

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{w_a^2}{r_a} + \frac{w_b^2}{r_b} + \frac{w_c^2}{r_c} \geq \frac{m_b m_c}{h_a} + \frac{m_c m_a}{h_b} + \frac{m_a m_b}{h_c}$$

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$$\begin{aligned} \sum_{cyc} \frac{w_a^2}{r_a} &= \sum_{cyc} \left(\frac{4bc}{(b+c)^2} \frac{s(s-a)^2}{rs} \right) = \\ &= \sum_{cyc} \left(\frac{bc}{r} \cdot \frac{(b+c)^2 + a^2 - 2a(b+c)}{(b+c)^2} \right) = \\ &= \frac{s^2 + 4Rr + r^2}{r} + \\ &\quad \frac{4Rs}{4s^2(s^2 + 2Rr + r^2)^2} \sum_{cyc} \left(a \left(a^4 + \left(\sum_{cyc} ab \right)^2 + 2a^2 \left(\sum_{cyc} ab \right) \right) \right) - \\ &\quad \frac{8Rs(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} = \frac{s^2 + 4Rr + r^2}{r} + \\ &\quad \frac{R}{s(s^2 + 2Rr + r^2)^2} \cdot \left(\begin{aligned} &4s(s^2 - 4Rr - r^2)(s^2 - 6Rr - 3r^2) + 4Rrs(s^2 + 4Rr + r^2) - \\ &2s((s^2 + 4Rr + r^2)^2 - 16Rrs^2) \\ &+ 2s(s^2 + 4Rr + r^2)^2 + 4s(s^2 + 4Rr + r^2)(s^2 - 6Rr - 3r^2) \end{aligned} \right) \\ &- \frac{4R(5s^2 + 4Rr + r^2)}{s^2 + 2Rr + r^2} \geq \sum_{cyc} \frac{a(2a^2 + bc)}{8rs} = \frac{4s(s^2 - 6Rr - 3r^2) + 12Rrs}{8rs} \Leftrightarrow \\ &\quad s^6 - (9Rr - 7r^2)s^4 - r^2(88R^2 + 50Rr - 11r^2)s^2 + \\ &\quad r^3(12R^3 + 24R^2r + 23Rr^2 + 5r^3) \stackrel{?}{\geq} 0 \text{ and } \therefore s^4(s^2 - 16Rr + 5r^2) \stackrel{Gerretsen}{\geq} 0 \\ &\therefore \text{ in order to prove (1), it suffices to prove : LHS of (1) } \stackrel{?}{\geq} s^4(s^2 - 16Rr + 5r^2) \\ &\quad \Leftrightarrow (7R + 2r)s^4 - r(88R^2 + 50Rr - 11r^2)s^2 + \\ &\quad r^3(12R^3 + 24R^2r + 23Rr^2 + 5r^3) \stackrel{?}{\geq} 0 \text{ and } \therefore s^2(s^2 - 16Rr + 5r^2) \stackrel{Gerretsen}{\geq} 0 \\ &\therefore \text{ in order to prove (2), it suffices to prove : LHS of (2) } \stackrel{?}{\geq} \\ &\quad (7R + 2r)s^2(s^2 - 16Rr + 5r^2) \Leftrightarrow (24R^2 - 53Rr + r^2)s^2 + \\ &\quad r(12R^3 + 24R^2r + 23Rr^2 + 5r^3) \stackrel{?}{\geq} 0 \text{ and it is trivially true if :} \end{aligned}$$

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$24R^2 - 53Rr + r^2 \geq 0$ and when : $24R^2 - 53Rr + r^2 < 0$, then :

LHS of (3) $\overset{\text{Gerretsen}}{\geq}$

$$(24R^2 - 53Rr + r^2)(4R^2 + 4Rr + 3r^2) + r(12R^3 + 24R^2r + 23Rr^2 + 5r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow 24t^4 - 26t^3 - 28t^2 - 33t + 2 \overset{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(24t^3 + 22t^2 + 16t - 1) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \frac{w_a^2}{r_a} \geq \sum_{\text{cyc}} \frac{a(2a^2 + bc)}{8rs} \geq \sum_{\text{cyc}} \frac{m_b m_c}{h_a} \left(\because m_b m_c \leq \frac{2a^2 + bc}{4} \text{ and analogs} \right)$$

$$\therefore \frac{w_a^2}{r_a} + \frac{w_b^2}{r_b} + \frac{w_c^2}{r_c} \geq \frac{m_b m_c}{h_a} + \frac{m_c m_a}{h_b} + \frac{m_a m_b}{h_c} \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)