

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} r_a \left(\cos \frac{A}{2} - \cos \frac{B}{2} \right) \leq 0$$

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$$\text{WLOG: } a \leq b \leq c \Rightarrow -a \geq -b \geq -c \Rightarrow$$

$$\Rightarrow s - a \geq s - b \geq s - c \Rightarrow \frac{1}{s - a} \leq \frac{1}{s - b} \leq \frac{1}{s - c} \Rightarrow$$

$$\Rightarrow \frac{F}{s - a} \leq \frac{F}{s - b} \leq \frac{F}{s - c} \Rightarrow r_a \leq r_b \leq r_c$$

$$a \leq b \leq c \Rightarrow \cos \frac{A}{2} \geq \cos \frac{B}{2} \geq \cos \frac{C}{2}$$

By rearrangements' inequality:

$$r_a \cos \frac{A}{2} + r_b \cos \frac{B}{2} + r_c \cos \frac{C}{2} \leq r_a \cos \frac{B}{2} + r_b \cos \frac{C}{2} + r_c \cos \frac{A}{2}$$

$$r_a \left(\cos \frac{A}{2} - \cos \frac{B}{2} \right) + r_b \left(\cos \frac{B}{2} - \cos \frac{C}{2} \right) + r_c \left(\cos \frac{C}{2} - \cos \frac{A}{2} \right) \leq 0$$

$$\sum_{cyc} r_a \left(\cos \frac{A}{2} - \cos \frac{B}{2} \right) \leq 0$$

Equality holds for $a = b = c$.