

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{9}{4}a^2 + h_a^2 \geq 2(h_b^2 + h_c^2)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x, c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n), y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2), y^5 + z^5 = x^5(m((m^2 - 2n)^2 + n^2 - nm^2)), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3),$ and now, $\frac{9}{4}a^2 + h_a^2 \stackrel{?}{\geq} 2(h_b^2 + h_c^2) \Leftrightarrow 9R^2a^2 + b^2c^2 \stackrel{?}{\geq} 2(c^2a^2 + a^2b^2) \Leftrightarrow 9a^4b^2c^2 + 16s(s-a)(s-b)(s-c)b^2c^2 \stackrel{?}{\geq} 32s(s-a)(s-b)(s-c)a^2(b^2 + c^2) \Leftrightarrow 9(z+x)^2(x+y)^2(y+z)^4 + 16xyz(x+y+z)(z+x)^2(x+y)^2 \stackrel{?}{\geq} 32xyz(x+y+z)(y+z)^2((z+x)^2 + (x+y)^2) \Leftrightarrow 16x^6.yz + 48x^5.yz(y+z) + 9x^4(y^4 + z^4) + 20x^4.yz(y^2 + z^2) + 54x^4.y^2z^2 + 18x^3(y^5 + z^5) - 22x^3.yz(y^3 + z^3) - 92x^3.y^2z^2(y+z) + 9x^2(y^6 + z^6) - 24x^2.yz(y^4 + z^4) - 81x^2.y^2z^2(y^2 + z^2) - 80x^2.y^3z^3 - 14x.yz(y^5 + z^5) - 6x.y^2z^2(y^3 + z^3) + 68x.y^3z^3(y+z) + 9y^2z^2(y+z)^4 \stackrel{?}{\geq} 0$ and via

set of relations "S", to prove (1), suffices to prove, following simplification :

$$(9m^4 + 64m^3 + 96m^2 + 64m + 32)n^2 - (14m^5 + 78m^4 + 112m^3 + 16m^2 - 48m - 16)n + 9m^4(m+1)^2 + 16(m+1)n^3 \stackrel{?}{\geq} 0$$

and $\because n(m+1)(4n - m^2)^2 \geq 0 \therefore$ in order to prove (2),

$$\text{it suffices to prove : LHS of (2)} \stackrel{?}{\geq} n(m+1)(4n - m^2)^2$$

$$\Leftrightarrow \overbrace{(9m^4 + 72m^3 + 104m^2 + 64m + 32)}^{\Omega_2} n^2 - \overbrace{(15m^5 + 79m^4 + 112m^3 + 16m^2 - 48m - 16)}^{\Omega_3} n + 9m^4(m+1)^2 \stackrel{?}{\geq} 0$$

and it's trivially true when : $\Omega_2 \leq 0$ and when : $\Omega_2 > 0$, then : $\Omega_2 =$

$$\left(3m^4 + \frac{88m^3}{5} + \frac{824m^2}{25} + \frac{2872m}{125} + \frac{2616}{625}\right)(5m - 3) - \frac{2152}{625} > 0 \Rightarrow$$

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$$\left(3m^4 + \frac{88m^3}{5} + \frac{824m^2}{25} + \frac{2872m}{125} + \frac{2616}{625}\right)(5m-3) > \frac{2152}{625} > 0 \Rightarrow m > \frac{3}{5} \rightarrow (*)$$

Now, LHS of ③ is a quadratic polynomial in "n" with discriminant,

$$\begin{aligned} \delta &= \Omega_2^2 - 4\Omega_1\Omega_3 = \\ &= -(m+1)^2(3m+2)(m-2)^2(33m^5 + 334m^4 + 472m^3 + 80m^2 - 112m - 32) \\ &= -(m+1)^2(3m+2)(m-2)^2 \left(\left(\frac{33m^4}{5} + \frac{1769m^3}{25} + \frac{17107m^2}{125} + \frac{61321m}{625} + 36 + \frac{1463}{3125} + 77 + \frac{1264}{3125} \right) (5m-3) \right) \end{aligned}$$

via (*)
 ≤ 0 and so, ③ $\Rightarrow$ ② $\Rightarrow$ ① is true $\therefore \frac{9}{4}a^2 + h_a^2 \geq 2(h_b^2 + h_c^2) \forall \Delta ABC$,
 " = " iff ΔABC is equilateral (QED)