ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC and for all $k \in \mathbb{R}$ such that $k \geq 0$ the following relationship holds :

$$\frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k} \geq \frac{1}{r+k}$$

$$\frac{1}{r+\frac{k}{3}} \geq \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \geq \frac{1}{r+\frac{k}{2}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k} \overset{?}{\geq} \frac{1}{r+k} \Leftrightarrow \frac{s^2+2k(4R+r)++3k^2}{rs^2+ks^2+k^2(4R+r)+k^3} \overset{?}{\geq} \frac{1}{r+k}$$

$$\Leftrightarrow 2k\left(k^2+2k(R+r)+r(4R+r)\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because k \geq 0$$

$$\therefore \frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k} \geq \frac{1}{r+k} \text{ and now, } \frac{1}{r+\frac{k}{3}} \overset{?}{\geq} \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k}$$

$$\Leftrightarrow \frac{3}{3r+k} \overset{?}{\geq} \frac{\frac{2rs^2}{R}+\frac{2k(s^2+4Rr+r^2)}{2R}+3k^2}{\frac{2r^2s^2}{R}+k.\frac{2rs^2}{R}+\frac{k^2(s^2+4Rr+r^2)}{2R}+k^3}$$

$$\Leftrightarrow k.2r\left(s^2-16Rr+5r^2+4r(R-2r)\right)+k^2\left(s^2-16Rr+5r^2+2r(R-2r)\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because s^2-16Rr+5r^2 \overset{\text{Gerretsen}}{\geq} 0 \text{ and } R-2r \overset{\text{Euler}}{\geq} 0$$

$$\therefore \frac{1}{r+\frac{k}{3}} \geq \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \text{ and finally, } \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \overset{?}{\geq} \frac{1}{r+\frac{k}{2}}$$

$$\Leftrightarrow \frac{\frac{2rs^2}{R}+\frac{2k(s^2+4Rr+r^2)}{2R}+3k^2}{\frac{2r^2s^2}{R}+k.\frac{2rs^2}{R}+\frac{k^2(s^2+4Rr+r^2)}{2R}+k^3} \overset{?}{\geq} \frac{2}{2r+k}$$

$$\Leftrightarrow 2k\left(Rk^2+6Rrk+2r^2(4R+r)\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because k \geq 0$$

$$\therefore \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \geq \frac{1}{r+\frac{k}{2}} \text{ and so,}$$

$$\frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k} \geq \frac{1}{r+k} \; \forall \Delta ABC \text{ and } \forall k \geq 0, '' = '' \text{ iff } k = 0 \text{ and}$$

$$\frac{1}{r+\frac{k}{3}} \geq \frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \; \forall \Delta ABC \; \& \; \forall k \geq 0, '' = '' \text{ iff } \Delta ABC \text{ is equilateral}$$

$$\frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \geq \frac{1}{r+\frac{k}{2}} \; \forall \Delta ABC \text{ and } \forall k \geq 0, '' = '' \text{ iff } k = 0 \; (\text{QED})$$