

ROMANIAN MATHEMATICAL MAGAZINE

In acute $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} \tan\left(\frac{A}{2}\right) \sec(A) \geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc} \tan\left(\frac{A}{2}\right) \sec(A) = \sum_{cyc} \tan\left(\frac{A}{2}\right) \cdot \frac{1 + \tan^2\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)} = \sum_{cyc} \frac{\tan\left(\frac{A}{2}\right) + \tan^3\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$$

$$\text{Let } \tan\left(\frac{A}{2}\right) = x, \quad x \in (0; 1), \quad f(x) = \frac{x + x^3}{1 - x^2} \rightarrow f''(x) = \frac{4x^3 + 12x}{(1 - x^2)^3} > 0$$

So $f(x)$ is convex function : By Jensen's Inequality

$$\begin{aligned} \sum_{cyc} \frac{\tan\left(\frac{A}{2}\right) + \tan^3\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)} &\geq 3 \left(\frac{\tan\frac{A+B+C}{6} + \tan^3\frac{A+B+C}{6}}{1 - \tan^2\frac{A+B+C}{6}} \right) = 3 \left(\frac{\left(\frac{\sqrt{3}}{3}\right) + \left(\frac{\sqrt{3}}{3}\right)^3}{1 - \left(\frac{\sqrt{3}}{3}\right)^2} \right) = \\ &= 3 \cdot \frac{36\sqrt{3}}{54} = 2\sqrt{3} \end{aligned}$$

Equality holds if $a = b = c$