{27}{\left(\frac{2s}{R} \right)}} \stackrel{\text{Mitrinovic}}{\geq} \sqrt{\frac{27}{3\sqrt{3}}} = \sqrt{3\sqrt{3}} \\ & \text{so, } \sum_{\text{cyc}} \left(\sqrt{\frac{\sin B + \sin C}{\sin A + \sin B}} \cdot \frac{\sqrt{\sin B + \sin C} + \sqrt{\sin A + \sin C}}{\sqrt{(\sin A + \sin B)(\sin B + \sin C) + \sin A + \sin C}} \right) \geq \sqrt{3\sqrt{3}} \\ & \forall \Delta ABC, '' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$