ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$\sum_{cyc} h_a\,(m_a - m_b) \geq 0$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

WLOG: $a \leq b \leq c \Rightarrow \begin{cases} h_a \geq h_b \geq h_c \\ m_a \geq m_b \geq m_c \end{cases}$

By rearrangements' inequality

$$h_a m_a + h_b m_b + h_c m_c \geq h_a m_b + h_b m_c + h_c m_a$$

$$h_a(m_a - m_b) + h_b(m_b - m_c) + h_c(m_c - m_a) \geq 0$$

$$\sum_{cyc} h_a\,(m_a - m_b) \geq 0$$

Equality holds for $a = b = c$.

Observation 1

$$a \leq b \Leftrightarrow \frac{1}{a} \geq \frac{1}{b} \Leftrightarrow \frac{2F}{a} \geq \frac{2F}{b} \Leftrightarrow h_a \geq h_b$$

Observation 2

$$a \leq b \Leftrightarrow m_a \geq m_b$$

$$m_a \geq m_b \Leftrightarrow m_a^2 \geq m_b^2 \Leftrightarrow \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2 \geq \frac{1}{2}(a^2 + c^2) - \frac{1}{4}b^2$$

$$\Leftrightarrow 2(b^2 + c^2) - a^2 \geq 2(a^2 + c^2) - b^2$$

$$\Leftrightarrow 2b^2 + 2c^2 - a^2 \geq 2a^2 + 2c^2 - b^2$$

$$\Leftrightarrow 3a^2 \leq 3b^2 \Leftrightarrow a^2 \leq b^2 \Leftrightarrow a \leq b$$