

# ROMANIAN MATHEMATICAL MAGAZINE

In any  $\Delta ABC$  the following relationship holds :

$$h_a + h_b + h_c \leq s - \frac{27Rr}{4s} + \left( \frac{18 - 3\sqrt{3}}{2} \right) r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum_{cyc} h_a &\stackrel{?}{\leq} s - \frac{27Rr}{4s} + \left( \frac{18 - 3\sqrt{3}}{2} \right) r \\ &\Leftrightarrow \frac{s^2 + 4Rr + r^2}{2R} - 9r + \frac{3\sqrt{3}}{2} r \stackrel{?}{\leq} \frac{4s^2 - 27Rr}{4s} \\ &\Leftrightarrow \frac{s^2 - 14Rr + r^2}{2R} + \frac{3\sqrt{3}}{2} r \stackrel{?}{\leq} \frac{4s^2 - 27Rr}{4s} \\ &\Leftrightarrow \frac{(s^2 - 14Rr + r^2)^2}{4R^2} + \frac{27r^2}{4} + \frac{3\sqrt{3} \cdot r(s^2 - 14Rr + r^2)}{2R} \stackrel{?}{\leq} \frac{(4s^2 - 27Rr)^2}{16s^2} \\ &\text{and } \because \frac{3\sqrt{3}}{2} < \frac{13}{5} \text{ and } \because s^2 - 14Rr + r^2 \stackrel{\text{Gerretsen}}{\geq} 2r(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \\ &\quad \therefore \text{it suffices to prove :} \\ &\frac{(s^2 - 14Rr + r^2)^2}{4R^2} + \frac{27r^2}{4} + \frac{13r(s^2 - 14Rr + r^2)}{5R} \stackrel{?}{\leq} \frac{(4s^2 - 27Rr)^2}{16s^2} \\ &\Leftrightarrow -20s^6 + (80R^2 + 352Rr - 40r^2)s^4 - \\ &\quad r(1080R^3 + 1548R^2r - 352Rr^2 + 20r^3)s^2 + 3645R^4r^2 \stackrel{?}{\geq} 0 \end{aligned}$$

Now, Rouché  $\Rightarrow s^2 - (m - n) \geq 0$  and  $s^2 - (m + n) \leq 0$ , where  $m = 2R^2 + 10Rr - r^2$  &  $n = 2(R - 2r)\sqrt{R^2 - 2Rr} \therefore (s^2 - (m + n))(s^2 - (m - n)) \leq 0$

$$\begin{aligned} \Rightarrow s^4 - s^2(2m) + m^2 - n^2 &\leq 0 \Rightarrow s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3 \stackrel{(*)}{\leq} 0 \\ \Rightarrow P &= -20s^2(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) \geq 0 \end{aligned}$$

$\therefore$  in order to prove ①, it suffices to prove : LHS of ①  $\stackrel{?}{\geq} P$

$$\Leftrightarrow -48s^4 + (200R^2 - 588Rr + 592r^2)s^2 + 3645R^3r \stackrel{?}{\geq} 0 \text{ and again,}$$

$$Q = -48(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) \stackrel{\text{via } (*)}{\geq} 0$$

$\therefore$  in order to prove ②, it suffices to prove : LHS of ②  $\stackrel{?}{\geq} Q$

$$\Leftrightarrow (8R^2 - 1548Rr + 688r^2)s^2 + r(6717R^3 + 2304R^2r + 576Rr^2 + 48r^3) \stackrel{?}{\geq} 0$$

and it's trivially true if :  $8R^2 - 1548Rr + 688r^2 \geq 0$  and when :

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$$\begin{aligned}
 & 8R^2 - 1548Rr + 688r^2 < 0, \text{ then : LHS of } \textcircled{3} \stackrel{\text{Gerretsen}}{\geq} \\
 & (8R^2 - 1548Rr + 688r^2)(4R^2 + 4Rr + 3r^2) + \\
 & r(6717R^3 + 2304R^2r + 576Rr^2 + 48r^3) \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow 32t^4 + 557t^3 - 1112t^2 - 1316t + 2112 \stackrel{?}{\geq} 0 \left( t = \frac{R}{r} \right) \\
 & \Leftrightarrow (t - 2)(32t^3 + 621(t^2 - 4) + 130t + 1428) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \\
 & \Rightarrow \textcircled{3} \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true and so,} \\
 & h_a + h_b + h_c \leq s - \frac{27Rr}{4s} + \left( \frac{18 - 3\sqrt{3}}{2} \right) r \forall \Delta ABC, \\
 & " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$