

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$\sum_{cyc} a(h_a - h_b) \leq 0$$

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WLOG: $a \leq b \leq c \Rightarrow h_a \geq h_b \geq h_c$.

By rearrangements' inequality:

$$\begin{aligned} ah_a + bh_b + ch_c &\leq ah_b + bh_c + ch_a \\ a(h_a - h_b) + b(h_b - h_c) + c(h_c - h_a) &\leq 0 \\ \sum_{cyc} a(h_a - h_b) &\leq 0 \end{aligned}$$

Equality holds for $a = b = c$.

Observation:

$$\begin{aligned} a \leq b &\Rightarrow h_a \geq h_b \\ h_a \geq h_b &\Leftrightarrow \frac{2F}{a} \geq \frac{2F}{b} \Leftrightarrow \frac{1}{a} \geq \frac{1}{b} \Leftrightarrow a \leq b \end{aligned}$$