ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC with $b = \text{mid}\{a, b, c\}$ the following relationship holds :

$$\text{If } k \geq \frac{a+c}{b}, k \in (1,2], \text{then} : s^2 \geq 2Rr.\frac{(k+1)^2}{k-1} - \left(\frac{k+1}{k-1}\right)^2 r^2$$

$$\text{If } \frac{a+c}{b} \geq m \geq 2, \text{then} : s^2 \leq 2Rr.\frac{(m+1)^2}{m-1} - \left(\frac{m+1}{m-1}\right)^2 r^2$$

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$$\text{Let } \cos\frac{A-C}{2} = C \text{ and } \sin\frac{B}{2} = S \text{ and now, } b = \text{mid}\{a,b,c\} \Rightarrow$$

$$(b-a)(b-c) \leq 0 \Rightarrow b^2 - b(a+c) + ac \leq 0$$

$$\Rightarrow 16R^2S^2(1-S^2) - 16R^2SC(1-S^2) + 4R^2(C^2-S^2) \leq 0$$

$$\Rightarrow C^2 - 4C(S-S^3) + 3S^2 - 4S^4 \leq 0 \text{ and } \because \text{discriminant, } \delta =$$

$$16(S-S^3)^2 - 4(3S^2-4S^4) = 4S^2(2S^2-1)^2 \therefore C \leq \frac{4(S-S^3)+2S(1-2S^2)}{2}$$

$$= 3S - 4S^3 \left(\because C \leq \frac{4(S-S^3)+2S(2S^2-1)}{2} \Rightarrow C \leq S \rightarrow \text{impossible}\right)$$

$$\therefore C \leq 3S - 4S^3 \rightarrow (1) \Rightarrow 1 < \frac{C}{S} \leq 3 - 4S^2 \Rightarrow 1 > 2S^2 \rightarrow (2) \text{ \& now, we proceed to}$$

$$\text{prove the 1st problem : We have : } s^2 \overset{?}{\geq} 2Rr.\frac{(k+1)^2}{k-1} - \left(\frac{k+1}{k-1}\right)^2 r^2 \Leftrightarrow$$

$$4R^2(1-S^2)(C+S)^2(k-1)^2 - 4R^2S(C-S)(k-1)(k+1)^2 +$$

$$4R^2S^2(C-S)^2(k+1)^2 \overset{?}{\geq} 0 \Leftrightarrow (4kS^2 + (k-1)^2)C^2 -$$

$$(4k^2S^3 + k^3S - k^2S + 4S^3 + 3kS - 3S)C + k^3S^2 + 4kS^4 + 2k^2S^2 - 3kS^2 \underset{(*)}{\overset{?}{\geq}} 0$$

Case 1 $4S^2 \leq 3 - k$ and now, $k \geq \frac{a+c}{b} \Rightarrow k \geq \frac{C}{S} \Rightarrow C \leq kS \rightarrow (3)$ and and $\because P =$

$(4kS^2 + (k-1)^2)(C-kS)^2 \geq 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{\geq} P \overset{\text{factoring out } (k^2-1)}{\Leftrightarrow} (k^2-1)(C-kS)(4S^2-3+k) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$\because 4S^2 \leq 3 - k, k > 1$ and via $(3) \Rightarrow$ **$(*)$ is true**

Case 2 $4S^2 > 3 - k \wedge S \geq \frac{1}{2}$ and now, in order to prove :

$$(1-2S^2)k^2 - (8S^4 - 10S^2 + 4)k + 3 - 4S^2 \overset{?}{\leq} 0, \text{it suffices to prove :}$$

$$2(1-2S^2)k \underset{(i)}{\overset{?}{\leq}} 8S^4 - 10S^2 + 4 + \sqrt{\delta} \text{ AND } 2(1-2S^2)k \underset{(ii)}{\overset{?}{\geq}} 8S^4 - 10S^2 + 4 - \sqrt{\delta};$$

$$\delta = 4(1-S^2)^2(1-4S^2)^2 \geq 0; \text{Indeed, } 2(1-2S^2)k - (8S^4 - 10S^2 + 4) \overset{k \leq 2}{\leq}$$

$$4(1-2S^2) - (8S^4 - 10S^2 + 4) \overset{?}{\leq} \sqrt{\delta} \Leftrightarrow 2S^2(4S^2-1) + \sqrt{\delta} \overset{?}{\geq} 0 \rightarrow \text{true} \because S \geq \frac{1}{2};$$

$$\text{also, } 2(1-2S^2)k - (8S^4 - 10S^2 + 4) \overset{k \geq 3-4S^2}{\geq} 2(3-4S^2)(1-2S^2) -$$

$$(8S^4-10S^2+4)\overset{?}{=}-\sqrt{\delta}\Leftrightarrow\sqrt{\delta}\overset{?}{=}2(1-S^2)(4S^2-1)$$

$$\Leftrightarrow 4(1-S^2)^2(1-4S^2)^2\overset{?}{=}4(1-S^2)^2(1-4S^2)^2\to\text{true}\therefore\text{(i) and (ii) are true}$$

$$\therefore 8kS^4+2k^2S^2-10kS^2-k^2+4S^2+4k-3\geq 0\to\textcircled{4};\ \text{So, }\textcircled{1}\text{ and }\textcircled{3}\Rightarrow Q_1=(4kS^2+(k-1)^2)(C-3S+4S^3)(C-kS)\geq 0\therefore\text{to prove }(*),\text{ it suffices to prove :}$$

$$\text{LHS of }(*)\overset{?}{\geq}Q_1\Leftrightarrow 8k^2S^5+2k^3S^3-10k^2S^3-k^3S+4kS^3+4k^2S-3kS\overset{(**)}{\overset{?}{\geq}}$$

$$(8kS^4+2k^2S^2-10kS^2-k^2+4S^2+4k-3)C;\ \&\ \text{RHS of }(**)\overset{\text{via }\textcircled{3}\text{ and }\textcircled{4}}{\leq}$$

$$(8kS^4+2k^2S^2-10kS^2-k^2+4S^2+4k-3)kS=\text{LHS of }(**)\Rightarrow\boxed{(*)\text{ is true}}$$

$$4S^2>3-k\wedge S<\frac{1}{2}\Rightarrow 1>4S^2>3-k\Rightarrow k>2\to\text{a contradiction}$$

$$\therefore 4S^2>3-k\wedge S<\tfrac{1}{2}\text{ is impossible and now, in order to prove :}$$

$$(1-2S^2)m^2-(8S^4-10S^2+4)m+3-4S^2\overset{?}{\leq}0,\text{ it suffices to prove :}$$

$$2(1-2S^2)m\overset{(m)}{\overset{?}{\leq}}8S^4-10S^2+4+\sqrt{\delta}\ \text{AND}\ 2(1-2S^2)m\overset{(n)}{\overset{?}{\geq}}8S^4-10S^2+4-\sqrt{\delta}$$

$$\text{Indeed, }2(1-2S^2)m-(8S^4-10S^2+4)\overset{m\leq\frac{C}{S}\leq 3-4S^2\text{ via }\textcircled{1}}{\leq}2(1-2S^2)(3-4S^2)-(8S^4-10S^2+4)\overset{?}{=}\sqrt{\delta}\Leftrightarrow 2(1-S^2)(4S^2-1)\overset{?}{=}\sqrt{\delta}\Leftrightarrow 4(1-S^2)^2(1-4S^2)^2\overset{?}{=}4(1-S^2)^2(1-4S^2)^2\to\text{true and also, }2(1-2S^2)m-(8S^4-10S^2+4)\overset{m\geq 2}{\geq}4(1-2S^2)-(8S^4-10S^2+4)\overset{?}{\geq}-\sqrt{\delta}\Leftrightarrow 2S^2(1-4S^2)+\sqrt{\delta}\overset{?}{\geq}0\to\text{true}$$

$$\because S\leq\frac{1}{2};\ as\ 2\leq m\leq\frac{C}{S}\leq 3-4S^2\therefore\text{(m) and (n) are true}$$

$$\therefore 8mS^4+2m^2S^2-10mS^2-m^2+4S^2+4m-3\geq 0\to\textcircled{6}$$

$$\text{We have : }s^2\overset{?}{\leq}2Rr.\frac{(m+1)^2}{m-1}-\left(\frac{m+1}{m-1}\right)^2r^2\Leftrightarrow(4mS^2+(m-1)^2)C^2-(4m^2S^3+m^3S-m^2S+4S^3+3mS-3S)C+m^3S^2+4mS^4+2m^2S^2-3mS^2\overset{(\bullet)}{\overset{?}{\geq}}0$$

$$\therefore\textcircled{1},\textcircled{6}\text{ and }C-mS\geq 0\Rightarrow Q_2=(4kS^2+(m-1)^2)(C-3S+4S^3)(C-mS)+2S(8mS^4+2m^2S^2-10mS^2-m^2+4S^2+4m-3)(C-mS)\leq 0$$

$$\therefore\text{in order to prove }(\bullet),\text{ it suffices to prove : LHS of }(\bullet)\overset{?}{\leq}Q_2\Leftrightarrow 0\leq 0\to\text{true}\Rightarrow\boxed{(\bullet)\text{ is true}};\ \text{hence both problems are proved, }''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$