

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} r_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) \geq 0$$

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$$\text{WLOG: } a \leq b \leq c \Rightarrow -a \geq -b \geq -c \Rightarrow s-a \geq s-b \geq s-c$$

$$\frac{1}{s-a} \leq \frac{1}{s-b} \leq \frac{1}{s-c} \Rightarrow \frac{F}{s-a} \leq \frac{F}{s-b} \leq \frac{F}{s-c} \Rightarrow r_a \leq r_b \leq r_c$$

$$a \leq b \leq c \Rightarrow \sin \frac{A}{2} \leq \sin \frac{B}{2} \leq \sin \frac{C}{2}$$

By rearrangements' inequality:

$$r_a \sin \frac{A}{2} + r_b \sin \frac{B}{2} + r_c \sin \frac{C}{2} \leq r_a \sin \frac{B}{2} + r_b \sin \frac{C}{2} + r_c \sin \frac{A}{2}$$

$$r_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) + r_b \left(\sin \frac{B}{2} - \sin \frac{C}{2} \right) + r_c \left(\sin \frac{C}{2} - \sin \frac{A}{2} \right) \leq 0$$

$$\sum_{cyc} r_a \left(\sin \frac{A}{2} - \sin \frac{B}{2} \right) \leq 0$$

Equality holds for $a = b = c$.