

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{s^2 - 2Rr - (23 - 12\sqrt{3})r^2}{2R} \leq s \leq 2R + r + \frac{9(3\sqrt{3} - 5)r^2}{7R - 5r}$$

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$$\begin{aligned} s &\stackrel{?}{\leq} 2R + r + \frac{9(3\sqrt{3} - 5)r^2}{7R - 5r} \Leftrightarrow s \leq \frac{14R^2 - 3Rr - 50r^2}{7R - 5r} + \frac{27\sqrt{3}.r^2}{7R - 5r} \stackrel{\text{squaring}}{\Leftrightarrow} \\ s^2 &\stackrel{?}{\leq} \frac{(14R^2 - 3Rr - 50r^2)^2}{(7R - 5r)^2} + \frac{2187r^4}{(7R - 5r)^2} + \frac{54\sqrt{3}.r^2(14R^2 - 3Rr - 50r^2)}{7R - 5r} \text{ and } \because \\ 54\sqrt{3} &> \frac{187}{2} \text{ and } \because 14R^2 - 3Rr - 50r^2 = (R - 2r)(14R + 25r) \stackrel{\text{Euler}}{\geq} 0 \\ \therefore \text{ it suffices to prove : } s^2 &\stackrel{?}{\leq} \frac{(14R^2 - 3Rr - 50r^2)^2}{(7R - 5r)^2} + \frac{2187r^4}{(7R - 5r)^2} + \\ &\frac{187r^2(14R^2 - 3Rr - 50r^2)}{2(7R - 5r)} \text{ and now, } s^2 \stackrel{\text{Rouche}}{\leq} \\ &2R^2 + 10Rr - r^2 + 2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \stackrel{?}{\leq} \text{RHS of } (*) \\ &\Leftrightarrow 196R^4 - 868R^3r + 1234R^2r^2 - 601Rr^3 + 74r^4 \stackrel{?}{\geq} \\ &4(R - 2r) \cdot \sqrt{R^2 - 2Rr} \cdot (7R - 5r)^2 \\ &\Leftrightarrow (R - 2r)(196R^3 - 476R^2r + 282Rr^2 - 37r^3) \stackrel{?}{\geq} \text{RHS of } (**) \\ &4(R - 2r) \cdot \sqrt{R^2 - 2Rr} \cdot (7R - 5r)^2 \text{ and } \because 196R^3 - 476R^2r + 282Rr^2 - 37r^3 = \\ &(R - 2r)(196R^2 - 84Rr + 114r^2) + 191r^3 \stackrel{\text{Euler}}{\geq} 191r^3 > 0 \\ \therefore \text{ in order to prove } (**) \text{, it suffices to prove : } \\ &(196R^3 - 476R^2r + 282Rr^2 - 37r^3)^2 \stackrel{?}{\geq} 16(R^2 - 2Rr)(7R - 5r)^4 \\ &\Leftrightarrow 8232t^3 - 7252t^2 - 868t + 1369 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \\ &\Leftrightarrow (t - 2)(8232t^2 + 9212t + 17556) + 36481 \stackrel{?}{\geq} 0 \rightarrow \text{true } \because t \stackrel{\text{Euler}}{\geq} 2 \\ \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore s &\leq 2R + r + \frac{9(3\sqrt{3} - 5)r^2}{7R - 5r} \text{ and now, } \\ \frac{s^2 - 2Rr - (23 - 12\sqrt{3})r^2}{2R} &\stackrel{?}{\leq} s \Leftrightarrow s^2 - 2Rr - 23r^2 + 12\sqrt{3}.r^2 \stackrel{?}{\leq} 2Rs \\ &\Leftrightarrow (s^2 - 2Rr - 23r^2)^2 + 432r^4 + 24\sqrt{3}.r^2(s^2 - 2Rr - 23r^2) \stackrel{?}{\leq} 4R^2s^2 \text{ and } \because \\ 24\sqrt{3} &< \frac{125}{3} \text{ and } s^2 - 2Rr - 23r^2 \stackrel{\text{Gerretsen}}{\geq} 14Rr - 28r^2 = 14r(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \end{aligned}$$

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∴ it suffices to prove :

$$(s^2 - 2Rr - 23r^2)^2 + 432r^4 + \frac{125}{3} \cdot r^2(s^2 - 2Rr - 23r^2) \stackrel{?}{\leq} 4R^2s^2$$

$$\Leftrightarrow -3s^4 + (12R^2 + 12Rr + 13r^2)s^2 - r^2(12R^2 + 26Rr + 8r^2) \stackrel{?}{\geq} 0 \quad (***)$$

Now, Rouché $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ & $n = 2(R - 2r)\sqrt{R^2 - 2Rr} \therefore (s^2 - (m + n))(s^2 - (m - n)) \leq 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \leq 0$$

$$\Rightarrow P = -3(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) \geq 0$$

∴ in order to prove (***) , it suffices to prove : LHS of (***) $\stackrel{?}{\geq} P$

$$\Leftrightarrow 192R^3 + 132R^2r + 10Rr^2 - 5r^3 \stackrel{?}{\geq} (48R - 19r)s^2 \text{ and finally,}$$

$$(48R - 19r)s^2 \stackrel{\text{Gerretsen}}{\leq} (48R - 19r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} \text{LHS of (****)}$$

$$\Leftrightarrow 2r(R - 2r)(8R - 13r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (****) \Rightarrow (***) \text{ is true}$$

$$\therefore s \geq \frac{s^2 - 2Rr - (23 - 12\sqrt{3})r^2}{2R} \text{ and so,}$$

$$\frac{s^2 - 2Rr - (23 - 12\sqrt{3})r^2}{2R} \leq s \leq 2R + r + \frac{9(3\sqrt{3} - 5)r^2}{7R - 5r} \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)