

In any ΔABC holds :

$$\frac{\sqrt{s^2 - 9Rr}}{r} \leq \frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \leq \frac{\sqrt{s^2 - 7Rr - 4r^2}}{r}$$

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$$\begin{aligned} \sum_{cyc} \frac{m_a^2}{r_a^2} &= \sum_{cyc} \left(\frac{2 \sum_{cyc} a^2 - 3a^2}{4r^2 s^2} \cdot (s-a)^2 \right) = \\ &= \frac{4(s^2 - 4Rr - r^2)(s^2 - 8Rr - 2r^2) - 3(s^2 \sum_{cyc} a^2 - 2s \sum_{cyc} a^3 + 2 \sum_{cyc} a^2 b^2 - 16r^2 s^2)}{4r^2 s^2} \\ &= \frac{4(s^2 - 4Rr - r^2)(s^2 - 8Rr - 2r^2) - 3(2s^2(s^2 - 4Rr - r^2) - 4s^2(s^2 - 6Rr - 3r^2) + 2(s^2 + 4Rr + r^2)^2 - 32Rrs^2 - 16r^2 s^2)}{4r^2 s^2} \quad \textcircled{1} \\ &= \sum_{cyc} \frac{m_a^2}{r_a^2}; \text{ Now, } 2m_b m_c \stackrel{?}{\geq} \frac{9}{4} a^2 + h_a^2 - m_b^2 - m_c^2 \\ &= \frac{a^2(5a^2 - b^2 - c^2) + 2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4}{4a^2} = \frac{a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2}{4a^2} \\ &\text{and it's trivially true if : } a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 \leq 0 \text{ and when :} \\ &\quad a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2 > 0, \text{ then we are to prove :} \\ &4a^4(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2) \stackrel{?}{\geq} (a^2(4a^2 + b^2 + c^2) - (b^2 - c^2)^2)^2 \Leftrightarrow \\ &(b^2 - c^2)^2(-a^4 + 2a^2(b^2 + c^2) - b^4 - c^4 + 2b^2 c^2) \stackrel{?}{\geq} 0 \Leftrightarrow (b^2 - c^2)^2 \cdot 16F^2 \stackrel{?}{\geq} 0 \\ &\rightarrow \text{true} \therefore 2m_b m_c \geq \frac{9}{4} a^2 + h_a^2 - m_b^2 - m_c^2 = \frac{6a^2 - \sum_{cyc} a^2}{4} + h_a^2 \text{ and analogs} \\ &\Rightarrow \sum_{cyc} \frac{2m_b m_c}{r_b r_c} \geq \sum_{cyc} \left(\frac{\frac{6a^2 - \sum_{cyc} a^2}{4} + h_a^2}{r^2 s^2} (-s^2 + sa + bc) \right) \\ &= \frac{6R^2 \left(-s^2 \sum_{cyc} a^2 + s \sum_{cyc} a^3 + 4Rrs \cdot 2s - 2(s^2 - 4Rr - r^2)(4Rr + r^2) \right) - s^2 \sum_{cyc} a^2 b^2 + 4Rrs^2(s^2 + 4Rr + r^2) + \sum_{cyc} a^3 b^3}{4R^2 r^2 s^2} \\ &= \frac{6}{4R^2 s^2} \left(\frac{-2s^2(s^2 - 4Rr - r^2) + 2s^2(s^2 - 6Rr - 3r^2) + 8Rrs^2}{2(s^2 - 4Rr - r^2)(4Rr + r^2)} \right) + \\ &\frac{1}{4R^2 r^2 s^2} \left(\frac{-s^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + 4Rrs^2(s^2 + 4Rr + r^2)}{(s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)} \right) \quad \textcircled{2} \sum_{cyc} \frac{2m_b m_c}{r_b r_c} \\ &\therefore \textcircled{1} + \textcircled{2} \xrightarrow{\text{upon simplification}} \left(\sum_{cyc} \frac{m_a}{r_a} \right)^2 = \sum_{cyc} \frac{m_a^2}{r_a^2} + \sum_{cyc} \frac{2m_b m_c}{r_b r_c} \geq \end{aligned}$$

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$$\frac{(4R^2 + r^2)s^4 - r(32R^3 + 32R^2r + 4Rr^2 - 2r^3)s^2 + r^2(64R^4 + 96R^3r + 52R^2r^2 + 12Rr^3 + r^4)}{4R^2r^2s^2}$$

$$\stackrel{?}{\geq} \frac{s^2 - 9Rr}{r^2} \Leftrightarrow rs^4 + (4R^3 - 32R^2r - 4Rr^2 + 2r^3)s^2 +$$

$$r(64R^4 + 96R^3r + 52R^2r^2 + 12Rr^3 + r^4) \stackrel{?}{\geq} 0 \text{ \& } \because (s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove (*), it suffices to prove : LHS of (*) $\stackrel{?}{\geq} r(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow \underbrace{(R^3 - 8R^2r + 7Rr^2 - 2r^3)}_Y s^2 + r \underbrace{(16R^4 + 24R^3r - 51R^2r^2 + 43Rr^3 - 6r^4)}_X$$

$$\stackrel{?}{\geq} 0; \text{ trivially true if : } Y \geq 0 \text{ \& when : } Y < 0, \text{ then : LHS of (**) } \stackrel{\text{Gerretsen}}{\geq}$$

$$(R^3 - 8R^2r + 7Rr^2 - 2r^3)(4R^2 + 4Rr + 3r^2) + X \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (t - 2)(2t^4 + 2t^3(t - 2) + 15t(t - 2) + 5t + 6) \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \geq \frac{\sqrt{s^2 - 9Rr}}{r} \text{ and again, } \sum_{cyc} \frac{2m_b m_c}{r_b r_c} \leq$$

$$\sum_{cyc} \frac{2(2a^2 + bc)(-s^2 + sa + bc)}{4r^2 s^2} =$$

$$\frac{-s^2(4s^2 + 6(s^2 - 4Rr - r^2)) + 8s^2(s^2 - 6Rr - 3r^2) + 24Rrs^2 + 32Rrs^2 + 2(s^2 + 4Rr + r^2)^2 - 32Rrs^2}{4r^2 s^2} \rightarrow \textcircled{3}$$

$$\therefore \textcircled{1} + \textcircled{3} \Rightarrow \left(\sum_{cyc} \frac{m_a}{r_a} \right)^2 \leq \frac{s^4 - (8Rr + 5r^2)s^2 + r^2(4R + r)^2}{r^2 s^2} \stackrel{?}{\leq} \frac{s^2 - 7Rr - 4r^2}{r^2}$$

$$\Leftrightarrow (R + r)s^2 \stackrel{?}{\geq} r(4R + r)^2 \rightarrow \text{true} \because (R + r)s^2 - r(4R + r)^2 \stackrel{\text{Gerretsen}}{\geq}$$

$$(R + r)(16Rr - 5r^2) - r(4R + r)^2 = 12r^2(R - 2r) \stackrel{\text{Euler}}{\geq} 0$$

$$\therefore \frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \leq \frac{\sqrt{s^2 - 7Rr - 4r^2}}{r}; \text{ QED, " = " iff } \Delta ABC \text{ is equilateral}$$