ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$3r\sum_{cyc} m_a \le w_a r_a + w_b r_b + w_c r_c$$

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$$\sum_{cyc} w_a r_a = \sum_{cyc} \frac{r_a h_a}{\frac{h_a}{w_a}} = \frac{rs^2}{2R}\cdot\sum_{cyc}\frac{\sec^2\frac{A}{2}}{\cos\frac{B-C}{2}} = \frac{rs^2}{2R}\cdot\sum_{cyc}\frac{\sec^4\frac{A}{2}}{\sec^2\frac{A}{2}\cdot\cos\frac{B-C}{2}} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{rs^2}{2R}\cdot\frac{\left(\sum_{cyc}\sec^2\frac{A}{2}\right)^2}{\sum_{cyc}\left(\sec^2\frac{A}{2}\cdot\cos\frac{B-C}{2}\right)} \overset{CBS}{\ge} \frac{rs^2}{2R}\cdot\frac{\left(\sum_{cyc}\sec^2\frac{A}{2}\right)^2}{\sqrt{\sum_{cyc}\sec^2\frac{A}{2}}\cdot\sqrt{\sum_{cyc}\left(\sec^2\frac{A}{2}\cos^2\frac{B-C}{2}\right)}}$$

$$\therefore \left(\sum_{cyc} w_a r_a\right)^2 \overset{(*)}{\ge} \frac{r^2s^4}{4R^2}\cdot\frac{\left(\sum_{cyc}\sec^2\frac{A}{2}\right)^3}{\sum_{cyc}\left(\sec^2\frac{A}{2}\cos^2\frac{B-C}{2}\right)}$$

$$\text{Now, } \sum_{cyc}\left(\sec^2\frac{A}{2}\cos^2\frac{B-C}{2}\right) = \sum_{cyc}\left(\frac{(b+c)^2}{16R^2\cos^2\frac{A}{2}\cdot\sin^2\frac{A}{2}}\cdot\frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}\right)$$

$$= \frac{1}{16R^2s^2}\cdot\sum_{cyc}\left(\frac{s^2+(s-a)^2+2s(s-a)}{(s-a)^2}\cdot b^2c^2\right) \to (1)$$

- $$\sum_{cyc}\frac{s^2b^2c^2}{(s-a)^2} = s^4\cdot\sum_{cyc}\sec^4\frac{A}{2} = s^4\cdot\left(\left(\sum_{cyc}\sec^2\frac{A}{2}\right)^2 - 2\cdot\frac{16R^2}{s^2}\cdot\sum_{cyc}\cos^2\frac{A}{2}\right)$$

$$= s^4\cdot\left(\frac{((4R+r)^2+s^2)^2}{s^4} - 2\cdot\frac{16R^2}{s^2}\cdot\frac{4R+r}{2R}\right) = \sum_{cyc}\frac{s^2b^2c^2}{(s-a)^2} \to (2)$$

- $$\sum_{cyc}\left(\frac{(s-a)^2}{(s-a)^2}\cdot b^2c^2\right) = (s^2+4Rr+r^2)^2 - 16Rrs^2 \to (3)$$

- $$\sum_{cyc}\frac{2s(s-a)b^2c^2}{(s-a)^2} = \frac{2s}{r^2s}\cdot\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right) =$$

$$\frac{2}{r^2}\left(\begin{matrix}-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)+(s^2+4Rr+r^2)^3\\ -24Rrs^2(s^2+2Rr+r^2)\end{matrix}\right)$$

$$\sum_{cyc}\frac{2s(s-a)b^2c^2}{(s-a)^2} = 2(s^4-(4Rr-2r^2)s^2+r(4R+r)^3) \to (4) \therefore (1),(2),(3),(4) \Rightarrow$$

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$$\sum_{cyc}\sec^2\frac{A}{2}\cos^2\frac{B-C}{2}\overset{(**)}{=}\frac{s^4-(8R^2+4Rr-2r^2)s^2+64R^4+96R^3r+52R^2r^2+12Rr^3+r^4}{4R^2s^2}$$

$$\therefore via\ (*)\ and\ (**),\left(\sum_{cyc}w_ar_a\right)^2\geq\frac{\frac{r^2s^4}{4R^2}\cdot\frac{\left((4R+r)^2+s^2\right)^3}{s^6}}{\frac{s^4-(8R^2+4Rr-2r^2)s^2+64R^4+96R^3r+52R^2r^2+12Rr^3+r^4}{4R^2s^2}}$$

$$\overset{?}{\geq}9r^2(4s^2-16Rr+5r^2)^2\Leftrightarrow-35s^6+(336R^2+312Rr-114r^2)s^4-$$
$$(1536R^4+3840R^3r+1800R^2r^2-84Rr^3+123r^4)s^2+$$
$$4\overbrace{(1024R^6+3840R^5r+3696R^4r^2+1112R^3r^3-93R^2r^4-93Rr^5-11r^6)}^{X}\overset{?}{\underset{(\bullet)}{\geq}}0\ and$$

$$\because P-35(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{Double-Rouche}{\geq}0\ \therefore to\ prove\ (\bullet),$$
$$it\ suffices\ to\ prove:LHS\ of\ (\bullet)\overset{?}{\geq}P\Leftrightarrow(49R^2-97Rr-11r^2)s^4-$$
$$(384R^4+400R^3r+30R^2r^2-126Rr^3+22r^4)s^2+X\overset{?}{\underset{(\bullet\bullet)}{\geq}}0$$

Case 1 $49R^2-97Rr-11r^2\geq0\ and\ then:since\ Q=$
$$(49R^2-97Rr-11r^2)(s^2-4R^2-4Rr-3r^2)^2\overset{Gerretsen}{\geq}0\ \therefore in\ order\ to\ prove\ (\bullet\bullet),$$
$$it\ suffices\ to\ prove:LHS\ of\ (\bullet\bullet)\overset{?}{\geq}Q$$
$$\Leftrightarrow\overbrace{(4R^4-392R^3r-300R^2r^2-272Rr^3-44r^4)}^{Y}s^2+$$
$$\overbrace{120R^6+1912R^5r+2508R^4r^2+2084R^3r^3+1117R^2r^4+522Rr^5+44r^6}^{Z}\overset{?}{\underset{(\bullet\bullet\bullet)}{\geq}}0\ and\ it's$$

$$trivially\ true\ if:Y\geq0\ and\ when:Y<0,then:LHS\ of\ (\bullet\bullet\bullet)\overset{Gerretsen}{\geq}$$
$$Y(4R^2+4Rr+3r^2)+Z\overset{?}{\geq}0\Leftrightarrow(t-2)\begin{pmatrix}136t^5+632t^4+1016t^3+\\652t^2+257t+44\end{pmatrix}\overset{?}{\geq}0$$
$$\left(t=\frac{R}{r}\right)\rightarrow true\because t\overset{Euler}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\ is\ true$$

Case 2 $49R^2-97Rr-11r^2<0\ and\ then:since\ T=$
$$(49R^2-97Rr-11r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)-$$
$$8R(94R^3-96R^2r+1056Rr^2-50r^3)(s^2-4R^2-4Rr-3r^2)\overset{Double Rouche,Gerretsen}{\geq}$$
$$0\ \therefore in\ order\ to\ prove\ (\bullet\bullet),it\ suffices\ to\ prove:LHS\ of\ (\bullet\bullet)\overset{?}{\geq}T\Leftrightarrow$$
$$(t-2)(136t^4+632t^3+918t^2+1042t-109)\overset{?}{\geq}0\rightarrow true\because t\overset{Euler}{\geq}2$$
$$\Rightarrow(\bullet\bullet)\ is\ true\ \therefore combining\ both\ cases,(\bullet\bullet)\Rightarrow(\bullet)\ is\ true\ \forall\ \Delta\ ABC\ \therefore\sum_{cyc}w_ar_a\geq$$

$$3r.\sqrt{4s^2-16Rr+5r^2}\overset{Chu-Yang}{\geq}3r.\sum_{cyc}m_a\ ''=''\ iff\ \Delta\ ABC\ is\ equilateral\ (QED)$$