ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC and for all $k \in \mathbb{R}$ such that $k \geq 0$, the following relationship holds :

$$\frac{1}{h_a+kR}+\frac{1}{h_b+kR}+\frac{1}{h_c+kR} \leq \frac{1}{r_a+2kr}+\frac{1}{r_b+2kr}+\frac{1}{r_c+2kr}$$

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$$\frac{1}{h_a+kR}+\frac{1}{h_b+kR}+\frac{1}{h_c+kR} \overset{?}{\leq} \frac{1}{r_a+2kr}+\frac{1}{r_b+2kr}+\frac{1}{r_c+2kr}$$

$$\Leftrightarrow \frac{\frac{2rs^2}{R}+\frac{2kR(s^2+4Rr+r^2)}{2R}+3k^2R^2}{\frac{2r^2s^2}{R}+k^3R^3+kR.\frac{2rs^2}{R}+\frac{k^2R^2(s^2+4Rr+r^2)}{2R}} \overset{?}{\leq} \frac{s^2+4kr(4R+r)+12k^2r^2}{rs^2+8k^3r^3+4k^2r^2(4R+r)+2krs^2}$$

$$\Leftrightarrow 2rs^2\left(\overbrace{(R-4r)s^2-r(4R^2-31Rr-8r^2)}^{X}\right)+$$

$$ks^2\left(\overbrace{(R^2-4Rr)s^2-r(2R^3-49R^2r+52Rr^2-32r^3)}^{Y}\right)+$$

$$k^2(2R^4+4R^3r-28R^2r^2+40Rr^3-32r^4)s^2+$$

$$k^2r^2(64R^4-96R^3r-60R^2r^2-8Rr^3)+$$

$$k^3Rr\left(\overbrace{32R^4-88R^3r+24R^2r^2-52Rr^3-16r^4+r(12R-16r)s^2}^{Z}\right) \overset{?}{\geq} 0 \quad (*)$$

Now, in order to prove : $X \geq 0$, we seggregate into 2 cases :

Case 1_X $R-4r \geq 0$ **and then :** $X \overset{\text{Gerretsen}}{\geq}$

$$(R-4r)(16Rr-5r^2)-r(4R^2-31Rr-8r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2r(6R-7r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow X \geq 0$$

Case 2_X $R-4r < 0$ **and then :** $X \overset{\text{Gerretsen}}{\geq}$

$$(R-4r)(4R^2+4Rr+3r^2)-r(4R^2-31Rr-8r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2t^3-8t^2+9t-2 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(2t(t-2)+1) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow X \geq 0 \therefore$ **combining both cases,** $X \geq 0 \; \forall \; \Delta ABC \rightarrow (1)$; **again, in order to prove :** $Y \geq 0$, **we seggregate into 2 cases :**

Case 1_Y $R^2-4Rr \geq 0$ **and then :** $Y \overset{\text{Gerretsen}}{\geq}$

$$(R^2-4Rr)(16Rr-5r^2)-r(2R^3-49R^2r+52Rr^2-32r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow 7t^3-10t^2-16t+16 \overset{?}{\geq} 0 \Leftrightarrow (t-2)\left(7t^2+4(t-2)\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow Y \geq 0$$

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$$\boxed{Case\ 2_Y}\ R^2-4Rr<0\ and\ then: Y\overset{Gerretsen}{\geq}$$

$$(R^2-4Rr)(4R^2+4Rr+3r^2)-r(2R^3-49R^2r+52Rr^2-32r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow 2t^4-7t^3+18t^2-32t+16\overset{?}{\geq}0\Leftrightarrow(t-2)(2t^3-3t^2+12t-8)\overset{?}{\geq}0\rightarrow true$$

$$\because t\overset{Euler}{\geq}2\Rightarrow Y\geq 0\therefore combining\ both\ cases, Y\geq 0\ \forall\Delta ABC\rightarrow (2)\ and\ also,$$

$$k^2(2R^4+4R^3r-28R^2r^2+40Rr^3-32r^4)s^2+$$
$$k^2r^2(64R^4-96R^3r-60R^2r^2-8Rr^3)$$

$$=k^2(R-2r)(R^3+4R^2r-6Rr^2+8r^3)+4Rk^2r^2(R-2r)(4R+r)^2\overset{Euler}{\geq}0\rightarrow (3)$$

$$and\ finally, 32R^4-88R^3r+24R^2r^2-52Rr^3-16r^4+r(12R-16r)s^2\overset{Gerretsen}{\geq}$$

$$32R^4-88R^3r+24R^2r^2-52Rr^3-16r^4+r(12R-16r)(16Rr-5r^2)\overset{?}{\geq}0$$

$$\Leftrightarrow 4t^4-11t^3+27t^2-46t+8\overset{?}{\geq}0\Leftrightarrow(t-2)(4t^3-3t^2+21t-4)\overset{?}{\geq}0\rightarrow true$$

$$\because t\overset{Euler}{\geq}2\Rightarrow Z\geq 0\ \forall\Delta ABC\rightarrow (4)\therefore (1)+(2)+(3)+(4)\ combined\ with\ k\geq 0$$
$$\Rightarrow (*)\ is\ true\ and\ so,$$

$$\frac{1}{h_a+kR}+\frac{1}{h_b+kR}+\frac{1}{h_c+kR}\leq\frac{1}{r_a+2kr}+\frac{1}{r_b+2kr}+\frac{1}{r_c+2kr}\ \forall\Delta ABC$$
$$and\ \forall k\geq 0,'' = ''\ iff\ \Delta ABC\ is\ equilateral\ (QED)$$