

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$w_a^2 r_a + w_b^2 r_b + w_c^2 r_c \leq m_a w_b w_c + m_b w_c w_a + m_c w_a w_b$$

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$$\begin{aligned} \sum_{\text{cyc}} (w_a^2 r_a) &= \sum_{\text{cyc}} \left(\frac{4b^2 c^2}{(b+c)^2} \cos^2 \frac{A}{2} \cdot s \tan \frac{A}{2} \right) = \sum_{\text{cyc}} \left(\frac{4b^2 c^2}{(b+c)^2} \cdot \frac{as}{4R} \right) \\ &= 4rs^2 \cdot \frac{1}{4s^2(s^2 + 2Rr + r^2)^2} \cdot \sum_{\text{cyc}} \left(bc \left(a^4 + \left(\sum_{\text{cyc}} ab \right)^2 + 2a^2 \left(\sum_{\text{cyc}} ab \right) \right) \right) \\ &= \frac{r \left(8Rrs^2(s^2 - 6Rr - 3r^2) + (s^2 + 4Rr + r^2)^3 + 16Rrs^2(s^2 + 4Rr + r^2) \right)}{(s^2 + 2Rr + r^2)^2} = \sum_{\text{cyc}} (w_a^2 r_a) \rightarrow \textcircled{1} \end{aligned}$$

$$\begin{aligned} \text{Again, } \sum_{\text{cyc}} (m_a w_b w_c) &\stackrel{\text{Gerretsen}}{\geq} \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \sum_{\text{cyc}} \frac{m_a}{w_a} \stackrel{\text{Lascu}}{\geq} \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \sum_{\text{cyc}} \frac{\frac{b+c}{2} \cdot \cos \frac{A}{2}}{\frac{2bc}{b+c} \cdot \cos \frac{A}{2}} \\ &= \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \sum_{\text{cyc}} \frac{a(b+c)^2}{4abc} = \\ &= \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \frac{1}{16Rrs} \cdot \left(\left(\sum_{\text{cyc}} a \right) \left(\sum_{\text{cyc}} ab \right) + 3abc \right) \\ &= \frac{16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \frac{2s}{16Rrs} \cdot (s^2 + 10Rr + r^2) \stackrel{?}{\geq} \sum_{\text{cyc}} (w_a^2 r_a) \stackrel{\text{via } \textcircled{1}}{=} \\ &= \frac{r \left(8Rrs^2(s^2 - 6Rr - 3r^2) + (s^2 + 4Rr + r^2)^3 + 16Rrs^2(s^2 + 4Rr + r^2) \right)}{(s^2 + 2Rr + r^2)^2} \end{aligned}$$

$$\Leftrightarrow s^6 - (12Rr - r^2)s^4 - r^2(24R^2 - 8Rr + r^2)s^2 - r^3(4R + r)^3 \stackrel{?}{\geq} 0 \text{ and } \therefore \textcircled{1}$$

$$\begin{aligned} s^4(s^2 - 16Rr + 5r^2) &\stackrel{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } \textcircled{1}, \text{ it suffices to prove :} \\ \text{LHS of } \textcircled{1} &\stackrel{?}{\geq} s^4(s^2 - 16Rr + 5r^2) \Leftrightarrow (4R - 4r)s^4 - r(24R^2 - 8Rr + r^2)s^2 - \\ r^2(4R + r)^3 &\stackrel{?}{\geq} 0 \text{ and } \therefore s^2(s^2 - 16Rr + 5r^2) \stackrel{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } \textcircled{2}, \end{aligned}$$

$$\text{it suffices to prove : LHS of } \textcircled{2} \stackrel{?}{\geq} s^2(s^2 - 16Rr + 5r^2)$$

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$$\Leftrightarrow (40R^2 - 76Rr + 19r^2)s^2 \stackrel{?}{\geq} r(4R + r)^3 \text{ and finally,}$$

$$(40R^2 - 76Rr + 19r^2)s^2 \stackrel{\text{Gerretsen}}{\geq} (40R^2 - 76Rr + 19r^2)(16Rr - 5r^2) \\ \stackrel{?}{\geq} r(4R + r)^3 \Leftrightarrow 24t^3 - 61t^2 + 28t - 4 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(24t^2 - 13t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{3} \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true} \\ \therefore w_a^2 r_a + w_b^2 r_b + w_c^2 r_c \leq m_a w_b w_c + m_b w_c w_a + m_c w_a w_b \quad \forall \Delta ABC, \\ \text{" = " iff } \Delta ABC \text{ is equilateral (QED)}$$