

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC holds :

$$g_b g_c \leq \min \left\{ w_a r_a, \frac{\sqrt{(4R + 19r)r}}{2} a \right\}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x, c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,

$$\begin{aligned} & \text{and via Bogdan Fustei} \\ & y^3 + z^3 = x^3(m^3 - 3mn), \text{ and now, } g_b g_c \leq r_a w_a \Leftrightarrow \\ & \left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b} \right) \left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c} \right) \leq \\ & \frac{s(s-b)(s-c)}{s-a} \cdot \frac{4bc}{(b+c)^2} \cdot s(s-a) \\ & \Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \geq (4zx + xy + yz)(4xy + zx + yz)(2x + y + z)^2 \\ & \Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4 \cdot yz - 32x^3 \cdot yz(y+z) - \\ & 5x^2 \cdot yz(y+z)^2 + 3x \cdot yz(y^3+z^3) + 13x \cdot y^2 z^2(y+z) + 3y^2 z^2(y+z)^2 \geq 0 \text{ and via} \end{aligned}$$

set of relations "S", to prove (1), it suffices to prove, following simplification :

$$\overbrace{(3m^2 + 4m)}^{\Omega_1} n^2 + \overbrace{(3m^3 - 5m^2 - 32m - 28)}^{\Omega_2} n + \overbrace{(8m^2 + 16m + 4)}^{\Omega_3} + \geq 0$$

and it's trivially true when : $\Omega_2 \geq 0$ and when : $\Omega_2 < 0$, then :

$$3m^3 - 5m^2 - 32m - 28 = (m-5)(3m^2 + 10m + 18) + 62 < 0$$

$$\Rightarrow (m-5)(3m^2 + 10m + 18) < -62 < 0 \Rightarrow m < 5 \rightarrow (i)$$

Now, LHS of (2) is a quadratic polynomial in "n" with discriminant, $\delta =$

$$\sigma_2^2 - 4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196) \text{ and when : } 9m^3 - 75m^2 + 40m + 196 \leq 0, \text{ then } \delta \leq 0 \Rightarrow (2) \text{ is trivially true and when :}$$

$$9m^3 - 75m^2 + 40m + 196 > 0, \text{ then : } 9m^3 - 75m^2 + 40m + 196$$

$$= \left(3m(m-5) - 3m - \frac{86}{3} \right) (3m-7) - \frac{14}{3} > 0$$

$$\Rightarrow \left(3m(m-5) - 3m - \frac{86}{3} \right) (3m-7) > \frac{14}{3} > 0 \text{ and } \therefore m \leq 5 \text{ via (i)}$$

$$\Rightarrow 3m(m-5) - 3m - \frac{86}{3} < 0 \therefore m < \frac{7}{3} \rightarrow (ii) \text{ and so, when : } \delta > 0,$$

$$\text{then it suffices to prove : } 2\Omega_1 \cdot n \leq -\Omega_2 - \sqrt{\delta} \Leftrightarrow \text{and } \therefore n \leq \frac{AM-GM}{4} m^2$$

$$\therefore \text{ suffices to prove : } \Omega_1 \cdot \frac{m^2}{2} \leq -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2 - 2m - 14) \geq 2 \cdot \sqrt{\delta}$$

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and $\therefore 3m^2 - 2m - 14 = \left(m + \frac{5}{3}\right)(3m - 7) - \frac{7}{3} \stackrel{?}{<} 0 \therefore$ it suffices to prove :

$$(m+2)^4(3m^2-2m-14)^2 \stackrel{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \stackrel{?}{\geq} 0} \therefore g_b g_c \leq r_a w_a \text{ and now,}$$

$$g_b g_c \stackrel{?}{\leq} \frac{\sqrt{(4R+19r)r}}{2} a \Leftrightarrow \left(\frac{(s-b)^2 + 4(s-a)(s-b)(s-c)}{b} \right) \left(\frac{(s-c)^2 + 4(s-a)(s-b)(s-c)}{c} \right)$$

$$\stackrel{?}{\leq} \frac{(abc + 19(s-a)(s-b)(s-c))^2}{4} \cdot a^2 \Leftrightarrow$$

$$\frac{((y+z)(z+x)(x+y) + 19xyz)(y+z)^2(z+x)(x+y)}{4(x+y+z)yz(y(z+x) + 4zx)(z(x+y) + 4xy)} \stackrel{?}{\geq}$$

$$4(x+y+z)yz(y(z+x) + 4zx)(z(x+y) + 4xy) \Leftrightarrow$$

$$(m^3 - m^2 - 56m - 36)n^2 + (2m^4 + 5m^3 + 3m^2)n + m^3(m+1)^2 \stackrel{?}{\geq} 4mn^3 + 4n^3$$

and $\therefore 4mn^3 + 4n^3 \leq 4mn^2 \cdot \frac{m^2}{4} + 4n^2 \cdot \frac{m^2}{4} \therefore$ to prove (*), it suffices to prove :

$$\text{LHS of } (*) \stackrel{?}{\geq} m^3 n^2 + m^2 n^2 \Leftrightarrow \overbrace{(2m^2 + 56m + 36)}^{\sigma_1} n^2 - \overbrace{(2m^4 + 5m^3 + 3m^2)}^{\sigma_2} n -$$

$$\overbrace{m^3(m+1)^2}^{\sigma_3} \stackrel{?}{\geq} 0; \text{ discriminant, } \delta_0 \text{ of LHS of } (**) = (2m^4 + 5m^3 + 3m^2)^2 +$$

$4(2m^2 + 56m + 36)m^3(m+1)^2 > 0 \therefore$ to prove (**), suffices to prove :

$$2\sigma_1 \cdot n \stackrel{?}{\geq} \sigma_2 + \sqrt{\delta_0} \text{ \& } 2\sigma_1 \cdot n \stackrel{?}{\geq} \sigma_2 - \sqrt{\delta_0}; \text{ now, } 2\sigma_1 \cdot n \stackrel{\text{AM-GM}}{\leq} 2\sigma_1 \cdot \frac{m^2}{4} \stackrel{?}{\leq} \sigma_2 + \sqrt{\delta_0}$$

$$\Leftrightarrow 2m^2(-m^2 + 23m + 15) \stackrel{?}{\leq} 2\sqrt{\delta_0}, \text{ trivially true if : } -m^2 + 23m + 15 \leq 0$$

$$\text{and when : } -m^2 + 23m + 15 > 0, \text{ then : } 2m^2(-m^2 + 23m + 15) \stackrel{?}{\leq} 2\sqrt{\delta_0} \Leftrightarrow$$

$$\delta_0 \stackrel{?}{\geq} m^4(-m^2 + 23m + 15)^2 \Leftrightarrow m^3(3m+2)(m-2)^2(m^2 + 28m + 18) \stackrel{?}{\geq} 0$$

$\rightarrow$ true $\therefore m > 0 \Rightarrow$ (i) is true; also, $2\sigma_1 \cdot n + \sqrt{\delta_0} > \sqrt{\delta_0} > \sqrt{(2m^4 + 5m^3 + 3m^2)^2}$
 $= \sigma_2 \Rightarrow$ (ii) is true So, (i), (ii) both are true $\Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore g_b g_c \leq \frac{\sqrt{(4R+19r)r}}{2} a \text{ and so, } g_b g_c \leq \min \left\{ w_a r_a, \frac{\sqrt{(4R+19r)r}}{2} a \right\} \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)