

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$(m_a + m_b + m_c)^2 + (w_a + w_b + w_c)^2 \leq 6(w_a^2 + w_b^2 + w_c^2)$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{4s(s-a)}{(b+c)^2} &= \sum_{\text{cyc}} \frac{(b+c)^2 - a^2}{(b+c)^2} = 3 - \sum_{\text{cyc}} \frac{(2s - (2s-a))^2}{(2s-a)^2} \\ &= 4s \sum_{\text{cyc}} \frac{1}{b+c} - 4s^2 \left(\left(\sum_{\text{cyc}} \frac{1}{b+c} \right)^2 - \frac{2}{2s(s^2 + 2Rr + r^2)} \cdot \sum_{\text{cyc}} (b+c) \right) \\ &= \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} - \frac{(5s^2 + 4Rr + r^2)^2}{(s^2 + 2Rr + r^2)^2} + \frac{16s^2}{s^2 + 2Rr + r^2} \Rightarrow \sum_{\text{cyc}} \frac{4s(s-a)}{(b+c)^2} \\ &= \frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R + r)}{(s^2 + 2Rr + r^2)^2} \therefore \sum_{\text{cyc}} w_a = \frac{\sqrt{bc}}{b+c} \cdot \sqrt{4s(s-a)} \stackrel{\text{CBS}}{\leq} \\ &\sqrt{\sum_{\text{cyc}} bc} \cdot \sqrt{\sum_{\text{cyc}} \frac{4s(s-a)}{(b+c)^2}} = \sqrt{(s^2 + 4Rr + r^2) \cdot \frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R + r)}{(s^2 + 2Rr + r^2)^2}} \\ &\quad \therefore \left(\sum_{\text{cyc}} m_a \right)^2 + \left(\sum_{\text{cyc}} w_a \right)^2 \stackrel{\text{Chu-Yang}}{\leq} 4s^2 - 16Rr + 5r^2 + \\ &\quad (s^2 + 4Rr + r^2) \cdot \frac{s^4 + (20Rr + 18r^2)s^2 + r^3(4R + r)}{(s^2 + 2Rr + r^2)^2} \stackrel{?}{\leq} 6 \left(\sum_{\text{cyc}} w_a^2 \right) \\ &= 6 \cdot \frac{s^6 + 3r^2s^4 + r^2s^2(32R^2 + 40Rr + 3r^2) + r^4(16R^2 + 8Rr + r^2)}{(s^2 + 2Rr + r^2)^2} \end{aligned}$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh – 112;
published at www.ssmrmh.ro)

$$\Leftrightarrow s^6 - (24Rr + 14r^2)s^4 + r^2(160R^2 + 140Rr - 15r^2)s^2 + Rr^3(64R^2 + 124Rr + 36r^2) \stackrel{?}{\geq} 0 \text{ and } \therefore (s^2 - 16Rr + 5r^2)^3 \stackrel{\text{Gerretsen}}{\geq} 0$$

$$\begin{aligned} \therefore \text{in order to prove } \textcircled{1}, \text{ it suffices to prove : LHS of } \textcircled{1} &\stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^3 \\ \Leftrightarrow (24R - 29r)s^4 - r(608R^2 - 620Rr + 90r^2)s^2 + \\ r^2(4160R^3 - 3716R^2r + 1236Rr^2 - 125r^3) &\stackrel{?}{\geq} 0 \text{ and } \therefore \end{aligned}$$

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Gerretsen
and
Euler

$$(24R - 29r)(s^2 - 16Rr + 5r^2)^2 \stackrel{?}{\geq} 0 \therefore \text{in order to prove } \textcircled{2},$$

it suffices to prove : LHS of $\textcircled{2} \stackrel{?}{\geq} (24R - 29r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (40R^2 - 137Rr + 50r^2)s^2 \stackrel{?}{\geq} r(496R^3 - 1887R^2r + 1001Rr^2 - 150r^3)$$

Case 1 $40R^2 - 137Rr + 50r^2 \geq 0$ and then : LHS of $\textcircled{3} \stackrel{\text{Gerretsen}}{\geq}$

$$(40R^2 - 137Rr + 50r^2)(16Rr - 5r^2) \stackrel{?}{\geq} \text{RHS of } \textcircled{3}$$

$$\Leftrightarrow 144t^3 - 505t^2 + 484t - 100 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(35t^2 + 109t(t - 2) + t + 50) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{3} \text{ is true}$$

Case 2 $40R^2 - 137Rr + 50r^2 < 0$ and then : LHS of $\textcircled{3} \stackrel{\text{Rouche}}{\geq}$

$$(40R^2 - 137Rr + 50r^2) \left(2R^2 + 10Rr - r^2 + 2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \right) \stackrel{?}{\geq} \text{RHS of } \textcircled{3}$$

$$\Leftrightarrow (R - 2r)(80R^3 - 210R^2r + 157Rr^2 - 50r^3) \stackrel{?}{\geq}$$

$$2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \cdot \left(-(40R^2 - 137Rr + 50r^2) \right)$$

$$\Leftrightarrow 80R^3 - 210R^2r + 157Rr^2 - 50r^3 \stackrel{?}{\geq}$$

$$2 \cdot \sqrt{R^2 - 2Rr} \cdot \left(-(40R^2 - 137Rr + 50r^2) \right) \left(\because R \stackrel{\text{Euler}}{\geq} 2r \right)$$

$$\Leftrightarrow (80R^3 - 210R^2r + 157Rr^2 - 50r^3)^2 \stackrel{?}{\geq} 4(R^2 - 2Rr)(40R^2 - 137Rr + 50r^2)^2$$

$$\left(\begin{array}{l} \because 80R^3 - 210R^2r + 157Rr^2 - 50r^3 = \\ (R - 2r)(55R^2 + 25R(R - 2r) + 57r^2) + 64r^3 > 0 \end{array} \right)$$

$$\Leftrightarrow (t - 2) \left((t - 2)(14352t^3 + 8688t^2(t - 2) + 1348t + 945) + 2688 \right) + 4096 \stackrel{?}{\geq} 0$$

$\rightarrow$ true (strict inequality) $\because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{3} \text{ is true} \therefore$ combining both cases,

$$\textcircled{3} \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true and so, } (m_a + m_b + m_c)^2 + (w_a + w_b + w_c)^2 \leq$$

$$6(w_a^2 + w_b^2 + w_c^2) \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$