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In any ΔABC the following relationship holds :

$$\sum_{cyc} \frac{h_b h_c}{h_a} \leq \sum_{cyc} m_a \leq \sum_{cyc} \frac{w_a h_a}{r_a}$$

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$$2F \sum_{cyc} \frac{m_a}{m_b m_c} \stackrel{?}{\leq} \sum_{cyc} a \Leftrightarrow 4F^2 \left(\sum_{cyc} m_a^2 \right)^2 \stackrel{?}{\leq} 4s^2 \cdot (m_a m_b m_c)^2 \Leftrightarrow$$

$$\frac{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3}{16} \stackrel{?}{\geq}$$

$$r^2 \cdot \frac{9}{4} (s^2 - 4Rr - r^2)^2$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh – 124; published at www.ssmrmh.ro) $\Leftrightarrow$

$$s^6 - (12Rr + 3r^2)s^4 - r^2(60R^2 - 168Rr - 39r^2)s^2 -$$

$$r^3(64R^3 + 624R^2r + 300Rr^2 + 37r^3) \stackrel{?}{\geq} 0$$

and $\because (s^2 - 16Rr + 5r^2)^3 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (2R - r)s^4 - r(46R^2 - 36Rr + 2r^2)s^2 +$$

$$r^2(224R^3 - 248R^2r + 50Rr^2 - 9r^3) \stackrel{?}{\geq} 0 \text{ and } \because (s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0$$

$\therefore$ to prove ②, it suffices to prove : LHS of ② $\stackrel{?}{\geq} (2R - r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (9R^2 - 8Rr + 4r^2)s^2 \stackrel{?}{\geq} r(144R^3 - 164R^2r + 80Rr^2 - 8r^3) \text{ and finally,}$$

$$\text{via Rouché + Euler, LHS of ③} \geq (9R^2 - 8Rr + 4r^2) \left(16Rr - 5r^2 + \frac{r^2(R - 2r)}{R - r} \right)$$

$$\stackrel{?}{\geq} \text{RHS of ③} \Leftrightarrow r^3(R - 2r)(7R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow ③ \Rightarrow ② \Rightarrow ①$$

is true $\therefore 2F \sum_{cyc} \frac{m_a}{m_b m_c} \leq \sum_{cyc} a$ and implementing it on a triangle XYZ with sides

$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, whose medians and area as a consequence of trivial calculations are

$$\frac{a}{2}, \frac{b}{2}, \frac{c}{2} \text{ and } \frac{F}{3} \text{ respectively, we arrive at : } \frac{2F}{3} \cdot \sum_{cyc} \frac{\left(\frac{a}{2}\right)}{\left(\frac{bc}{4}\right)} \leq \frac{2}{3} \sum_{cyc} m_a \Rightarrow 2F \frac{\frac{h_a}{4F^2}}{h_b h_c} \leq$$

$$\sum_{cyc} m_a \Rightarrow \sum_{cyc} \frac{h_b h_c}{h_a} \leq \sum_{cyc} m_a \text{ and now, } \sum_{cyc} \frac{w_a h_a}{r_a} = \sum_{cyc} \left(\frac{w_a}{2Rr} \cdot 4F^2 \right)$$

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$$\begin{aligned}
 &= \frac{1}{2Rr} \sum_{\text{cyc}} (w_a(bc - 4Rr)) \stackrel{\text{Chebyshev}}{\geq} \frac{1}{6Rr} \left(\sum_{\text{cyc}} w_a \right) \cdot \sum_{\text{cyc}} (bc - 4Rr) \\
 &(\because \text{WLOG assuming } a \geq b \geq c \Rightarrow w_a \leq w_b \leq w_c \text{ and } bc \leq ca \leq ab) \\
 &\stackrel{\text{Dang Ngoc Minh}}{\geq} \frac{\sqrt{s^2 + 12Rr + 30r^2}}{6Rr} \cdot (s^2 - 8Rr + r^2) \\
 &\left(\text{Reference : Inequality in Triangle by Dang Ngoc Minh - 90;} \right) \stackrel{?}{\geq} \sqrt{4s^2 - 16Rr + 5r^2} \\
 &\quad \text{published at } \text{www.ssmrmh.ro} \\
 &\Leftrightarrow s^6 - (4Rr - 32r^2)s^4 - r^2(272R^2 + 472Rr - 61r^2)s^2 + \\
 &r^3(1344R^3 + 1548R^2r - 468Rr^2 + 30r^3) \stackrel{?}{\geq} 0 \text{ \& } \because (s^2 - 16Rr + 5r^2)^3 \stackrel{\text{Gerretsen}}{\geq} 0 \\
 &\quad \text{in order to prove } (*), \text{ it suffices to prove : LHS of } (*) \stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^3 \\
 &\quad \Leftrightarrow (44R + 17r)s^4 - r(1040R^2 - 8Rr + 14r^2)s^2 + \\
 &r^2(5440R^3 - 22928R^2r + 732Rr^2 - 95r^3) \stackrel{?}{\geq} 0; \because (s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \\
 &\quad \text{in order to prove } (**), \text{ it suffices to prove : LHS of } (**) \stackrel{?}{\geq} \\
 &\quad (44R + 17r)(s^2 - 16Rr + 5r^2)^2 \\
 &\Leftrightarrow (92R^2 + 28Rr - 46r^2)s^2 \stackrel{?}{\geq} r(1456R^3 - 99R^2r - 588Rr^2 + 130r^3) \text{ and} \\
 &\text{finally, } (92R^2 + 28Rr - 46r^2)s^2 \stackrel{\text{Gerretsen}}{\geq} (92R^2 + 28Rr - 46r^2)(16Rr - 5r^2) \\
 &\stackrel{?}{\geq} \text{RHS of } (***) \Leftrightarrow 16t^3 + 87t^2 - 288t + 100 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \\
 &\Leftrightarrow (t - 2)(16t^2 + 119t - 50) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \\
 &\therefore \sum_{\text{cyc}} \frac{w_a h_a}{r_a} \geq \sqrt{4s^2 - 16Rr + 5r^2} \stackrel{\text{Chu-Yang}}{\geq} \sum_{\text{cyc}} m_a \Rightarrow \boxed{\sum_{\text{cyc}} m_a \leq \sum_{\text{cyc}} \frac{w_a h_a}{r_a}} \text{ and so,} \\
 &\sum_{\text{cyc}} \frac{h_b h_c}{h_a} \leq \sum_{\text{cyc}} m_a \leq \sum_{\text{cyc}} \frac{w_a h_a}{r_a} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$