ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC and for all $k \in \mathbb{R}$ such that $k \geq 0$ the following relationship holds :

$$\frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \geq \frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \overset{?}{\geq} \frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k}$$

$$\Leftrightarrow \frac{\frac{2rs^2}{R}+\frac{2k(s^2+4Rr+r^2)}{2R}+3k^2}{\frac{2r^2s^2}{R}+k.\frac{2rs^2}{R}+\frac{k^2(s^2+4Rr+r^2)}{2R}+k^3} \overset{?}{\geq} \frac{s^2+2k(4R+r)++3k^2}{rs^2+ks^2+k^2(4R+r)+k^3}$$

$$4k^3s^2(R-2r)+k^4\left(4R^2+4Rr+3r^2-s^2+2(R-2r)(2R+r)\right) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because s^2-16Rr+5r^2 \overset{\text{Gerretsen}}{\geq} 0, R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } 4R^2+4Rr+3r^2-s^2 \overset{\text{Gerretsen}}{\geq} 0$$

and $\because k \geq 0$ and so, $\frac{1}{h_a+k}+\frac{1}{h_b+k}+\frac{1}{h_c+k} \geq \frac{1}{r_a+k}+\frac{1}{r_b+k}+\frac{1}{r_c+k}$ $\forall \Delta ABC$

and $\forall k \geq 0, '' = ''$ iff ΔABC is equilateral (QED)