

In ΔABC holds :

$$\frac{\sqrt{3}}{6} \cdot \sum_{cyc} (am_b m_c) \geq \frac{1}{8} \cdot \prod_{cyc} (m_a + m_b) \geq \frac{1}{4} \sum_{cyc} (a^2 m_a) \geq m_a m_b m_c$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \left(\sum_{cyc} \left(\frac{1}{3\sqrt{3}} m_a bc \right) \right)^2 = \\ & \frac{1}{27} \left(\frac{1}{4} \sum_{cyc} \left(b^2 c^2 \left(2 \sum_{cyc} a^2 - 3a^2 \right) \right) + 4Rrs \sum_{cyc} (a \cdot 2m_b m_c) \right) \geq \\ & \frac{1}{27} \left(\frac{(s^2 - 4Rr - r^2)((s^2 + 4Rr + r^2)^2 - 16Rrs^2) +}{4Rrs \sum_{cyc} \left(a \left(\frac{9}{4} a^2 + h_a^2 - \frac{(3a^2 + \sum_{cyc} a^2)}{4} \right) \right)} \right) \\ & \left(\text{Reference : Solution to Inequality in Triangle by Dang Ngoc Minh – 393;} \right. \\ & \quad \left. \text{published at } \text{www.ssmrmh.ro} \right) \\ & = \frac{1}{27} \left(\frac{(s^2 - 4Rr - r^2)((s^2 + 4Rr + r^2)^2 - 16Rrs^2) +}{4Rrs^2 \cdot \frac{3R(s^2 - 6Rr - 3r^2) - R(s^2 - 4Rr - r^2) + r(s^2 + 4Rr + r^2)}{R}} \right) \\ & \geq \left(\frac{1}{16} \prod_{cyc} (b + c) \right)^2 = \frac{s^2(s^2 + 2Rr + r^2)^2}{64} \\ & \Leftrightarrow 37s^6 - (364Rr - 266r^2)s^4 - r^2(2924R^2 + 620Rr - 165r^2)s^2 - r^3(4R + r)^3 \\ & \quad \stackrel{?}{\geq} 0 \text{ and } \because (s^2 - 16Rr + 5r^2)^3 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore \text{to prove } \textcircled{1}, \text{ it suffices to prove :} \end{aligned}$$

$$\begin{aligned} & \text{LHS of } \textcircled{1} \stackrel{?}{\geq} 37(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (1412R - 289r)s^4 - \\ & \quad 10r(3134R^2 - 1714Rr + 261r^2)s^2 + \\ & \quad r^2(16(9216R^3 - 9072R^2r + 2727Rr^2 - 293r^3) - r^3) \stackrel{?}{\geq} 0 \text{ and} \\ & \because (1412R - 289r)(s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } \textcircled{2}, \\ & \text{it suffices to prove : LHS of } \textcircled{2} \stackrel{?}{\geq} (1412R - 289r)(s^2 - 16Rr + 5r^2)^2 \\ & \quad \Leftrightarrow (3461R^2 - 1557Rr + 70r^2)s^2 \stackrel{?}{\geq} \\ & \quad r(16(3344R^3 - 2418R^2r + 592Rr^2 - 40r^3) + 5Rr^2 + 6r^3) \text{ and finally,} \end{aligned}$$

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$$\begin{aligned}
 & (3461R^2 - 1557Rr + 70r^2)s^2 \stackrel{\text{Gerretsen}}{\geq} (3461R^2 - 1557Rr + 70r^2)(16Rr - 5r^2) \\
 & \stackrel{?}{\geq} \text{RHS of } \textcircled{3} \Leftrightarrow 1872t^3 - 3529t^2 - 572t + 284 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \\
 & \Leftrightarrow (t-2)(1872t^2 + 215t - 142) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \textcircled{3} \Rightarrow \textcircled{2} \Rightarrow \textcircled{1} \text{ is true} \\
 & \therefore \sum_{\text{cyc}} \left(\frac{1}{3\sqrt{3}} m_a bc \right) \geq \frac{1}{16} \prod_{\text{cyc}} (b+c) \text{ \& implementing it on } a \Delta XYZ \text{ with sides} \\
 & \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}, \text{ whose medians as a consequence of trivial calculations are } \frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \\
 & \text{we arrive at : } \frac{4}{54\sqrt{3}} \cdot \sum_{\text{cyc}} (am_b m_c) \geq \frac{1}{16} \cdot \frac{8}{27} \cdot \prod_{\text{cyc}} (m_a + m_b) \\
 & \therefore \boxed{\frac{\sqrt{3}}{6} \cdot \sum_{\text{cyc}} (am_b m_c) \geq \frac{1}{8} \cdot \prod_{\text{cyc}} (m_a + m_b)}; \text{ again, } \frac{1}{32} \prod_{\text{cyc}} (b+c) \stackrel{?}{\geq} \frac{1}{9} \sum_{\text{cyc}} (am_a^2) \\
 & = \frac{1}{36} \sum_{\text{cyc}} \left(a \left(2 \sum_{\text{cyc}} a^2 - 3a^2 \right) \right) \Leftrightarrow \frac{2s(s^2 + 2Rr + r^2)}{32} \stackrel{?}{\geq} \\
 & \frac{1}{36} (4(s^2 - 4Rr - r^2)(2s) - (2s)3(s^2 - 6Rr - 3r^2)) \\
 & \Leftrightarrow (s^2 - 16Rr + 5r^2) + 18r(R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true via Gerretsen and Euler} \\
 & \therefore \frac{1}{32} \prod_{\text{cyc}} (b+c) \geq \frac{1}{9} \sum_{\text{cyc}} (am_a^2) \text{ and implementing it on triangle XYZ,} \\
 & \text{we get : } \boxed{\frac{1}{8} \cdot \prod_{\text{cyc}} (m_a + m_b) \geq \frac{1}{4} \sum_{\text{cyc}} (a^2 m_a)} \text{ and finally, } \frac{1}{9} \sum_{\text{cyc}} (am_a^2) \stackrel{?}{\geq} \frac{abc}{4} \\
 & \Leftrightarrow \frac{1}{36} (4(s^2 - 4Rr - r^2)(2s) - (2s)3(s^2 - 6Rr - 3r^2)) \stackrel{?}{\geq} Rrs \Leftrightarrow \\
 & s^2 - 16Rr + 5r^2 \stackrel{?}{\geq} 0 \rightarrow \text{true via Gerretsen} \therefore \frac{1}{9} \sum_{\text{cyc}} (am_a^2) \geq \frac{abc}{4} \text{ \& implementing} \\
 & \text{it on triangle XYZ, we get : } \boxed{\frac{1}{4} \sum_{\text{cyc}} (a^2 m_a) \geq m_a m_b m_c} \text{ and so,} \\
 & \frac{\sqrt{3}}{6} \cdot \sum_{\text{cyc}} (am_b m_c) \geq \frac{1}{8} \cdot \prod_{\text{cyc}} (m_a + m_b) \geq \frac{1}{4} \sum_{\text{cyc}} (a^2 m_a) \geq m_a m_b m_c \forall \Delta ABC, \\
 & \text{" = " iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$