ADVANCED MATHEMATICS III

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1. Calculate $\int\int_D xdxdy$, where D is bounded by the parabola $y = x^2 - 2$ and the line $y = x$.

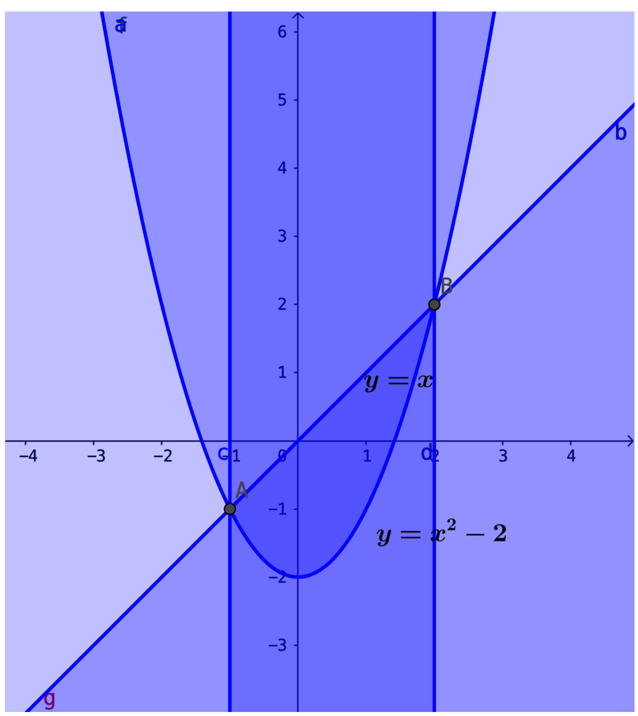
Solution.

$$P : y = x^2 - 2, \qquad d : y = x, \qquad P \cap d : x = x^2 - 2$$

$$x^2 - x - 2 = 0, \qquad \Delta = 1 + 8 = 9, \qquad x_1 = \frac{1+3}{2} = 2 \Rightarrow A(2,2)$$

$$x_2 = \frac{1-3}{2} = -1 \Rightarrow B(-1,1) \Rightarrow x \in [-1,2]$$

$$\int\int_D xdxdy = \int_{-1}^{2}\left(\int_{x^2-2}^{x} xdy\right)dx = \int_{-1}^{2}\left(x \cdot y\Big|_{x-2}^{x}\right)dx =$$



$$= \int_{-1}^{2}(x(x - x^2 + 2))dx = \int_{-1}^{2}(x^2 - x^3 + 2x)dx = \frac{x^3}{3}\Big|_{-1}^{2} - \frac{x^4}{4}\Big|_{-1}^{2} =$$

$$= \frac{2^3}{3} - \frac{(-1)^3}{3} - \left(\frac{2^4}{4} - \frac{(-1)^4}{4}\right) + 2\left(\frac{2^2}{2} - \frac{(-1)^2}{2}\right) =$$

$$= \frac{8}{3} + \frac{1}{3} - \left(4 - \frac{1}{4}\right) + 4 - 1 = 3 - \frac{15}{4} + 3 = 6 - \frac{15}{4} = \frac{24-15}{4} = \frac{9}{4}$$

□

2. Find:
$$\int\int_D \sin 2x \cdot y^2 dxdy, \text{ where } D = \left[0, \frac{\pi}{6}\right] \times [0,3].$$

Solution.
$$\int\int_D \sin 2x \cdot y^2 dxdy = \left(\int_0^{\frac{\pi}{6}} \sin 2x dx\right) \cdot \left(\int_0^3 y^2 dy\right) =$$
$$D = \left[0, \frac{\pi}{6}\right] \times [0,3]$$
$$= \left(-\frac{1}{2}\cos 2x\Big|_0^{\frac{\pi}{6}}\right) \cdot \left(\frac{y^3}{3}\Big|_0^3\right) = \left(-\frac{1}{2}\cos\frac{\pi}{3} + \frac{1}{2}\cos 0\right) \cdot \left(\frac{27}{3} - \frac{0}{3}\right) =$$
$$= 9\left(-\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}\right) = 9\left(\frac{1}{2} - \frac{1}{4}\right) = 9\frac{2-1}{4} = \frac{9}{4}$$

□

3. Calculate $\int\int_D xdxdy$, where D is bounded by the parabola $y = 4 - x^2$ and the x - axis Ox.

Solution.
$$P : y = 4 - x^2, \qquad Ox : y = 0, \qquad 4 - x^2 = 0$$
$$x^2 = 4, \qquad x = \pm 2, \qquad -2 \le x \le 2, \qquad 0 \le y \le 4 - x^2$$
$$\int\int_D xdxdy = \int_{-2}^2 \left(\int_0^{4-x^2} xdy\right) dx = \int_{-2}^2 \left(xy\Big|_0^{4-x^2}\right) dx = \int_{-2}^2 (x(4 - x^2 - 0))dx =$$
$$= \int_{-2}^2 (4x - x^3)dx = 4\int_{-2}^2 xdx - \int_{-2}^2 x^3 dx = 4 \cdot \frac{x^2}{2}\Big|_{-2}^2 - \frac{x^4}{4}\Big|_{-2}^2 =$$
$$= 4\left(\frac{2^2}{2} - \frac{(-2)^2}{2}\right) - \left(\frac{2^4}{4} - \frac{(-2)^4}{4}\right) = 4\left(\frac{4}{2} - \frac{4}{2}\right) - \left(\frac{16}{4} - \frac{16}{4}\right) = 0$$
$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \qquad z = z$$
$$J = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$
(Jacobian for cylinder coordinates)

□

4. Calculate $\int\int\int_\Omega \sqrt{x^2 + y^2} dxdydz$, where Ω is bounded by the paraboloid $x^2 + y^2 = 3z$ and the plane $z = 3$.

Solution.
$$P : x^2 + y^2 = 3z, \qquad p : z = 3$$
We use the substitution: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$
$$J = \rho, \qquad P \cap p : x^2 + y^2 = 9, \qquad \rho \in [0,3], \qquad \theta \in [0, 2\pi], \qquad z \in \left[\frac{\rho^2}{3}, 3\right]$$
$$\sqrt{x^2 + y^2} = \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \sqrt{\rho^2(\cos^2\theta + \sin^2\theta)} = \sqrt{\rho^2} = \rho$$

$$I = \int\int\int_\Omega \sqrt{x^2+y^2}dxdydz = \int_0^{2\pi} d\theta\left(\int_0^3 \rho\left(\int_{\frac{\rho^2}{3}}^3 \theta dz\right)\right)d\rho = \int_0^{2\pi} d\theta \int_0^3 \rho\cdot\left(\rho z\Big|_{\frac{\rho^2}{3}}^3\right)d\rho =$$

$$= \theta\Big|_0^{2\pi} \cdot \int_0^3 \rho\left(\rho\left(3-\frac{\rho^2}{3}\right)\right)d\rho = 2\pi\rho_0^3\left(3\rho^2-\frac{\rho^4}{3}\right)d\rho = 2\pi\left(\frac{\rho^3}{3}\Big|_0^3 - \frac{1}{3}\cdot\frac{\rho^5}{5}\Big|_0^3\right) =$$

$$= 2\pi\left(3\cdot\frac{27}{3} - \frac{1}{3}\cdot\left(\frac{3^5}{5}-\frac{0^5}{5}\right)\right) = 2\pi\left(27-\frac{81}{5}\right) = 2\pi\cdot\frac{135-81}{5} = 2\pi\cdot\frac{54}{5} = \frac{108\pi}{5}$$

□

5. Calculate $\int\int\int_\Omega xydxdydz$, where Ω is bounded $x^2+y^2=4, z=0, z=2,$ $x\geq 0, y\geq 0$.

Solution.

$$\int\int\int_\Omega xydxdydz = \left(\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta\right)\cdot\cdot\left(\int_0^2 \rho^3 d\rho\right)\cdot\left(\int_0^2 dz\right) = \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}}\cdot\frac{\rho^4}{4}\Big|_0^2\cdot z\Big|_0^2 =$$

$$= \left(\frac{\sin^2\frac{\pi}{2}}{2} - \frac{\sin^2 0}{0}\right)\cdot\left(\frac{2^4}{4}-\frac{0^4}{4}\right)\cdot(2-0) = \frac{1}{2}\left(\frac{16}{4}\right)\cdot 2 = 4$$

$$x^2+y^2=4, \qquad z=0; z=2, \qquad x\geq 0, y\geq 0$$

We use cylinder coordinates:

$$x=\rho\cos\theta, y=\rho\sin\theta, z=z, J=\rho$$

$$xy=\rho^2\sin\theta\cos\theta, \qquad x^2+y^2\leq 4 \Rightarrow \rho\in[0,2]$$

$$x\geq 0; y\geq 0 \Rightarrow \theta\in\left[0,\frac{\pi}{2}\right), \qquad z\in[0,2]$$

□

6. Calculate $\int\int_D (x^2y-2xy+y^3+2)dxdy$, where $D=[0,1]\times[0,2]$.

Solution.

$$x\in[0,1]; y\in[0,2]$$

$$\int\int_D (x^2y-2xy+y^3+2)dxdy = \int_0^1\left(\int_0^2 (x^2y-2xy+y^3+2)dy\right)dx =$$

$$= \int_0^1\left(x^2\cdot\frac{y^2}{2}\Big|_0^2 - 2x\cdot\frac{y^2}{2}\Big|_0^2 + \frac{y^4}{4}\Big|_0^2 + 2y\Big|_0^2\right)dx =$$

$$= \int_0^1\left(x^2\left(\frac{2^2}{2}-\frac{0^2}{2}\right) - 2x\left(\frac{2^2}{2}-\frac{0^2}{2}\right) + \frac{2^4}{4}-\frac{0^4}{4} + 2\cdot 2 - 2\cdot 0\right)dx =$$

$$= \int_0^1 (2x^2-4x+4+4)dx \int_0^1 (2x^2-4x+8)dx =$$

$$= 2\int_0^1 x^2dx - 4\int_0^1 xdx + 8\int_0^1 dx = 2\cdot\frac{x^3}{3}\Big|_0^1 - 4\cdot\frac{x^2}{2}\Big|_0^1 + 8x\Big|_0^1 =$$

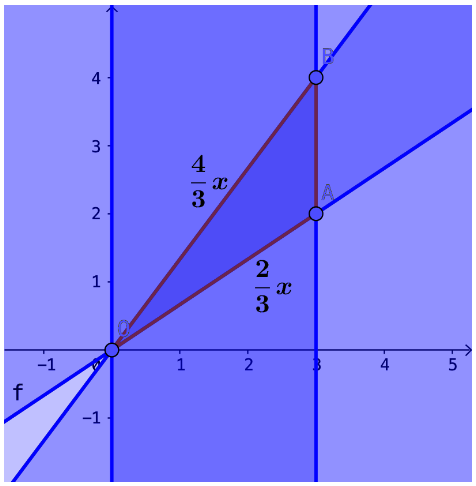
$$= 2\left(\frac{13}{3}-\frac{0^3}{3}\right) - 4\left(\frac{1^2}{2}-\frac{0^2}{2}\right) + 8(1-0) = \frac{2}{3}-2+8 = \frac{2}{3}+6 = \frac{2+18}{3} = \frac{20}{3}$$

□

7. Calculate $\int\int_D (x-2y)dxdy$, where D is the interior triangle OAB with $O(0,0), A(3,2)$ and $B(3,4)$.

Solution.

$$0(0,0); A(3,2); B(3,4) \Rightarrow x \in [0,3]$$



$$OA : \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0, \qquad OA : 3y - 2x = 0, \qquad OA : y = \frac{2}{3}x$$

$$OB : \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 3 & 4 & 1 \end{vmatrix} = 0, \qquad OB : 3y - 4x = 0, \qquad OB : y = \frac{4x}{3}$$

$$I = \int\int_D (x-2y)dxdy = \int_0^3 \left(\int_{\frac{2x}{3}}^{\frac{4x}{3}} (x-2y)dy \right) dx = \int_0^3 \left(xy \Big|_{\frac{2x}{3}}^{\frac{4x}{3}} - 2 \cdot \frac{y^2}{2} \Big|_{\frac{2x}{3}}^{\frac{4x}{3}} \right) dx =$$

$$= \int_0^3 \left(x\Big(\frac{4x}{3} - \frac{2x}{3}\Big) - \Big(\Big(\frac{4x}{3}\Big)^2 - \Big(\frac{2x}{3}\Big)^2\Big) \right) dx =$$

$$= \int_0^3 \Big(\frac{2x^2}{3} - \frac{16x^2}{9} + \frac{4x^2}{9}\Big) dx = \int_0^3 -\frac{6x^2}{9} dx = -\frac{2}{3} \cdot \frac{x^3}{3}\Big|_0^3 = -\frac{2}{3}\Big(\frac{3^3}{3} - \frac{0^3}{3}\Big) =$$

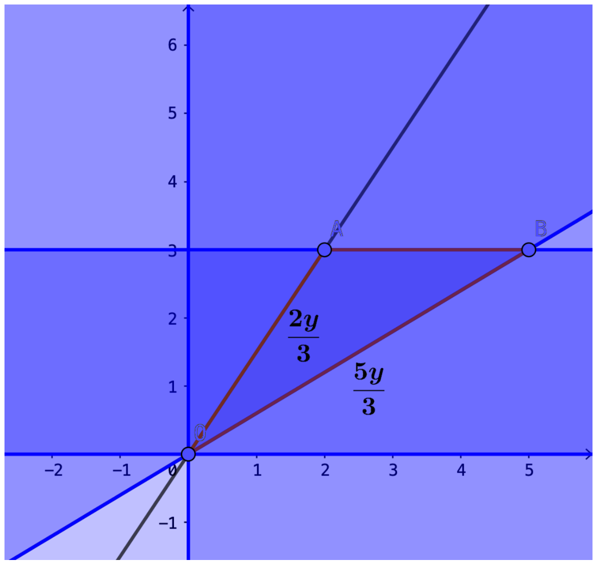
$$= -\frac{2}{3} \cdot 9 = -6$$

□

8. Calculate $\int\int_D (2x - y)dxdy$, where D is the interior triangle OAB with $O(0,0), A(2,3)$ and $B(5,3)$.

Solution.

$$O(0,0); A(2,3); B(5,3), \qquad y \in [0,3]$$



$$OA : \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0, \quad OA : 0 + 2y - 3x = 0, \quad OA : x = \frac{2y}{3}$$

$$OB : \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 5 & 3 & 1 \end{vmatrix} = 0, \quad OB : 3x - 5y = 0, \quad OB : x = \frac{5y}{3}, \quad \frac{2y}{3} \le x \le \frac{5y}{3}$$

$$I = \int\int_D (2x - y)dxdy = \int_0^3 \left(\int_{\frac{2y}{3}}^{\frac{5y}{3}} (2x - y)dx \right) dy =$$

$$= \int_0^3 \left(2 \cdot \frac{x^2}{2} \Big|_{\frac{2y}{3}}^{\frac{5y}{3}} - yx \Big|_{\frac{2y}{3}}^{\frac{5y}{3}} \right) dy = \int_0^3 \left(\frac{25y^2}{9} - \frac{4y^2}{9} - y^2 \right) dy =$$

$$= \int_0^3 \left(\frac{21y^2}{9} - y^2 \right) dy = \int_0^3 \frac{21y^2 - 9y^2}{9} dy = \int_0^3 \frac{12y^2}{9} dy = \frac{12}{9} \cdot \frac{y^3}{3} \Big|_0^3 = \frac{4}{3} \cdot 9 = 12$$

□

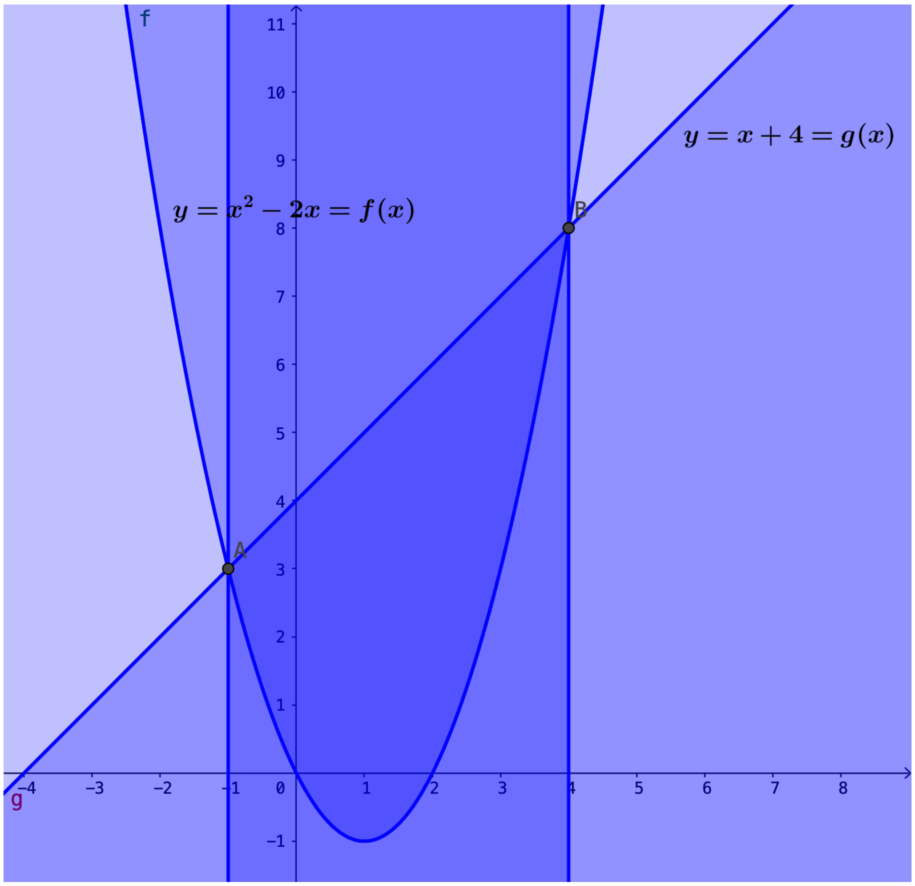
9. Calculate the coordinates of the centroid G of domain D, where D is bounded by the parabola $y = x^2 - 2x$ and the line $y = x + 4$.

Solution.

$$P : y = x^2 - 2x, \qquad d : y = x + 4, \qquad P \cap d : x^2 - 2x = x + 4$$

$$x^2 - 3x - 4 = 0, \qquad \Delta = 9 + 16 = 25, \qquad x_1 = \frac{3+5}{2} = 4, \qquad x_2 = \frac{3-5}{2} = -1$$

$$x = 4 \Rightarrow y = 8, x = -1 \Rightarrow y = 3, B(4,8), A(-1,3)$$



$$A = \int_{-1}^{4}(g(x) - f(x))dx = \int_{-1}^{4}(x + 4 - (x^2 - 2x))dx =$$

$$= \int_{-1}^{4}(x + 4 - x^2 + 2x)dx = \int_{-1}^{4}(-x^2 + 3x + 4)dx = -\frac{x^3}{3}\Big|_{-1}^{4} + 3\frac{x^2}{2}\Big|_{-1}^{4} + 4x\Big|_{-1}^{4} =$$

$$= -\frac{1}{3}(64 + 1) + 3\left(\frac{16}{2} - \frac{1}{2}\right) + 4(4 + 1) = -\frac{65}{3} + \frac{45}{2} + 20 =$$

$$= \frac{-130 + 135}{6} + 20 = \frac{5}{6} + 20 = \frac{125}{6}$$

$$G(x_G, y_G)$$

$$x_G = \frac{1}{A}\int_{-1}^{4} x(g(x) - f(x))dx; y_G = \frac{1}{2A}\int_{-1}^{4}(g^2(x) - f^2(x))dx$$

$$\int_{-1}^{4} x(g(x) - f(x))dx = \int_{-1}^{4} x(-x^2 + 3x + 4)dx = -\int_{-1}^{4} x^3 dx + 3\int_{-1}^{4} x^2 dx + 4\int_{-1}^{4} dx =$$

$$= -\frac{x^4}{4}\Big|_{-1}^{4} + 3 \cdot \frac{x^3}{3}\Big|_{-1}^{4} + \frac{4x^2}{2}\Big|_{-1}^{4} = -\frac{1}{4}(4^4 - 1) + 64 + 1 + 2(16 - 1) =$$

$$= -4^3 + \frac{1}{4} + 65 + 20 = -64 + 65 + 30 + \frac{1}{4} = 31 + \frac{1}{4} = \frac{125}{4}$$

$$x_G = \frac{1}{125} \cdot \frac{85}{4} = \frac{6}{125} \cdot \frac{125}{4} = \frac{3}{2}$$

$$\int_{-1}^{4}(g^2(x)-f^2(x))dx=\int_{-1}^{4}((x+4)^2-(x^2-2x)^2)dx=$$
$$=\int_{-1}^{4}(x^2+8x+16-x^4-4x^2+4x^3)dx=$$
$$=\int_{-1}^{4}(-x^4+4x^3-3x^2+8x+16)dx=$$
$$=-\frac{x^5}{5}\Big|_{-1}^{4}+4\frac{x^4}{4}\Big|_{-1}^{4}-3\frac{x^3}{3}\Big|_{-1}^{4}+8\frac{x^2}{2}\Big|_{-1}^{4}+16x\Big|_{-1}^{4}=$$
$$=-\frac{1}{5}(4^5+1)+4^4-1-4^3+1+4(16-1)+16\cdot 5=$$
$$=-\frac{64\cdot 16}{6}+\frac{1}{5}+64\cdot 4-1-64+1+60+80=$$
$$=64\cdot 3+140+\frac{1-64\cdot 16}{5}=192+140+\frac{(1-8\cdot 4)(1+8\cdot 4)}{5}$$
$$=332+\frac{33\cdot(-31)}{5}=\frac{1660-31\cdot 33}{5}=125$$
$$y_G=\frac{1}{2\cdot\frac{125}{6}}\cdot 125=3\Rightarrow G\Big(\frac{3}{2},3\Big)$$

□

10. Calculate the mass of the body Ω, with density $\rho(x,y,z)=x$, bounded by the plane $x+2y-z=2$ and the coordinate plane $x=0,y=0,z=0$.

Solution.

$$M=\int\int\int_{\Omega}\rho(x,y,z)dxdydz$$
$$\rho(x,y,z)=x$$
$$x+2y-z=2\Rightarrow z=x+2y-2$$
$$y=0,z=0\Rightarrow x=2,x\in[0,2],\qquad y\in\Big[0,1-\frac{x}{2}\Big]$$
$$M=\int_0^2dx\int_0^{1-\frac{x}{2}}dy\int_{x+2y-2}^{0}xdz=\int_0^2\left(\int_0^{1-\frac{x}{2}}\left(\int_{x+2y-2}^{0}xdz\right)dy\right)dx=$$
$$=-\int_0^2\left(\int_0^{1-\frac{x}{2}}\left(xz\Big|_0^{x+2y-2}\right)dy\right)dx=-\int_0^2\left(\int_0^{1-\frac{x}{2}}(x(x+2y-2))dy\right)dx=$$
$$=-\int_0^2\left(\Big(x^2y+2x\cdot\frac{y^2}{2}-2x\Big)\Big|_0^{1-\frac{x}{2}}\right)dx=$$
$$=-\int_0^2x^2\Big(1-\frac{x}{2}\Big)+x\Big(1-\frac{x}{2}\Big)^2-2x\Big(1-\frac{x}{2}\Big)dx$$
$$M=\int_0^2\Big(x^2-\frac{x^3}{2}+x+\frac{x^3}{4}-x^2-2x+x^2\Big)dx=-\int_0^2\Big(-\frac{x^3}{4}+x^2-2\Big)dx=$$
$$=\int_0^2\Big(\frac{x^3}{4}-x^2+x\Big)dx=\frac{1}{4}\cdot\frac{x^4}{4}\Big|_0^2-\frac{x^3}{3}\Big|_0^2+\frac{x^2}{2}\Big|_0^2=$$
$$=\frac{16}{16}-\frac{8}{3}+\frac{4}{2}=1+2-\frac{8}{3}=\frac{9-8}{3}=\frac{1}{3}$$

□

11. Calculate the area of the domain bounded by the parabola $y = x^2$ and the line $y = 5x - 6$.

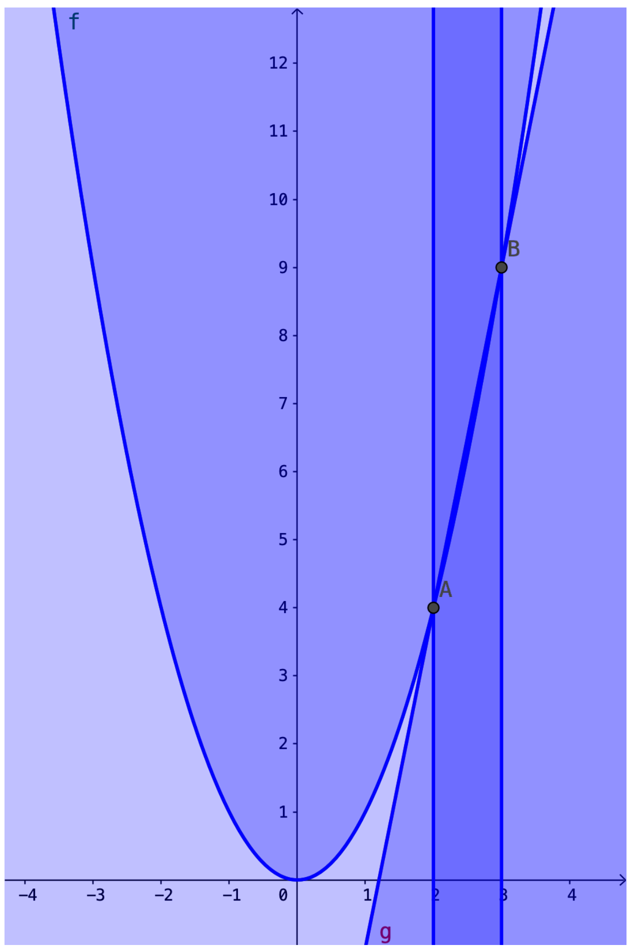
Solution.

$$P : y = x^2, \qquad d : y = 5x - 6, \qquad d \cap P : x^2 = 5x - 6$$

$$x^2 - 5x + 6 = 0, \qquad \Delta = 25 - 24 = 1, \qquad x_1 = \frac{5+1}{2} = 3, \qquad x_2 = \frac{5-1}{2} = 2$$

$$x = 2 \Rightarrow y = 4$$

$$x = 3 \Rightarrow y = 9$$



$$\text{Area } = \left|\int_2^3 (5x - 6 - x^2)dx\right| = \left|5\int_2^3 xdx - 6\int_2^3 dx - \frac{x^3}{3}\Big|_2^3\right| =$$

$$= \left|5 \cdot \frac{x^2}{2}\Big|_2^3 - 6 \cdot \frac{x^2}{2}\Big|_2^3 - \left(\frac{3^3}{3} - \frac{2^3}{3}\right)\right| = \left|5\left(\frac{3^2}{2} - \frac{2^2}{2}\right) - 3(3^2 - 2^2) - \frac{27-8}{3}\right| =$$

$$= \left|5 \cdot \frac{9-4}{2} - 3(9-4) - \frac{19}{3}\right| = \left|\frac{25}{2} - 15 - \frac{19}{3}\right| = \left|\frac{25-30}{2} - \frac{19}{3}\right| = \left|-\frac{5}{2} - \frac{19}{3}\right| =$$

$$= \left|\frac{-15-38}{6}\right| = \frac{53}{6}$$

$\square$

Note: CENTROID

$$G(x_G, y_G), x_G = \frac{1}{A}\int_{-1}^{4} x(g(x) - f(x))dx, y_G = \frac{1}{2A}\int_{-1}^{4} (g^2(x) - f^2(x))dx$$

MASS

$$M = \int\int\int_{\Omega} \rho(x, y, z)dxdydz, \quad \rho(x, y, z) \text{ - density}$$

References

[1] *Romanian Mathematical Magazine.* Interactive Journal, www.ssmrmh.ro

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