ADVANCED MATHEMATICS II

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1. Calculate $\int_C (y+z)ds$, where the curve C is the line segment joining the points $A(2,3,3)$ and $B(1,-1,2)$.

Solution.

$$A(2,3,3); B(1,-1,2)$$

$$x_A y_A z_A \quad x_B y_B z_B$$

$$AB : \frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_A} = \frac{z-z_A}{z_B-z_A} = t, \qquad AB : \frac{x-2}{1-2} = \frac{y-3}{-1-3} = \frac{z-3}{2-3} = t$$

$$AB : \begin{cases} x-2=-t \\ y-3=-4t \\ z-3=-t \end{cases} \Rightarrow \begin{cases} x=-t+2 \\ y=-4t+3 \\ z=-t+3 \end{cases} ; t\in[0,1]$$

$$x(t)=2-t; x'(t)=-1$$

$$y(t)=3-4t; y'(t)=-4$$

$$z(t)=3-t, z'(t)=-1$$

$$I=\int_C (y+z)ds; f(x,y,z)=y+z$$

$$f(x(t);y(t);z(t)) = y(t)+z(t) = 3-4t+3-t = 6-5t$$

$$\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2} = \sqrt{(-1)^2+(-4)^2+(-1)^2} = \sqrt{18} = 3\sqrt{2}$$

$$I=\int_C (y+z)ds = \int_0^1 f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt =$$

$$= \int_0^1 (6-5t)\cdot 3\sqrt{2}dt = 18\sqrt{2}\int_0^1 dt - 15\sqrt{2}\int_0^1 tdt =$$

$$= 18\sqrt{2}t\Big|_0^1 - 15\sqrt{2}\cdot\frac{t^2}{2}\Big|_0^1 = 18\sqrt{2} - \frac{15\sqrt{2}}{2} = \frac{36\sqrt{2}-15\sqrt{2}}{2} = \frac{21\sqrt{2}}{2}$$

□

2. Calculate the area of the region bounded by the parabola $y=x^2-2x$ and the line $y=3x-6$.

Solution.

$$P : y=x^2-2x, y=3x-6$$

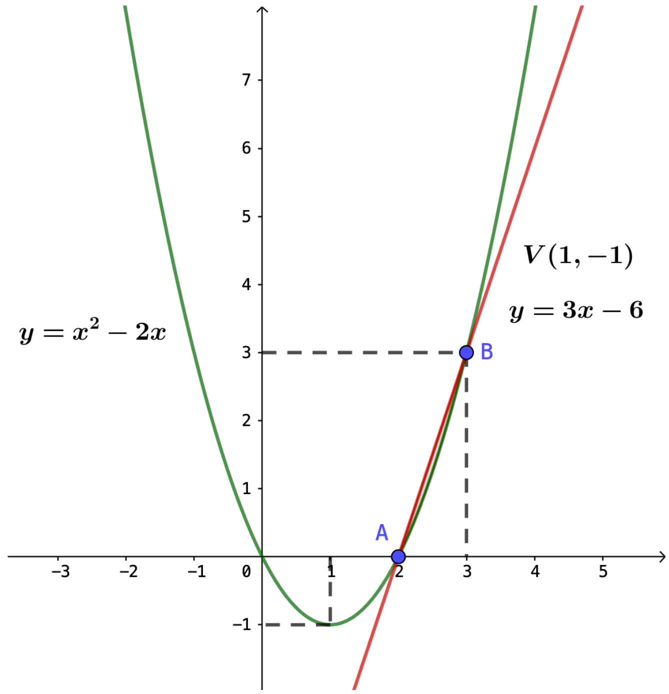
$$x=2 \Rightarrow y=3\cdot2-6=0, \qquad A(2,0)$$

$$x=3 \Rightarrow y=3\cdot3-6=3, \qquad B(3,3)$$

$$P\cap d : x^2-2x=3x-6$$

$$x^2-5x+6=0, \Delta=25-24=1, x_1=2, x_2=3$$

Vertex of the parabola $V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) = \left(-\frac{-2}{2}; -\frac{4}{4}\right)$



$$\text{Area } = \int_2^3 (3x - 6 - (x^2 - 2x))dx = \int_2^3 (3x - 6 - x^2 + 2x)dx =$$

$$= \int_2^3 (-x^2 + 5x - 6)dx = -\frac{x^3}{3}\Big|_2^3 + 5\frac{x^2}{2}\Big|_2^3 - 6x\Big|_2^3 = -\frac{27}{3} + \frac{8}{3} + 5\left(\frac{9}{2} - \frac{4}{2}\right) - 6(3-2) =$$

$$= -9 + \frac{8}{3} + \frac{25}{2} - 6 = \frac{8}{3} + \frac{25}{2} - 15 = \frac{16 + 75 - 90}{6} = \frac{1}{6}$$

□

3. Calculate $\int_C (x+y)dx + zxdy + ydz$, where the curve $C : x = t^2 + t$, $y = t + 2, z = t^3 + 2t, t \in [0, 1]$.

Solution.

$$\int_C (x+y)dx + zxdy - ydz$$

$$C : x = t^2 + t; y = t + 2; z = t^3 + 2t; t \in [0, 1]$$

$$P(x, y, z) = x + y; Q(x, y, z) = xz; R(x, y, z) = -y$$

$$P(x(t), y(t), z(t)) = x(t) + y(t) = t^2 + t + t + 2 = t^2 + 2t + 2$$

$$Q(x(t), y(t), z(t)) = x(t)z(t) = (t^2 + t)(t^3 + 2t) = t^5 + 2t^3 + t^4 + 2t^2$$

$$R(x(t), y(t), z(t)) = -y(t) = -t - 2$$

$$x'(t) = 2t + 1; y'(t) = 1; z'(t) = 3t^2 + 2$$

$$\int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz =$$

$$= \int_0^2 [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]dt =$$

$$= \int_0^1 ((t^2+2t+2)(2t+1) + (t^5+2t^3+t^4+2t^2)\cdot 1 + (-t-2)(3t^2+2))dt =$$

$$= \int_0^1 (2t^3+t^2+4t^2+2t+4t+2+t^5+2t^3+t^4+2t^2-3t^3-2t-6t^2-4)dt =$$

$$= \int_0^1 (t^5+t^4+t^3+4t-2)dt = \frac{t^6}{6}\Big|_0^1 + \frac{t^5}{5}\Big|_0^1 + \frac{t^4}{4}\Big|_0^1 + 4\cdot\frac{t^2}{2}\Big|_0^1 - 2t\Big|_0^1 =$$

$$= \frac{1}{6}+\frac{1}{5}+\frac{1}{4}+2-2 = \frac{1}{6}+\frac{1}{5}+\frac{1}{4} = \frac{10+12+15}{60} = \frac{37}{60}$$

□

4. Calculate $\int\int_{\sum}(2x+y)dydx + (y-2z)dzdx + (z-3x)dxdy$, where $\sum$ is the surface of the solid bounded by the paraboloid $z = 1-x^2-y^2$ and the plane $z=0$, and $\overrightarrow{n}_e$ is the outward normal to the surface.

Solution.

$$I = \int\int_{\sum}(2x+y)dxdy + (y-2z)dzdx + (z-3x)dxdy$$

$$P(x,y,z) = 2x+y; Q(x,y,z) = y-2z; R(x,y,z) = z-3x$$

(divergence)

$$\text{div } \overrightarrow{F} = 2+1+1 = 4 \text{ because:}$$

$$\frac{\partial P}{\partial x} = 1; \frac{\partial Q}{\partial y} = 1; \frac{\partial R}{\partial z} = 1$$

$$\int\int_{\sum} Pdxdy + Qdzdx + Rdxdy = \int\int\int_V \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\Big)dxdydz =$$

(Gauss - Ostrogradski formula)

$$= \int\int\int_V \text{div } \overrightarrow{F} dxdydz = \int\int\int_V 4dxdydz = 4\text{Vol }(V)$$

Calculation of the volume of the solid V :

$z = 1-x^2-y^2$ (paraboloid), $z=0$ (plan)

The intersection has the base circle: $x^2+y^2=1$

$$\begin{cases} x = r\cos\theta; \theta\in[0,2\pi] \\ y = r\sin\theta; r\in[0,1] \\ z = z; z\in[0,1-r^2] \end{cases}$$

$$\text{Vol }(V) = \int_0^{2\pi} d\theta \int_0^1 r\left(\int_0^{1-r^2} dz\right)dr = \theta\Big|_0^{2\pi}\cdot\int_0^1\Big(r\cdot z\Big|_0^{1-r^2}\Big)dr =$$

$$= 2\pi\cdot\int_0^1 (r(1-r^2))dr = 2\pi\int_0^1 (r-r^3)dr = 2\pi\left(\int_0^1 rdr - \int_0^1 r^3dr\right) =$$

$$= 2\pi\Big(\frac{r^2}{2}\Big|_0^1 - \frac{r^4}{4}\Big|_0^1\Big) = 2\pi\Big(\frac{1}{2}-\frac{1}{4}\Big) = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}$$

Initial integral:

$$I = 4\text{Vol }(V) = 4\cdot\frac{\pi}{2} = 2\pi$$

□

5. Determine the general solution of the equations:
a. $y' - \frac{1}{x}y = x^2\sqrt{y}$
b. $y'' - 3y' + 2y = 2xe^{2x}$
c. $y'' - 4y = e^{2x}$

Proof.
a. $y' - \frac{1}{x}y = x^2\sqrt{y}| : \sqrt{y} \Rightarrow \frac{y'}{\sqrt{y}} - \frac{\sqrt{y}}{x} = x^2$
Let be $u = \sqrt{y} \Rightarrow u' = \frac{y'}{2\sqrt{y}} \Rightarrow \frac{y'}{\sqrt{y}} = 2u' \Rightarrow 2u' - \frac{1}{x}u = x^2 \Rightarrow u' - \frac{u}{2x} = \frac{x^2}{2}$
Homogeneous equation: $u' - \frac{1}{2x}u = 0 \Rightarrow \frac{u'}{u} = \frac{1}{2x} \Rightarrow \int \frac{u'}{u}du = \int \frac{1}{2x} = \frac{1}{2}\ln|x| + \ln|c|$

$$\Rightarrow \ln|u| = \ln\sqrt{|x|} \cdot C \Rightarrow u = c\sqrt{x}$$

Solution of the homogeneous equation. We seek a particular solution:

$$u_p(x) = c(x)\sqrt{x}, \qquad u_p'(x) = c'(x)\sqrt{x} + c(x) \cdot \frac{1}{2\sqrt{x}}$$

$$u' - \frac{u}{2x} = \frac{x^2}{2}, \qquad c'(x)\sqrt{x} + c(x)\frac{1}{2\sqrt{x}} - \frac{1}{2x} \cdot c(x)\sqrt{x} = \frac{x^2}{2}$$

$$c'(x)\sqrt{x} = \frac{x^2}{2} \Rightarrow c'(x) = \frac{x\sqrt{x}}{2} = \frac{x^{\frac{3}{2}}}{2}$$

$$c(x) = \int \frac{x^{\frac{3}{2}}}{2}dx = \frac{1}{2} \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{1}{5}\sqrt{x^5} + c$$

$$u(x) = c(x)\sqrt{x} = \left(\frac{1}{5}\sqrt{x^5} + c\right)\sqrt{x} = \frac{x^3}{5} + c\sqrt{x}$$

$$y_{(x)} = u^2(x) = \left(\frac{x^3}{5} + c\sqrt{x}\right)^2$$

5.b. $y'' - 3y' + 2y = 2xe^{2x}$
Homogeneous equation: $y'' - 3y' + 2y = 0$. Characteristic equation: $r^2 - 3r + 2 = 0$
$r_1 = 1; r_2 = 2; S = \{e^x; e^{2x}\}$ - fundamental system of solutions
Solutions of the homogeneous equation: $y_h(x) = c_1e^x + c_2e^{2x}$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{2x} = 0 \\ c_1'(x)(e^x)' + c_2'(x)(e^{2x})' = 2xe^{2x} \end{cases}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{2x} = 0 \\ c_1'(x)e^x + 2c_2'(x)e^{2x} = 2xe^{2x} \end{cases}$$

$$w(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^{2x} \\ 2xe^{2x} & 2e^{2x} \end{vmatrix}}{w(x)} = \frac{-2xe^{4x}}{e^{3x}} = -2xe^x$$

$$c_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 2xe^{2x} \end{vmatrix}}{w(x)} = \frac{2xe^{3x}}{e^{3x}} = 2x$$

$$c_1'(x) = -xe^x$$

$$c_1(x) = \int(-2xe^x)dx = \int(-2x)(e^x)'dx =$$

$$= -2xe^x - \int (-2) \cdot e^x dx = -2xe^x + 2e^x = e^x(2-2x)$$

$$c_2'(x) = 2x, \qquad c_2(x) = \int 2x dx = x^2$$

y_p - particular solution

$$y_p(x) = e^x(2-2x)e^x + x^2e^{2x} = (x^2-2x+2)e^{2x}$$

General solution:

$$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{2x} + (x^2-2x+2)e^{2x}$$

5.c. $y'' - 4y = e^{2x}$
Characteristic equation: $r^2 - 4 = 0 \Rightarrow r_1 = 2; r_2 = -2$
Solution of the homogeneous equation: $y_h(x) = c_1e^{2x} + c_2e^{-2x}$
$S = \{e^{2x}; e^{-2x}\}$ - fundamental system of solutions
We seek a particular solution:

$$y_p(x) = c_1(x)e^{2x} + c_2(x)e^{-2x}$$

$$\begin{cases} c_1'(x)e^{2x} + c_2'(x)e^{-2x} = 0 \\ c_1'(x)(e^{2x})' + c_2'(x)(e^{-2x})' = e^{2x} \end{cases}$$

$$\begin{cases} c_1'(x)e^{2x} + c_2'(x)e^{-2x} = 0 \\ 2c_1'(x)e^{2x} - 2c_2'(x)e^{-2x} = e^{2x} \end{cases}$$

$$w(x) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2 = -4$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^{-2x} \\ e^{2x} & -2e^{-2x} \end{vmatrix}}{w(x)} = \frac{-e^{2x} \cdot e^{-2x}}{-4} = \frac{1}{4}$$

$$c_2'(x) = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & e^{2x} \end{vmatrix}}{w(x)} = \frac{e^{4x}}{-4}$$

$$y_p(x) = c_1(x)e^{2x} + c_2(x)e^{-2x}, \qquad c_1'(x) = \frac{1}{4} \Rightarrow c_1(x) = \frac{x}{4}$$

$$c_2'(x) = -\frac{1}{4}e^{4x} \Rightarrow c_2(x) = -\frac{1}{4}\int e^{4x}dx = -\frac{e^{4x}}{16}$$

$$y_p(x) = \frac{x}{4}e^{2x} - \frac{e^{4x}}{16}e^{-2x}, \qquad y_p(x) = \frac{x}{4}e^{2x}\left(1-\frac{1}{4}\right) = \frac{3x}{16}e^{2x}$$

General solution:

$$y(x) = y_h(x) + y_p(x)$$

$$y(x) = c_1e^{2x} + c_2e^{-2x} + \frac{3x}{16}e^{2x}$$

$\square$

6. Determine the general solution of the following systems of differential equations:

a. $\begin{cases} y_1' = 2y_1 + y_2 + e^x \\ y_2' = y_1 + 2y_2 \end{cases}$

b. $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + y_2 + y_3 \\ y_3' = y_2 \end{cases}$

Solution.

a. $\begin{cases} y_1' = 2y_1 + y_2 + e^x \\ y_2' = y_1 + 2y_2 \end{cases} \Rightarrow y_1 = y_2' - 2y_2, y_1' = y_2'' - 2y_2', y_1' - 2y_1 - y_2 = e^x$

$$y_2'' - 2y_2' - 2(y_2 - 2y_2) - y)2 = e^x$$
$$y_2'' - 2y_2' - 2y_2' + 4y_2 - y_2 = e^x$$
$$y_2'' - 4y_2' + 3y_2 = e^x$$

Homogeneous equation: $y_2'' - 4y_2' + 3y_2 = 0$. Characteristic equation: $r^2 - 4r + 3 = 0$
$r_1 = 1; r_2 = 3$; Fundamental system of solutions: $S = \{e^x; e^{3x}\}$
Solutions of the homogeneous equation: $y_h(x) = c_1e^x + c_2e^{3x}$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{3x} = 0 \\ c_1'(x)(e^x)' + c_2'(x)(e^{3x})' = e^x \end{cases}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{3x} = 0 \\ c_1'(x)e^x + 3c_2'(x)e^{3x} = e^x \end{cases}$$

$$w(x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 3e^{4x} - e^{4x} = 2e^{4x}$$

$$c_1' = \frac{\begin{vmatrix} 0 & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix}}{w(x)} = \frac{e^{4x}}{2e^{4x}} = \frac{1}{2}$$

$$c_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \end{vmatrix}}{w(x)} = \frac{e^{2x}}{2e^{4x}} = \frac{1}{2e^{2x}}$$

$$y_p(x) = c_1(x)e^x + c_2(x)e^{2x}$$
$$c_1'(x) = \frac{1}{2} \Rightarrow c_1(x) = \frac{x}{2}$$
$$c_2'(x) = \frac{1}{2}e^{-2x} \Rightarrow c_2(x) = -\frac{1}{4}e^{-2x}$$
$$y_p(x) = \frac{x}{2}e^x - \frac{1}{4}e^{-2x}$$

General solution:

$$y(x) = y_h(x) + y_p(x)$$
$$y(x) = c_1e^x + c_2e^{3x} + \frac{x}{2}e^x - \frac{1}{4}e^{-2x}$$

□

6.b. $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + y_2 + y_3 \\ y_3' = y_2 \end{cases}$

Matrix form of the system:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Calculation of the eigenvalues:

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2(1-\lambda) + 0 + 0 - 0 + \lambda + \lambda = 0$$

$$\lambda^2 - \lambda^3 + 2\lambda = 0 \Rightarrow -\lambda(\lambda^2 - \lambda - 2) = 0$$

$$\lambda_1 = 0; \lambda_2 = -1; \lambda_3 = 2$$

Calculation of the eigenvectors:

For $\lambda = 0$: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_2 \\ v_1 + v_2 + v_3 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow v_2 = 0; v_1 = -v_3 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

For $\lambda = -1$: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 + v_2 \\ v_1 + 2v_2 + v_3 \\ v_2 + 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} v_1 + v_2 = 0 \\ v_2 = -2v_3 \\ v_1 = 2v_3 \end{matrix}$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_3 \\ -2v_3 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

For $\lambda = 2$: $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2v_1 + v_2 \\ v_1 - v_2 + v_3 \\ v_2 - 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{matrix} v_2 = 2v_1 \\ v_2 = 2v_3 \end{matrix} \Rightarrow v_1 = v_3 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \\ v_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

Solution.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = c_1 \cdot e^{0 \cdot x} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot e^{-1 \cdot x} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_3 \cdot e^{2 \cdot x} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} -c_1 + 2c_2e^{-x} + c_3e^{2x} \\ -c_2e^{-x} + 2c_3e^{2x} \\ c_2e^{-x} + c_1 + c_3e^2x \end{pmatrix}, \qquad \begin{cases} y_1(x) = -c_1 + 2c_2e^{-x} + c_3e^{2x} \\ y_2(x) = -c_2e^{-x} + 2c_3e^{2x} \\ y_3(x) = c_1 + c_2e^{-x} + c_3e^{2x} \end{cases}$$

□

References

[1] *Romanian Mathematical Magazine.* Interactive Journal, www.ssmrmh.ro

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