ADVANCED MATHEMATICS IV

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1. Find $\int_\gamma x^2 z ds$, where γ is the line segment joining the points $A(2,1,2)$ and $B(1,1,-1)$.

Solution.

$$A(2,1,2); B(1,1,-1)$$
$$x_A y_A z_A \quad x_B y_B z_B$$
$$AB : \frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_B} = \frac{z-z_A}{z_B-z_A} = t; t \in [0,1]$$
$$AB : \frac{x-2}{1-2} = \frac{y-1}{1-1} = \frac{z-2}{-1-2} = t$$
$$\begin{cases} x-2=-t \\ y-1=0 \\ z-2=-3t \end{cases} \Rightarrow \begin{cases} x(t)=2-t \\ y(t)=1 \\ z(t)=2-3t \end{cases} \Rightarrow \begin{cases} x'(t)=-1 \\ y'(t)=0 \\ z'(t)=-3 \end{cases}$$
$$ds = \sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt = \sqrt{(-1)^2+0^2+(-3)^2}dt = \sqrt{10}dt$$
$$\int_\gamma x^2 z ds = \int_0^1 (2-t)^2 \cdot (2-3t) \cdot \sqrt{10} dt =$$
$$= \sqrt{10}\int_0^1 (4-4t+t^2)(2-3t)dt = \sqrt{10}\int_0^1 (8-12t-8t+12t^2+2t^2-3t^3)dt =$$
$$= \sqrt{10}\int_0^1 (-3t^3+14t^2-20t+8)dt =$$
$$= \sqrt{10}\Big(-3\frac{t^4}{4}\Big|_0^1 + 14\frac{t^3}{3}\Big|_0^1 - 20\frac{t^2}{2}\Big|_0^1 + 8t\Big|_0^1\Big) =$$
$$= \sqrt{10}\Big(-\frac{3}{4}+\frac{14}{3}-10+8\Big) = \sqrt{10}\Big(\frac{-9+56}{12}-2\Big) = \sqrt{10}\Big(\frac{47-24}{12}\Big) = \frac{23\sqrt{10}}{12}$$

□

2. Calculate $\int_\gamma z ds$, where γ is the curve arc with parametric equations $x = 2t\cos t, y = 2t\sin t, z = 4t, t \in [2\pi, 5\pi]$.

Solution.

$$x = 2t\cos t, \qquad y = 2t\sin t, \qquad z = 4t, \qquad t \in [2\pi, 5\pi]$$
$$x'(t) = 2\cos t - 2t\sin t, \qquad y'(t) = 2\sin t + 2t\cos t, \qquad z'(t) = 4$$
$$\sqrt{x'^2(t)+y'^2(t)+z'^2(t)} =$$
$$= \sqrt{(2\cos t - 2t\sin t)^2 + (2\sin t + 2t\cos t)^2 + 4^2} =$$
$$= \sqrt{4\cos^2 t + 4t^2\sin^2 t - 8t\sin t\cos t + 4\sin^2 t + 4t^2\cos^2 t + 8t\sin t\cos t + 16} =$$
$$= \sqrt{4(\sin^2 + \cos^2 t) + 4t^2(\sin^2 t + \cos^2 t) + 16} =$$

$$= \sqrt{4+4t^2+16} = \sqrt{4t^2+20} = 2\sqrt{t^2+5}$$

$$\int_\gamma zdz = \int_{2\pi}^{5\pi} 4t \cdot 2\sqrt{t^2+5}dt = 8\int_{2\pi}^{5\pi} t(t^2+5)^{\frac{1}{2}}dt =$$

$$= 4\int_{2\pi}^{5\pi}(t^2+5)'(t^2+5)^{\frac{3}{2}-1}dt = 4\cdot\frac{1}{\frac{3}{2}}(t^2+5)^{\frac{3}{2}}\Big|_{2\pi}^{5\pi} = \frac{8}{3}\left((25\pi^2+5)^{\frac{3}{2}} - (4\pi^2+5)^{\frac{3}{2}}\right)$$

□

3. Calculate $\int_C xydx + zdy - 5ydz$, where the curve $C : x = \sqrt{t}, y = t^2$, $z = t^3 - 1, t \in [0, 1]$.

Solution.

$$x(t) = \sqrt{t}; x'(t) = \frac{1}{\sqrt{t}}, \qquad y(t) = t^2; y'(t) = 2t, \qquad z(t) = t^3 - 1; z'(t) = 3t^2$$

$$t \in [0, 1]$$

$$\int_C xydx + zdy - 5ydz = \int_0^1 \left(\sqrt{t}\cdot t^2 \cdot \frac{1}{2\sqrt{t}} + (t^3-1)\cdot 2t - 5t^2\cdot 3t^2\right)dt =$$

$$= \int_0^1\left(\frac{t^2}{2} + 2t^4 - 2t - 15t^4\right)dt = \frac{t^3}{6}\Big|_0^1 + 2\frac{t^5}{5}\Big|_0^1 - 2\frac{t^2}{2}\Big|_0^1 - 15\frac{t^5}{5}\Big|_0^1 =$$

$$= \frac{1}{6} + \frac{2}{5} - 1 - 3 = \frac{5+12}{30} - 4 = \frac{17-120}{30} = \frac{-130}{30}$$

□

4. Calculate $\int_C zds$, where the curve C has the parametric representations $C : x = 4\cos t, y = 4\sin t, z = 3t, t \in [0, \frac{\pi}{2}]$.

Solution.

$$x = 4\cos t \qquad x'(t) = -4\sin t$$

$$y = 4\sin t \qquad y'(t) = 4\cos t$$

$$z = 3t \qquad z'(t) = 3, \qquad t \in \left[0, \frac{\pi}{2}\right]$$

$$ds = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2 dt} =$$

$$= \sqrt{16\sin^2 t + 16\cos^2 t + 9dt} = \sqrt{16(\sin^2 t\cos^2 t) + 9}dt = \sqrt{25}dt = 5dt$$

$$\int_C zds = \int_0^{\frac{\pi}{2}} 3t\cdot 5dt = 15\frac{t^2}{2}\Big|_0^{\frac{\pi}{2}} = \frac{15}{2}\left(\frac{\pi}{2}\right)^2 = \frac{15\pi^2}{8}$$

□

5. Calculate $\int\int_{\sum}(x+y-z)d\sigma$, where

$$\sum = \left\{(x,y,z) \in \frac{\mathbb{R}^3}{x} = 2u + v, y = u - 2v, z - 1 + u + v, u \in [1,2], v \in [0,1]\right\}$$

Solution.

$$\int\int_{\sum}(x+y-z)d\sigma$$

$$\sum=\{(x,y,z)\in\mathbb{R}^3|x=2u+v,y=u-2v,z=1+u+v,u\in[1,2];v\in[0,1]\}$$

$$x=\varphi(u,v)=2u+v;\frac{\partial\varphi}{\partial u}=2;\frac{\partial\varphi}{\partial v}=1$$

$$y=\psi(u,v)=u-2v;\frac{\partial\psi}{\partial u}=1;\frac{\partial\psi}{\partial v}=-2$$

$$z=\lambda(u,v)=1+u+v;\frac{\partial\lambda}{\partial u}=1;\frac{\partial\lambda}{\partial v}=1$$

$$E=\Big(\frac{\partial\varphi}{\partial u}\Big)^2+\Big(\frac{\partial\psi}{\partial u}\Big)^2+\Big(\frac{\partial\lambda}{\partial u}\Big)^2=2^2+1^2+1^2=6$$

$$F=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial\varphi}{\partial v}+\frac{\partial\psi}{\partial u}\cdot\frac{\partial\psi}{\partial v}+\frac{\partial\lambda}{\partial u}\cdot\frac{\partial\lambda}{\partial v}=2\cdot1+1\cdot(-2)+1\cdot1=1$$

$$G=\Big(\frac{\partial\varphi}{\partial v}\Big)^2+\Big(\frac{\partial\psi}{\partial v}\Big)^2+\Big(\frac{\partial\lambda}{\partial v}\Big)^2=1^2+(-2)^2+1^2=6$$

$$\sqrt{EG-F^2}=\sqrt{6\cdot6-1^2}=\sqrt{35}$$

$$I=\int\int_{\sum}(x+y-z)d\sigma=\int\int_D(2u+v+u+2v-1-u-v)\sqrt{35}dudv$$

$$=\int\int_D(2u-2v-1)\sqrt{35}dudv=\int_1^2\left(\int_0^1(2u-2v-1)\sqrt{35}dv\right)du=$$

$$=\sqrt{35}\int_1^2\left(2uv\Big|_0^1-2\frac{v^2}{2}\Big|_0^1-v\Big|_0^1\right)du=\sqrt{35}\int_1^2(2u-1-1)du=\sqrt{35}\Big(u^2\Big|_1^2-2u\Big|_1^2\Big)=$$

$$=\sqrt{35}(4-1-2(2-1))=\sqrt{35}(3-2)=\sqrt{35}$$

□

6. Calculate $\int\int_{\sum}d\sigma$, where $\sum$ is the portions of the plane $x+y+z=2$ cut by coordinate planes.

Solution.

$$P:x+y+z=2\Rightarrow z=2-x-y,\qquad\frac{\partial z}{\partial x}=-1;\frac{\partial z}{\partial y}=-1$$

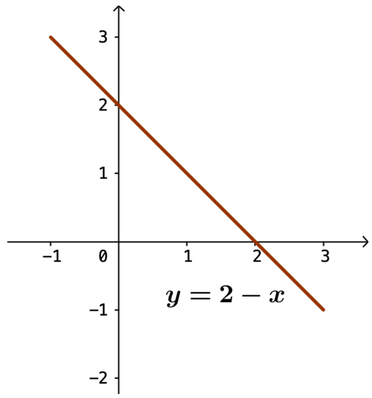
$$d\sigma=\sqrt{1+\Big(\frac{\partial z}{\partial x}\Big)^2+\Big(\frac{\partial z}{\partial y}\Big)^2}dxdy=\sqrt{1+(-1)^2+(-1)^2}dxdy=\sqrt{3}dxdy$$

$$z=0\Rightarrow2-x-y=0\Rightarrow x+y=2\Rightarrow y=2-x$$

$$D=\{(x,y)\in\mathbb{R}^2|0\le x\le2;0\le y\le2-x\}$$

$$I=\int\int_{\sum}d\sigma=\int\int_D\sqrt{3}dxdy=\sqrt{3}\qquad\underbrace{\int\int_D dxdy}_{\text{area of the rightangle triangle}=\frac{2\cdot2}{2}=2}$$

$$I=2\sqrt{3}$$



GREEN'S FORMULA - LINE INTEGRALS

$$\int_C Pdx + Qdy = \int\int_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

$\square$

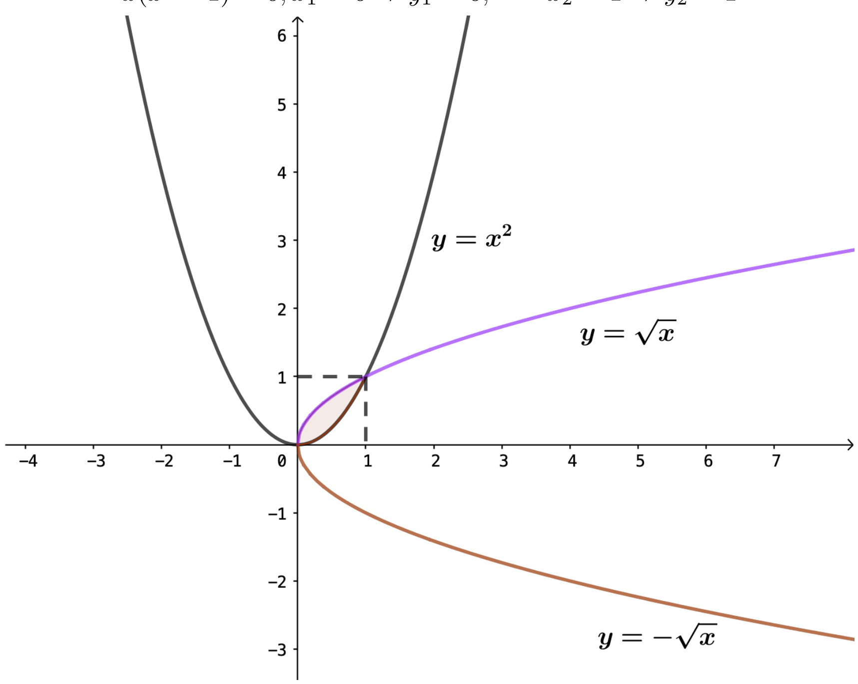
7. Calculate $\int_C (2xy - x^2)dx + (x + y^2)dy$, where the curve C is boundary of the planar region bounded by the curves $x^2 = y, y^2 = x$.

Solution.

$$C : \begin{cases} x^2 = y \\ y^2 = x \end{cases}, \qquad P(x,y) = 2xy - x^2; \frac{\partial P}{\partial y} = 2x, \qquad Q(x,y) = x + y^2; \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2x, \qquad C : \begin{cases} x^2 = y \\ y^2 = x \end{cases} \Rightarrow (x^2)^2 = x \Rightarrow x^4 - x = 0$$

$$x(x^3 - 1) = 0, x_1 = 0 \Rightarrow y_1 = 0, \qquad x_2 = 1 \Rightarrow y_2 = 1$$



$$D = \{(x,y) \in \mathbb{R}^2 | 0 \le x \le 1; x^2 \le y \le \sqrt{x}\}$$

$$\int_C (2xy - x^2)dx + (x+y^2)dy = \int\int_D (1-2x)dxdy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (1-2x)dy \right) dx =$$

$$= \int_0^1 \left((1-2x) \cdot y \Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 (1-2x)(\sqrt{x} - x^2)dx =$$

$$= \int_0^1 (\sqrt{x} - x^2 - 2x\sqrt{x} + 2x^3)dx =$$

$$= \int_0^1 x^{\frac{1}{2}} dx - \int_0^1 x^2 dx - 2\int_0^1 x^{\frac{3}{2}} dx + 2\int_0^1 x^3 dx =$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^1 - \frac{x^3}{3}\Big|_0^1 - 2 \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \Bigg|_0^1 + 2 \cdot \frac{x^4}{4}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} - 2 \cdot \frac{2}{5} + 2 \cdot \frac{1}{4} = \frac{1}{3} - \frac{4}{5} + \frac{1}{2} =$$

$$= \frac{10 - 24 + 15}{30} = \frac{1}{30}$$

□

8. Using the Gauss - Ostrogradski formula, calculate $\int\int_{\sum} (2y+x)dydz + (3z+y)dzdx + zdxdy, \sum$ where the surface is bounded by the paraboloid $x^2 + y^2 = z$ and the plane $z = 4$.

Solution.

$$\int\int_{\sum} (2y+x)dydz + (3z+y)dzdx + zdxdy$$

$$P(x,y,z) = 2y + x; Q(x,y,z) = y + 3z, R(x,y,z) = z$$

$$\frac{\partial P}{\partial x} = 1; \frac{\partial Q}{\partial y} = 1; \frac{\partial R}{\partial z} = 1$$

$$\text{div } \overrightarrow{F} = 1 + 1 + 1 = 3$$

$$I = \int\int_{\sum} (2y+x)dydz + (3z+y)dzdx + zdxdy =$$

$$= \int\int\int_V \text{div } \overrightarrow{F} dxdydz = \int\int\int_V 3dxdydz = 3\int\int\int_V dxdydz = 3 \text{ Vol } (V)$$

Calculation of the volume of the solid V:

$$x^2 + y^2 = z, \qquad z = 4$$

Base circle: $x^2 + y^2 = 4$

$$\begin{cases} x = r\cos\theta; \theta \in [0, 2\pi] \\ y = r\sin\theta; r \in [0, 1] \\ z = z; z \in [0, 4 - r^2] \end{cases}$$

$$\text{Vol } V = \int_0^{2\pi} d\theta \int_0^1 r \left(\int_0^{4-r^2} dz \right) dr =$$

$$= \theta \Big|_0^{2\pi} \cdot \int_0^1 \left(r \cdot z \Big|_0^{4-r^2} \right) dr = 2\pi \int_0^1 r(4 - r^2)dr =$$

$$= 2\pi \left(4\int_0^1 rdr - \int_0^1 r^3 dr \right) = 2\pi \left(4 \cdot \frac{r^2}{2}\Big|_0^1 - \frac{r^4}{4}\Big|_0^1 \right) =$$

$$= 2\pi\left(4\cdot\left(\frac{1^2}{2} - \frac{0^2}{2}\right) - \left(\frac{1^4}{4} - \frac{0^4}{4}\right)\right) = 2\pi\left(2 - \frac{1}{4}\right) = 2\pi\cdot\frac{8-1}{4} = \frac{7\pi}{2}$$

□

9. Using the Gauss - Ostrogradski formula, calculate $\int\int_{\sum}(2y+x)dydz + (3z+y)dzdx + zdxdy, \sum$ where the surface is bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $z=2$.

Solution.

$$I = \int\int_{\sum}(2y+x)dydz + (3z+y)dzdx + zdxdy$$

$$C: x^2+y^2=4; z=0; z=2$$

$$P(x,y,z) = 2y+x \Rightarrow \frac{\partial P}{\partial x} = 1$$

$$Q(x,y,z) = 3z+y \Rightarrow \frac{\partial Q}{\partial y} = 1$$

$$R(x,y,z) = z \Rightarrow \frac{\partial R}{\partial z} = 1$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1+1+1 = 3$$

$$I = \int\int_{\sum}(2y+x)dydz + (3z+y)dzdx + zdxdy = \int\int\int_{\Omega} 3dxdydz = 3 \text{ Vol } (\Omega)$$

$$A_b \text{ (area of the cylinder's base) } = \pi r^2 = 4\pi$$

$$\text{Vol } (\Omega) = A_b\cdot h = 4\pi\cdot(2-0) = 8\pi$$

$$I = 3 \text{ Vol } (\Omega) = 3\cdot 8\pi = 24\pi$$

□

10. Calculate $\int\int_{\sum}\sqrt{17-4z}d\sigma$, where $\sum$ is the portion of the surface $x^2+y^2+z=4$ cut by coordinate planes.

Solution.

$$x^2+y^2+z=4 \Rightarrow z = 4-x^2-y^2, \qquad \frac{\partial z}{\partial x} = -2x, \qquad \frac{\partial z}{\partial y} = -2y$$

$$d\sigma = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy, \qquad d\sigma = \sqrt{1+(-2x)^2+(-2y)^2}dxdy$$

$$d\sigma = \sqrt{1+4x^2+4y^2}dxdy$$

$$\sqrt{17-4z} = \sqrt{17-4(4-x^2-y^2)} = \sqrt{17-16+4x^2+4y^2} = \sqrt{1+4x^2+4y^2}$$

$$I = \int\int_{\sum}\sqrt{17-4z}d\sigma = \int\int_D \sqrt{1+4x^2+4y^2}\cdot\sqrt{1+4x^2+4y^2}dxdy =$$

$$= \int\int_D (1+4x^2+4y^2)dxdy$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \text{ - polar coordinates, } r\in[0,2], \theta\in\left[0,\frac{\pi}{2}\right]$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} =$$

(Jacobian)

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r\cdot 1 = r$$

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_0^2 (1+4r^2)\cdot rdr = \theta\Big|_0^{\frac{\pi}{2}} \cdot \int_0^2 (r+4r^3)dr = \frac{\pi}{2}\Big(\frac{r^2}{2}\Big|_0^2 + 4\cdot\frac{r^4}{4}\Big|_0^2\Big) =$$

$$= \frac{\pi}{2}\Big(\frac{2^2}{2} - \frac{0^2}{2} + 2^4 - 0^4\Big) = \frac{\pi}{2}(2+16) = \frac{18\pi}{2} = 9\pi$$

□

11. $\int\int_{\sum} \sqrt{x^2+y^2}d\sigma, (\sum)$ is the portion of the conic surface $z = \sqrt{x^2+y^2}$ bounded by the plane $z = 1$.

Solution.

$$\int\int_{\sum} \sqrt{x^2+y^2}d\sigma$$

$$\Big(\sum\Big): z = \sqrt{x^2+y^2}; z = 1$$

$$\frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The element can be written:

$$d\sigma = \sqrt{1+\Big(\frac{\partial z}{\partial x}\Big)^2+\Big(\frac{\partial z}{\partial y}\Big)^2}dxdy =$$

$$= \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}dxdy = \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}dxdy = \sqrt{2}dxdy$$

$$\Big(\sum\Big): z = \sqrt{x^2+y^2}; z = 1 \Rightarrow \sqrt{x^2+y^2} = 1$$

$$x^2+y^2 \le 1$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}; J = r \text{ (Jacobian)}$$

$$r \in [0,1]; \theta \in [0, 2\pi]$$

$$I = \int\int_{\sum} \sqrt{x^2+y^2}d\sigma = \int\int_D \sqrt{x^2+y^2}\sqrt{2}dxdy =$$

$$= \int\int_D \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\cdot\sqrt{2}rdr =$$

$$= \sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r\cdot rdr = \sqrt{2}\theta\Big|_0^{2\pi}\cdot\frac{r^3}{3}\Big|_0^1 = \sqrt{2}\cdot 2\pi\cdot\frac{1}{3^2} = 2\pi\frac{\sqrt{2}}{3} = \frac{2\pi\sqrt{2}}{3}$$

□

12. Using the Gauss - Ostrogradski formula, calculate
$\int\int_{\sum} xdydz + ydzdx + zdxdy, \sum$ being the surface of the ellipsoid $\frac{x^2}{9} + y^2 + \frac{z^2}{4} = 1$.

Proof.

$$I = \int\int_{\sum} xdydz + ydzdx + zdxdy$$

$$\sum : \frac{x^2}{9} + \frac{y^2}{1} + \frac{z^2}{4} = 1$$

$$a^2 = 9 \quad b^2 = 1 \quad c^2 = 4 \Rightarrow a = 3; b = 1; c = 2$$

$$V = V \text{ (elipsoid) } = \frac{4\pi}{3}abc = \frac{4\pi}{3} \cdot 3 \cdot 1 \cdot 2 = 8\pi$$

$$P(x,y,z) = x, \qquad Q(x,y,z) = y, \qquad R(x,y,z) = z$$

$$\frac{\partial P}{\partial x} = 1, \qquad \frac{\partial Q}{\partial y} = 1, \qquad \frac{\partial R}{\partial z} = 1$$

$$\text{div } \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 1 + 1 = 3 \text{ (divergence)}$$

$$I = \int\int_{\sum} xdydz + ydzdx + zdxdy =$$

$$= \int\int\int_{\Omega} \text{div } \overrightarrow{F} dxdydz = \int\int\int_{\Omega} 3dxdydz = 3 \text{ Vol } (V) = 3 \cdot 8\pi = 24\pi$$

□

13. Using the Gauss - Ostrogradski formula, calculate $\int\int_{\sum}(x^2+y^2)dydz + z^2dxdy, \sum$ being the surface of the spere $x^2+y^2+z^2=4$.

Solution.

$$I = \int\int_{\sum}(x^2+y^2)dydz + z^2dxdy$$

$$S : x^2 + y^2 + z^2 = 4$$

$$r^2 = 4 \Rightarrow r = 2 \Rightarrow V_{sphere} = \frac{4\pi r^3}{3} = \frac{4\pi \cdot 2^3}{3} = \frac{32\pi}{3}$$

$$P(x,y,z) = z^2, Q(x,y,z) = 0, R(x,y,z) = x^2 + y^2$$

$$\frac{\partial P}{\partial z} = 2z; \frac{\partial Q}{\partial y} = 0; \frac{\partial R}{\partial x} = 2x$$

$$I = \int\int_{\sum}(x^2+y^2)dydz + z^2dxdy = \int\int\int_{V}\Big(\frac{\partial P}{\partial z} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial x}\Big)dxdydz =$$

$$= \int\int\int_{V}(2x+2z)dxdydz = 2\int\int\int_{V} xdxdydz + 2\int\int\int_{V} zdxdydz = 0$$

$$f_1(x,y,z) = 2x \text{ odd on } D \text{ symmetric } \Rightarrow \int\int\int_{V} f_1(x,y,z)dxdydz = 0$$

$$f_2(x,y,z) = 2z \text{ odd on } D \text{ symmetric } \Rightarrow \int\int\int_{V} f_2(x,y,z)dxdydz = 0$$

□

14. Find $\int\int_{\sum}\Big(2x + \frac{4}{3}y + z\Big)d\sigma, (\sum) : \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x \geq 0, y \geq 0, z \geq 0$

Solution.

$$\int\int_{\sum}\Big(2x+\frac{4y}{3}+z\Big)d\sigma\left(\sum\right):\ \frac{x}{2}+\frac{y}{3}+\frac{z}{3}=1$$

$$x\geq 0; y\geq 0; z\geq 0$$

$$\frac{z}{4}=1-\frac{x}{2}-\frac{y}{3}$$

$$z=4-2x-\frac{4}{3}y$$

$$\frac{\partial z}{\partial x}=-2;\frac{\partial z}{\partial y}=-\frac{4}{3}$$

$$d\sigma=\sqrt{1+\Big(\frac{\partial z}{\partial x}\Big)^2+\Big(\frac{\partial z}{\partial y}\Big)^2}dxdy=\sqrt{1+(-2)^2+(-\frac{4}{3}\Big)^2}dxdy=$$

$$=\sqrt{1+4+\frac{16}{9}}dxdy=\sqrt{5+\frac{16}{9}}dxdy=\sqrt{\frac{45+16}{9}}dxdy=$$

$$=\frac{\sqrt{61}}{3}dxdy=\frac{\sqrt{61}}{3}dxdy$$

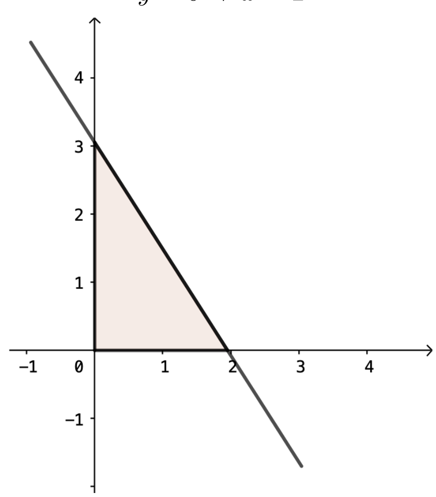
$$\left(\sum\right):\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$$

$$z=0\Rightarrow\frac{x}{2}+\frac{y}{3}=1\Rightarrow\frac{y}{3}=1-\frac{x}{2}$$

$$y=3-\frac{3x}{2}$$

$$x=0\Rightarrow y=3$$

$$y=0\Rightarrow x=2$$



$$D=\Big\{(x,y)\in\mathbb{R}^2|0\leq x\leq 2; 0\leq y\leq 3-\frac{3}{2}x\Big\}$$

$$\text{Area}_D=\frac{2\cdot 3}{2}=3$$

$$\int\int_{\sum}\Big(2x+\frac{4y}{3}+z\Big)d\sigma=\int\int_{\sum}\Big(2x+\frac{4y}{3}+4-2x-\frac{4y}{3}\Big)d\sigma=\int\int_{\sum}4d\sigma=$$

$$= \int\int_D 4 \cdot \frac{\sqrt{61}}{3} dxdy = \frac{4\sqrt{61}}{3} \int\int_D dxdy = \frac{4\sqrt{61}}{3} \text{ Area } (D) = \frac{4\sqrt{61}}{3} \cdot 3 = 4\sqrt{61}$$

□

References

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