ADVANCED MATHEMATICS I

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1. Calculate $\int_C (x+z)ds$, where the curve C is the line segment joining the points $A(3,2,0)$ and $B(1,1,2)$.

Solution.

$$A(3,2,0); B(1,1,2)$$
$$x_A y_A z_A \quad x_B y_B z_B$$
$$AB : \frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_A} = \frac{z-z_A}{z_B-z_A} = t, \qquad AB : \frac{x-3}{1-3} = \frac{y-2}{1-2} = \frac{z-0}{2-0} = t$$
$$AB : \begin{cases} x-3=-2t \\ y-2=-t \\ z=2t \end{cases} \Rightarrow AB : \begin{cases} x=3-2t \\ y=2-t \\ z=2t \end{cases} ; t \in [0,1]$$
$$x(t) = 3-2t; x'(t) = -2$$
$$y(t) = 2-t; y'(t) = -1$$
$$z(t) = 2t; z'(t) = 2$$
$$I = \int_C (x+z)dz; f(x,y,z) = x+z$$
$$f(x(t);y(t);z(t)) = x(t)+z(t) = 3-2t+2t = 3$$
$$\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2} = \sqrt{(-2)^2+(-1)^2+2^2} = 3$$
$$\int_C f(x,y,z) = \int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt$$
$$I = \int_C (x+z)ds = \int_0^1 3\cdot 3dt = 9t\Big|_0^1 = 9(1-0) = 9$$

□

2. Calculate $\int\int_{\sum} \sqrt{2x^2+2y^2+1}d\sigma$, where $\sum$ is the surface given by

$$\sum = \{(x,y,z) \in R^3 / x = u\cos v, y = u\sin v, z = u+v, u \in [0,2], v \in [0,\pi]\}$$

Solution.

$$x = \varphi(u,v) = u\cos v$$
$$y = \psi(u,v) = u\sin v$$
$$z = \chi(u,v) = u+v$$
$$E = \left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\chi}{\partial u}\right)^2 = \cos^2 v + \sin^2 v + 1 = 2$$
$$F = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial\varphi}{\partial v} + \frac{\partial\psi}{\partial u}\cdot\frac{\partial\psi}{\partial v} + \frac{\partial\chi}{\partial u}\cdot\frac{\partial\chi}{\partial v} = \cos v(-u\sin v) + \sin v\cdot(u\cos v) + 1\cdot 1 = 1$$
$$G = \left(\frac{\partial\psi}{\partial v}\right)^2 + \left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\chi}{\partial v}\right)^2 = (-u\sin v)^2 + (u\cos v)^2 + 1^2 = u^2+1$$

$$\sqrt{EG-F^2}=\sqrt{2(u^2+1)-1}=\sqrt{2u^2+1}$$

$$f(x,y,z)=\sqrt{2x^2+2y^2+1}$$

$$f(x(u,v),y(u,v),z(u,v))=\sqrt{2u^2\cos^2 v+2u^2\sin^2 v+1}=$$

$$=\sqrt{2u^2(\cos^2 v+\sin^2 v)+1}=\sqrt{2u^2+1}$$

$$\int\int_{\Sigma} f(x,y,z)d\tau=\int\int_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}dudv=$$

$$=\int\int_D \sqrt{2u^2+1}\cdot\sqrt{2u^2+1}dudv=\int_0^2(2u^2+1)du\cdot\int_0^\pi dv=$$

$$=2\left(\frac{u^3}{3}\Big|_0^2+u\Big|_0^2\right)\cdot\pi=2\pi\left(\frac{8}{3}+2\right)=\frac{28\pi}{3}$$

$\square$

3. Calculate the area of the region bounded by the parabola $y=x^2$ and the line $y=5x-6$.

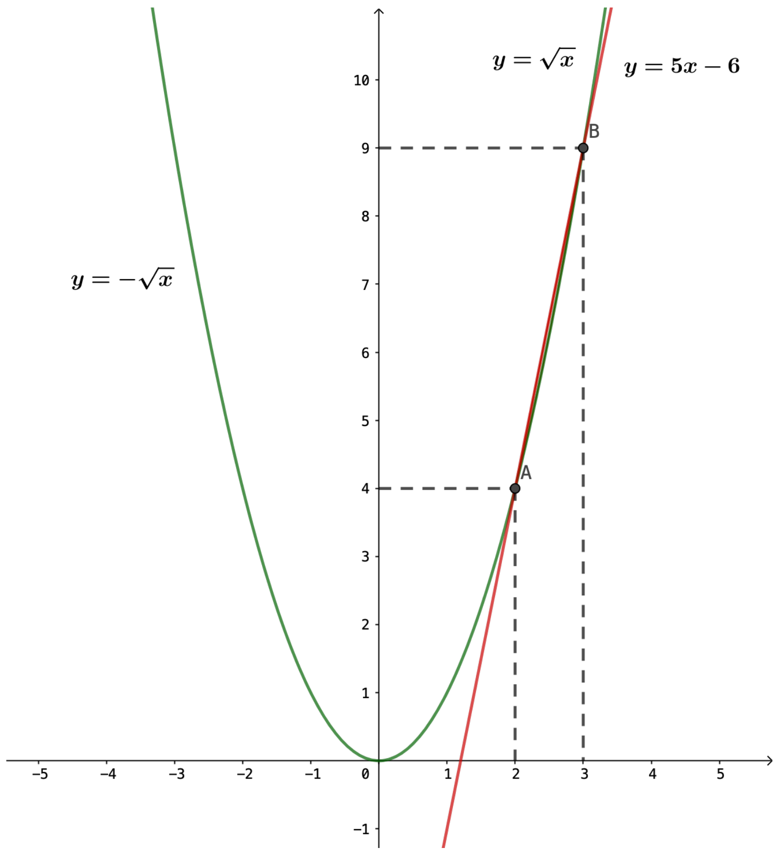
Solution.

$$P:y=x^2,d:y=5x-6,d\cap P:x^2=5x-6$$

$$x^2-5x+6=0,\qquad \Delta=25-24=1,\qquad x_1=2,x_2=3$$

$$x=2\Rightarrow y=2^2=4,\qquad A(2,4)$$

$$x=3\Rightarrow y=3^2=9,\qquad B(3,9)$$



$$\text{Area } = \int_2^3 (5x - 6 - \sqrt{x})dx = 5\int_2^3 xdx - 6\int_2^3 dx - \int_2^3 \sqrt{x}dx =$$

$$= 5 \cdot \frac{x^2}{2}\Big|_2^3 - 6x\Big|_2^3 - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_2^3 = 5\Big(\frac{9}{2} - \frac{4}{2}\Big) - 6(3-2) - \frac{2}{3}(3\sqrt{3} - 2\sqrt{2}) =$$

$$= \frac{25}{2} - 6 - \frac{2}{3}(3\sqrt{3} - 2\sqrt{2}) = \frac{13}{2} - \frac{2}{3}(3\sqrt{3} - 2\sqrt{2})$$

□

4. Calculate $\int_C (x+z)dx + zdy - dz$, where the curve $C : x = 2t, y = t^2$, $z = t^3 + t, t \in [0, 1]$.

Solution.

$$\int_C (x+z)dx + zdy - dz$$
$$C : x = 2t; y = t^2; z = t^3 + t; t \in [0, 1]$$
$$P(x, y, z) = x + z; Q(x, y, z) = z; R(x, y, z) = -1$$
$$P(x(t), y(t), z(t)) = x(t) + z(t) = 2t + t^3 + t = t^3 + 3t$$
$$Q(x(t), y(t), z(t)) = z|t| = t^3 + t$$
$$R(x(t), y(t), z(t)) = -1$$
$$x(t) = 2t \Rightarrow x'(t) = 2$$
$$y(t) = t^2 \Rightarrow y'(t) = 2t$$

$$z(t) = t^3 + t \Rightarrow z'(t) = 3t^2 + 1$$

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int_0^1 [P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t)]dt =$$

$$= \int_0^1 (2t^3 + 6t + 2t^4 + 2t^2 - 3t^2 - 1)dt = \int_0^1 (2t^4 + 2t^3 - t^2 + 6t - 1)dt =$$

$$= 2 \cdot \frac{t^5}{5}\Big|_0^1 + 2 \cdot \frac{t^4}{4}\Big|_0^1 - \frac{t^3}{3}\Big|_0^1 + 6 \cdot \frac{t^2}{2}\Big|_0^1 - t\Big|_0^1 = 2\Big(\frac{1}{5} - \frac{0}{5}\Big) + \frac{1}{2} - \frac{1}{3} + 3 - 1 =$$

$$= \frac{2}{5} + \frac{1}{2} - \frac{1}{3} + 2 = \frac{12 + 15 - 10 + 60}{30} = \frac{27 + 50}{30} = \frac{77}{30}$$

$\square$

5. Find the general solution of the equations:
a) $y' - \frac{1}{x}y = 2x\sqrt{y}$
b) $y'' - 4y + 3y = 2xe^{2x}$
c) $y'' - y = e^x$

Solution.
Bernoulli-type equation:
a. $y' - \frac{1}{x}y = 2x\sqrt{y}| : y, \qquad \frac{y'}{\sqrt{y}} - \frac{1}{x}\sqrt{y} = 2x$

Make the substitution $u = \sqrt{y}, u' = \dfrac{1}{2\sqrt{y}} \cdot y' \Rightarrow \dfrac{y'}{\sqrt{y}} = 2u' \Rightarrow 2u' - \dfrac{1}{x} \cdot u = 2x$

$$u' - \frac{1}{2x} \cdot u = x \text{ - linear equation}$$

Homogeneous equation: $u' - \dfrac{u}{2x} = 0 \Rightarrow \dfrac{u'}{u} = \dfrac{1}{2x} \Rightarrow \displaystyle\int \frac{u'}{u}du = \int \frac{1}{2x} = \frac{1}{2}\ln|x| + c$

$$\ln|u| = \frac{1}{2}\ln|x| + \ln|c|$$

$$u_h(x) = c\sqrt{x} \text{ - solution of the homogeneous equation}$$

Find a particular solution:

$$u_p(x) = c(x)\sqrt{x}, \qquad u_p' = c'(x)\sqrt{x} + c(x) \cdot \frac{1}{2\sqrt{x}}$$

$$u'(x) - \frac{1}{2x}u(x) = x, \qquad c'(x)\sqrt{x} + c(x) \cdot \frac{1}{2\sqrt{x}} - \frac{1}{2x} \cdot c(x)\sqrt{x} = x$$

$$c'(x)\sqrt{x} = x, \qquad c'(x) = \frac{x}{\sqrt{x}} \Rightarrow c'(x) = \sqrt{x} \Rightarrow c(x) = \int \sqrt{x}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x\sqrt{x} + c$$

$$u(x) = c(x)\sqrt{x} = \Big(\frac{2}{3}x\sqrt{x} + c\Big)\sqrt{x}, \qquad u = \sqrt{y} \Rightarrow y = u^2$$

$$y(x) = \left(\Big(\frac{2}{3}x\sqrt{x} + c\Big)\sqrt{x}\right)^2 = \Big(\frac{2}{3}x^2 + c\sqrt{x}\Big)^2$$

5. b. $y'' - 4y' + 3y = 2xe^{2x}$

Homogeneous equation: $y'' - 4y' + 3y = 0$. Characteristic equation: $r^2 - 4r + 3 = 0$

$$r_1 = 1; r_2 = 3. S = \{e^x, e^{3x}\} \text{ fundamental system of solutions}$$

General solution of the homogeneous equation:

$$y_h(x) = c_1 e^x + c_2 e^{3x}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{3x} = 0 \\ c_1'(x)(e^x)' + c_2'(x)(e^{3x})' = 2xe^{2x} \end{cases}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{3x} = 0 \\ c_1'(x)e^x + 3c_2'(x)e^{3x} = 2xe^{2x} \end{cases}$$

$$w(x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 3e^{4x} - e^{4x} = 2e^{4x}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^{3x} \\ 2xe^{2x} & 3e^{3x} \end{vmatrix}}{w(x)} = \frac{-2xe^{5x}}{2e^{4x}} = -xe^x$$

$$c_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 2xe^{2x} \end{vmatrix}}{w(x)} = \frac{2xe^{3x}}{2e^{4x}} = xe^{-x}$$

$$c_1'(x) = -xe^x$$

$$c_1(x) = \int (-xe^x)dx = -\int x(e^x)'dx = -xe^x + \int x'e^x dx = -xe^x + e^x = (1-x)e^x$$

$$c_2'(x) = xe^{-x}$$

$$c_2(x) = \int xe^{-x}dx = \int (-x)(e^{-x})'dx = -xe^{-x} + \int e^{-x}dx =$$

$$= -xe^{-x} - e^{-x} = (-1-x)e^{-x}$$

$$y_p(x) = (1-x)e^x e^x + (-1-x)e^{-x}e^{3x}$$

Particular solution

$$y_p(x) = (1-x)e^{2x} + (-1-x)e^{2x} = (1-x-1-x)e^{2x} = -2xe^2x$$

General solution:

$$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2e^{3x} - 2xe^{2x}$$

5.c. $y'' - y = e^x$

Characteristic equation: $r^2 - 1 = 0 \Rightarrow r_1 = 1; r_2 = -1$

$S = \{e^x; e^{-x}\}$ - fundamental system of solutions

Solution of the homogeneous equation:

$$y_h(x) = c_1e^x + c_2e^{-x}$$

Find a particular solution:

$$\begin{cases} c_1'(x)e^x + c_2'(e^{-x}) = 0 \\ c_1'(x)(e^x)' + c_2'(x)(e^{-x})' = e^x \end{cases}$$

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{-x} = 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} = e^x \end{cases}$$

$$w(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2e^x \cdot e^{-x} - e^x e^{-x} = 3$$

$$c_1'(x)=\frac{\begin{vmatrix}0 & e^{-x}\\ e^x & -e^{-x}\end{vmatrix}}{w(x)}=\frac{-e^xe^{-x}}{-3}=\frac{1}{3}$$

$$c_2'(x)=\frac{\begin{vmatrix}e^x & 0\\ e^x & e^x\end{vmatrix}}{w(x)}=\frac{e^{2x}}{3}$$

$$c_1'(x)=\frac{1}{3}\Rightarrow c_1(x)=\frac{x}{3}$$

$$c_2'(x)=\frac{e^{2x}}{3}\Rightarrow c_2(x)=\frac{1}{3}\int e^{2x}=\frac{1}{6}e^{2x}$$

$$y_p(x)=\frac{x}{3}e^x+\frac{1}{6}e^{2x}\cdot e^{-x}=\frac{x}{3}e^x+\frac{1}{6}e^x=\frac{e^x}{6}(2x+1)=\frac{(2x+1)e^x}{6}$$

General solution:

$$y(x)=y_h(x)+y_p(x)=c_1e^x+c_2e^{-x}+\frac{(2x+1)e^x}{6}$$

□

6. Find the general solution of the following systems of differential equations:

a) $\begin{cases}y_1'=2y_2\\ y_2'=y_1-y_2+e^x\end{cases}$

b) $\begin{cases}y_1'=2y_1-y_2+2y_3\\ y_2'=y_1+y_2+y_3\\ y_3'=y_2\end{cases}$

a. $\begin{cases}y_1'=2y_2\\ y_2'=y_1-y_2+e^x\end{cases}\Rightarrow y_2=\frac{1}{2}y_1'\Rightarrow y_2'=\frac{1}{2}y_1''$

$$\frac{1}{2}y_1''=y_1-\frac{1}{2}y_1'+e^x\Rightarrow y_1''=2y_1-y_1'+2e^x$$

$y_1''+y_1'-2y_1=2e^x$. Homogeneous equation: $y_1''+y_1'-2y_1=0$
Characteristic equation: $r^2+r-2=0, r_1=1; r_2=-2$
Fundamental system of solutions: $\{e^x; e^{-2x}\}$
Solution of the homogeneous equation: $y_h(x)=c_1e^x+c_2e^{-2x}$
Find the particular solution:

$$\begin{cases}c_1'(x)e^x+c_2'(x)e^{-2x}=0\\ c_1'(x)(e^x)'+c_2'(x)(e^{-2x})'=2e^x\end{cases}$$

$$\begin{cases}c_1'(x)e^x+c_2'(x)e^{-2x}=0\\ c_1'(x)e^x-2c_2'(x)e^{-2x}=2e^x\end{cases}$$

$$w(x)=\begin{vmatrix}e^x & e^{-2x}\\ e^x & -2e^{-2x}\end{vmatrix}=e^x(-2e^{-2x})-e^xe^{-2x}=-2e^{-x}-e^{-x}=-3e^{-x}$$

$$c_1'(x)=\frac{\begin{vmatrix}0 & e^{-2x}\\ 2e^x & -2e^{-2x}\end{vmatrix}}{w(x)}=\frac{-2e^xe^{-2x}}{-3e^{-x}}=\frac{-2e^{-x}}{3e^{-x}}=\frac{2}{3}$$

$$c_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 2e^x \end{vmatrix}}{w(x)} = \frac{e^x \cdot 2e^x}{-3e^{-x}} = -\frac{2}{3}e^{3x}$$

$$y_p(x) = \frac{2}{3}e^x + \Big(-\frac{2}{3}\Big)e^{3x}e^{-2x} = \frac{2}{3}e^x - \frac{2}{3}e^x = 0$$

$$y_1(x) = c_1e^x + c_2e^{-2x}, \qquad y_2 = \frac{1}{2}(c_1e^x - 2c_2e^{-2x})$$

6.b. $\begin{cases} y_1' = 2y_1 - y_2 + 2y_3 \\ y_2' = y_1 + y_2 + y_3 \\ y_3' = y_2 \end{cases}$

Matrix form of the system:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Calculation of the eigenvalues:

$$\det\left(\begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 0, \qquad \begin{vmatrix} 2-\lambda & -1 & 2 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)(-\lambda) + 2 - 0 - (2-\lambda) - \lambda = 0$$

$$(2-\lambda)(-\lambda+\lambda^2) + 2 - 2 + \lambda - \lambda = 0$$

$$(2-\lambda)(1-\lambda)(-\lambda) = 0 \left\langle \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{matrix} \right.$$

Calculation of the eigenvectors:

For $\lambda = 0$: $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2v_1 - v_2 + 2v_3 \\ v_1 + v_2 + v_3 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{matrix} v_2 = 0 \\ v_1 + v_3 = 0 \end{matrix} \Rightarrow v_1 = -v_3 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ 0 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

For $\lambda = 1$: $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 - v_2 + 2v_3 \\ v_1 + v_3 \\ v_2 - v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{matrix} v_1 + v_3 = 0 \\ v_2 - v_3 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1 = -v_3 \\ v_2 = v_3 \end{matrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ v_3 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

For $\lambda = 2$: $\begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -v_2 + 2v_3 \\ v_1 - v_2 + v_3 \\ v_2 - 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$v_1 - v_2 + v_3 = 0$$

$$v_2 - 2v_3 = 0 \Rightarrow v_2 = 2v_3; v_1 - 2v_3 + v_3 = 0; v_1 = v_3 \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ 2v_3 \\ v_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ - eigenvector}$$

Solution.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = c_1 \cdot e^{0 \cdot x} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \cdot e^{1 \cdot x} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 \cdot e^{2x} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 e^x + c_3 e^{2x} \\ 0 + c_2 e^x + 2c_3 e^{2x} \\ c_1 + c_2 e^x + c_3 e^{2x} \end{pmatrix}, \quad \begin{cases} y_1(x) = -c_1 - c_2 e^x + c_3 e^{2x} \\ y_2(x) = c_2 e^x + 2c_3 e^{2x} \\ y_3(x) = c_1 + c_2 e^x + c_3 e^{2x} \end{cases}$$

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References

[1] *Romanian Mathematical Magazine.* Interactive Journal, www.ssmrmh.ro

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