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In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} h_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) \leq 0$$

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$$\text{WLOG: } a \leq b \leq c \Rightarrow \begin{cases} h_a \geq h_b \geq h_c \\ \sin \frac{A}{2} \leq \sin \frac{B}{2} \leq \sin \frac{C}{2} \end{cases}$$

By rearrangements' inequality:

$$\begin{aligned} h_a \sin \frac{A}{2} + h_b \sin \frac{B}{2} + h_c \sin \frac{C}{2} &\leq h_a \sin \frac{B}{2} + h_b \sin \frac{C}{2} + h_c \sin \frac{A}{2} \\ h_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) + h_b \left(\sin \frac{B}{2} - \sin \frac{C}{2} \right) + h_c \left(\sin \frac{C}{2} - \sin \frac{A}{2} \right) &\leq 0 \\ \sum_{cyc} h_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) &\leq 0 \end{aligned}$$

Equality holds for $a = b = c$.