

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$w_a + w_b + w_c \leq \frac{h_a r_a}{w_a} + \frac{h_b r_b}{w_b} + \frac{h_c r_c}{w_c}$$

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$$\begin{aligned} \left(\sum_{\text{cyc}} \frac{h_a r_a}{w_a} \right)^2 &= \sum_{\text{cyc}} \left(\cos^2 \frac{B-C}{2} \cdot \frac{r^2 s^2}{(s-a)^2} \right) + \\ &\quad 2 \sum_{\text{cyc}} \left(\cos \frac{B-C}{2} \cos \frac{C-A}{2} \cdot r_a r_b \right) \\ &= \sum_{\text{cyc}} \left(\frac{(s^2 + (s-a)^2 + 2s(s-a)) \sin^2 \frac{A}{2} \cdot r^2 s^2}{16R^2 \cdot \sin^2 \frac{A}{2} \cdot \frac{s(s-a)}{bc}} \cdot \frac{r^2 s^2}{(s-a)^2} \right) + \\ &\quad 2 \cos \frac{B-C}{2} \cos \frac{C-A}{2} \cos \frac{A-B}{2} \cdot \sum_{\text{cyc}} \frac{r_a r_b}{\cos \frac{A-B}{2}} \\ &\geq r^2 s^2 \cdot \frac{s}{16R^2} \cdot \sum_{\text{cyc}} \frac{bc}{(s-a)^3} + r^2 s^2 \cdot \frac{1}{16R^2 s} \cdot \sum_{\text{cyc}} \frac{bc}{s-a} + \frac{r^2 s^2}{8R^2} \cdot \sum_{\text{cyc}} \frac{bc}{(s-a)^2} + \\ &\quad 2 \cdot \frac{2s(s^2 + 2Rr + r^2)}{4Rrs} \cdot \frac{r}{4R} \cdot \sum_{\text{cyc}} r_b r_c \left(\because 0 < \cos \frac{A-B}{2} \leq 1 \right) \rightarrow \textcircled{1} \\ &\quad \text{and analogs} \\ \text{Now, } \sum_{\text{cyc}} \frac{bc}{(s-a)^3} &= \frac{1}{r^6 s^3} \cdot \sum_{\text{cyc}} (bc(-s^2 + sa + bc)^3) = \\ &\quad \frac{1}{r^6 s^3} \cdot \sum_{\text{cyc}} \left(bc \left(a^3 s^3 + 3a^2 bcs^2 - 3a^2 s^4 + 3ab^2 c^2 s - 6abcs^3 + 3as^5 + b^3 c^3 - \right) \right. \\ &\quad \left. 3b^2 c^2 s^2 + 3bcs^4 - s^6 \right) \\ &= \frac{8Rrs^4(s^2 - 4Rr - r^2) + 144R^2 r^2 s^4 - 24Rrs^6 + 12Rrs^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 24Rrs^4(s^2 + 4Rr + r^2) + 36Rrs^6 + ((s^2 + 4Rr + r^2)^2 - 16Rrs^2)^2 - 64R^2 r^2 s^2(s^2 - 4Rr - r^2) - 3s^2((s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)) + 3s^4((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - s^6(s^2 + 4Rr + r^2)}{r^6 s^3} \\ &\rightarrow \textcircled{2}; \text{ again, } \sum_{\text{cyc}} \frac{bc}{(s-a)^2} = \frac{1}{r^4 s^2} \cdot \sum_{\text{cyc}} (bc(-s^2 + sa + bc)^2) \\ &= \frac{1}{r^4 s^2} \cdot \sum_{\text{cyc}} (bc(a^2 s^2 + 2abcs - 2as^3 + b^2 c^2 - 2bcs^2 + s^4)) \end{aligned}$$

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$$= \frac{1}{r^4 s^2} \cdot \left(8Rrs^4 + 8Rrs^2(s^2 + 4Rr + r^2) - 24Rrs^4 + (s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2) - 2s^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + s^4(s^2 + 4Rr + r^2) \right)$$

$$\rightarrow \textcircled{3}; \text{ also, } \sum \frac{bc}{s-a} = \frac{1}{r^2 s} \cdot \sum_{cyc} (bc(-s^2 + sa + bc))$$

$$= \frac{-s^2(s^2 + 4Rr + r^2) + 12Rrs^2 + (s^2 + 4Rr + r^2)^2 - 16Rrs^2}{r^2 s} \rightarrow \textcircled{4} \text{ and so,}$$

via $\textcircled{1}$ and via $\textcircled{2} + \textcircled{3} + \textcircled{4}$, we get, via simplification : $\left(\sum_{cyc} \frac{h_a r_a}{w_a} \right)^2 \geq$

$$\frac{s^4 - (12R^2 + 4Rr - 2r^2)s^2 + 64R^4 + 96R^3r + 52R^2r^2 + 12Rr^3 + r^4}{4R^2} \geq$$

$$s^2 + 21Rr + 12r^2 \Leftrightarrow$$

$$s^4 - (16R^2 + 4Rr - 2r^2)s^2 + 64R^4 + 12R^3r + 4R^2r^2 + 12Rr^3 + r^4 \stackrel{?}{\geq} 0$$

and $\because (s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow$$

$$16R^4 + 3R^3r - 63R^2r^2 + 43Rr^3 - 6r^4 \stackrel{?}{\geq} (4R^2 - 7Rr + 2r^2)s^2$$

Finally, RHS of $(**)$ $\stackrel{\text{Gerretsen}}{\leq} (4R^2 - 7Rr + 2r^2)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} \text{LHS of } (**)$

$$\Leftrightarrow 15t^3 - 55t^2 + 56t - 12 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \Leftrightarrow (t-2)(15t^2 - 25t + 6) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \left(\sum_{cyc} \frac{h_a r_a}{w_a} \right)^2 \geq s^2 + 21Rr + 12r^2 \geq \left(\sum_{cyc} w_a \right)^2$$

(Reference : Inequality in Triangle by Dang Ngoc Minh - 90;) and so,
published at www.ssmrmh.ro

$$w_a + w_b + w_c \leq \frac{h_a r_a}{w_a} + \frac{h_b r_b}{w_b} + \frac{h_c r_c}{w_c} \forall \triangle ABC, " = " \text{ iff } \triangle ABC \text{ is equilateral (QED)}$$