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In any ΔABC the following relationship holds :

$$\frac{\csc^2 \frac{A}{2} + \csc \frac{B}{2} \csc \frac{C}{2}}{\csc \frac{B}{2} + \csc \frac{C}{2}} + \frac{\csc^2 \frac{B}{2} + \csc \frac{C}{2} \csc \frac{A}{2}}{\csc \frac{C}{2} + \csc \frac{A}{2}} + \frac{\csc^2 \frac{C}{2} + \csc \frac{A}{2} \csc \frac{B}{2}}{\csc \frac{A}{2} + \csc \frac{B}{2}} \geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{\csc^2 \frac{A}{2} + \csc \frac{B}{2} \csc \frac{C}{2}}{\csc \frac{B}{2} + \csc \frac{C}{2}} &= \frac{\frac{1}{\sin^2 \frac{A}{2}} + \frac{1}{\sin \frac{B}{2} \sin \frac{C}{2}}}{\frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}}} = \sum_{\text{cyc}} \frac{\left(\frac{1}{\sin \frac{A}{2}}\right)^2}{\frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}}} + \sum_{\text{cyc}} \frac{1}{\sin \frac{B}{2} + \sin \frac{C}{2}} \\ &\stackrel{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}} \frac{1}{\sin \frac{A}{2}}\right)^2}{2 \sum_{\text{cyc}} \frac{1}{\sin \frac{A}{2}}} + \frac{9}{2 \sum_{\text{cyc}} \sin \frac{A}{2}} \stackrel{\text{Jensen}}{\geq} \frac{3}{2} \cdot \csc \frac{\pi}{6} + \frac{9}{6 \sin \frac{\pi}{6}} = 3 + 3 = 6 \text{ and so,} \\ \sum_{\text{cyc}} \frac{\csc^2 \frac{A}{2} + \csc \frac{B}{2} \csc \frac{C}{2}}{\csc \frac{B}{2} + \csc \frac{C}{2}} &\geq 6 \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)} \end{aligned}$$