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In any acute ΔABC the following relationship holds :

$$s.\sqrt{\frac{2R}{r}-1}\geq\frac{w_a r_a}{h_a}+\frac{w_b r_b}{h_b}+\frac{w_c r_c}{h_c}\geq\frac{9}{2}R$$

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$$\sum_{cyc}\left(r_a\cos^2\frac{B-C}{2}\right)=\frac{1}{abc}.\sum_{cyc}\left(\frac{s^2+(s-a)^2+2s(s-a)}{a}(s-b)(s-c)r_a\right)\rightarrow(1)$$

$$\bullet\ \frac{s^2}{4Rrs}.\sum_{cyc}\left((-s^2+sa+bc)\frac{r_a}{a}\right)=\frac{s}{4Rr}\left(-s^2.\frac{s}{4R}.\frac{s^2+(4R+r)^2}{s^2}+s(4R+r)\right)+$$

$$\frac{s}{4Rr}.\frac{rs}{4Rrs.r^2s}.\begin{pmatrix}-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)+\\(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\end{pmatrix}$$

$$=\frac{(4R+r)^3-(8R-r)s^2}{16R^2}$$

$$\therefore\frac{1}{abc}.\sum_{cyc}\frac{s^2(s-b)(s-c)r_a}{a}=\frac{(4R+r)^3-(8R-r)s^2}{16R^2}\rightarrow(2)$$

$$\bullet\ \frac{1}{abc}.\sum_{cyc}\frac{(s-a)^2(s-b)(s-c)r_a}{a}=\frac{r^2s}{4Rrs.4Rrs}.\sum_{cyc}\left((s-a)\frac{rs}{s-a}.bc\right)$$

$$\therefore\frac{1}{abc}.\sum_{cyc}\frac{(s-a)^2(s-b)(s-c)r_a}{a}=\frac{r(s^2+4Rr+r^2)}{16R^2}\rightarrow(3)$$

$$\bullet\ \frac{1}{abc}.\sum_{cyc}\frac{2s(s-a)(s-b)(s-c)r_a}{a}=\frac{2r(s^2+(4R+r)^2)}{16R^2}\rightarrow(4)$$

$$\therefore(1)\text{ combined with }(2)+(3)+(4)\Rightarrow\sum_{cyc}\left(r_a\cos^2\frac{B-C}{2}\right)=$$

$$\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)s^2}{4R^2}\rightarrow(5)\ \text{and now,}\ \sum_{cyc}\frac{w_a r_a}{h_a}=\sum_{cyc}\frac{r_a}{\cos\frac{B-C}{2}}=$$

$$\sum_{cyc}\frac{r_a^2}{r_a\cos\frac{B-C}{2}}\overset{\text{Bergstrom}}{\geq}\frac{(4R+r)^2}{\sum_{cyc}\left(r_a\cos\frac{B-C}{2}\right)}\overset{\text{CBS}}{\geq}\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\sum_{cyc}\left(r_a\cos^2\frac{B-C}{2}\right)}}\overset{\text{via }(5)}{=}$$

$$\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)s^2}{4R^2}}}\overset{\text{Walker}}{\geq}\frac{(4R+r)^2}{\sqrt{4R+r}.\sqrt{\frac{16R^3+20R^2r+8Rr^2+r^3-(2R-r)(2R^2+8Rr+3r^2)}{4R^2}}}$$

$$\overset{?}{\geq}\frac{9}{2}R\Leftrightarrow 26t^3+141t^2-309t-154\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

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$$\Leftrightarrow (t-2)(26t^2+193t+77)\overset{?}{\geq}0\rightarrow true\ \because t\overset{Euler}{\geq}2$$

$$\Rightarrow \boxed{\frac{w_a r_a}{h_a}+\frac{w_b r_b}{h_b}+\frac{w_c r_c}{h_c}\geq\frac{9}{2}R\ \forall\ acute\ \Delta\ ABC}\ and\ again,\ \sum_{cyc}\frac{w_a r_a}{h_a}=\sum_{cyc}\frac{r_a}{\cos\frac{B-C}{2}}$$

$$\overset{CBS}{\leq}\sqrt{\sum_{cyc}\frac{r_a}{\cos^2\frac{B-C}{2}}}\cdot\sqrt{4R+r}=\sqrt{\sum_{cyc}\frac{r_a\cdot a^2bc}{(b+c)^2(s-b)(s-c)}}\cdot\sqrt{4R+r}\overset{AM-GM}{\leq}$$

$$\sqrt{\sum_{cyc}\frac{\frac{rs}{s-a}\cdot a^2(s-a)}{4(s-b)(s-c)(s-a)}}\cdot\sqrt{4R+r}=\sqrt{\frac{(s^2-4Rr-r^2)(4R+r)}{2r}}\overset{?}{\leq}s\cdot\sqrt{\frac{2R}{r}-1}$$

$$\Leftrightarrow (4R+r)^2\overset{?}{\geq}3s^2\rightarrow true\ via\ Doucet\ (or\ Trucht)$$

$$\therefore \boxed{\frac{w_a r_a}{h_a}+\frac{w_b r_b}{h_b}+\frac{w_c r_c}{h_c}\leq s\cdot\sqrt{\frac{2R}{r}-1}\ \forall\ \Delta\ ABC}\ and\ so,$$

$$s\cdot\sqrt{\frac{2R}{r}-1}\geq\frac{w_a r_a}{h_a}+\frac{w_b r_b}{h_b}+\frac{w_c r_c}{h_c}\geq\frac{9}{2}R\ \forall\ acute\ \Delta\ ABC,$$

$$''=''\ iff\ \Delta\ ABC\ is\ equilateral\ (QED)$$