A Sharp Weighted Generalization of a Triangle Inequality

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Abstract

We establish a weighted, multivariable, and parametric extension of a rational inequality for the side lengths of a triangle. For positive weights of unit sum, let m and V denote the weighted arithmetic mean and variance of positive variables $a_1, \ldots, a_n$. Under the assumptions $\lambda \geq 1$ and $(1+\lambda)m - \lambda a_i > 0$, we prove that the weighted sum of $a_i^2/((1+\lambda)m - \lambda a_i)$ is bounded below by the weighted third moment divided by m^2, together with the variance correction $(\lambda - 1)V/m$. An explicit nonnegative remainder identity yields an elementary proof and a complete characterization of equality. With the coefficient of the normalized third moment fixed at unity, the variance coefficient $\lambda - 1$ is best possible for every fixed number of variables when the positive weights are allowed to vary. Consequences include a refinement involving the power mean of order three, a quantitative strengthening of the motivating triangle inequality, and an unweighted multivariable analogue.

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1 Introduction

Inequalities between means provide a natural framework for comparing homogeneous expressions in positive variables. Classical background on inequalities is given by Hardy, Littlewood, and Pólya [2]; a systematic treatment of means and their inequalities can be found in Bullen [1]. In the present paper, we study a rational expression whose denominators depend on the weighted arithmetic mean and show that its comparison with the third moment admits a sharp variance correction.

The motivating problem, attributed to Le Anh Tu, asks for a proof of

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geq 3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}, \tag{1.1}$$

where a, b, c are the side lengths of a nondegenerate triangle. The strict triangle inequalities ensure that all denominators are positive. The right-hand side is three times the power mean of order three, and equality holds for an equilateral triangle.

We extend this estimate to arbitrary positive weights and any number of variables, introducing a parameter $\lambda \geq 1$ in the denominators. The main result is stronger than a direct power-mean comparison: it contains both a normalized third moment and a variance term. The proof rests on an exact algebraic decomposition with a nonnegative remainder, which also identifies all equality cases. A two-point limiting family then determines the best possible variance coefficient.

The scope of the sharpness statement is essential. The coefficient is optimal for the third-moment estimate over the full class of positive weights. This does not, by itself, determine the optimal coefficient for the power-mean estimate or for its specialization to three equal weights.

2 Main result

Let $n \ge 2$, and let $w_1, \dots, w_n > 0$ satisfy $\sum_{i=1}^n w_i = 1$. For positive variables $a_1, \dots, a_n$, write

$$m = \sum_{i=1}^n w_i a_i, \qquad M_3 = \left(\sum_{i=1}^n w_i a_i^3 \right)^{1/3}, \qquad V = \sum_{i=1}^n w_i (a_i - m)^2. \tag{2.1}$$

For a fixed parameter $\lambda \ge 1$, assume that

$$d_i = (1+\lambda)m - \lambda a_i > 0 \qquad (i = 1, \dots, n). \tag{2.2}$$

Equivalently, $0 < a_i < (1 + 1/\lambda)m$ for every i. Define

$$F_\lambda(\boldsymbol{a}; \boldsymbol{w}) = \sum_{i=1}^n w_i \frac{a_i^2}{(1+\lambda)m - \lambda a_i}. \tag{2.3}$$

Theorem 2.1. *Under the assumptions above,*

$$F_\lambda(\boldsymbol{a}; \boldsymbol{w}) \ge \frac{\sum_{i=1}^n w_i a_i^3}{m^2} + (\lambda - 1)\frac{V}{m}. \tag{2.4}$$

Consequently,

$$F_\lambda(\boldsymbol{a}; \boldsymbol{w}) \ge M_3 + (\lambda - 1)\frac{V}{m} \ge M_3. \tag{2.5}$$

Equality in (2.4)*, in the first inequality of* (2.5)*, or in the estimate $F_\lambda \ge M_3$ holds if and only if*

$$a_1 = \cdots = a_n. \tag{2.6}$$

For every fixed $n \ge 2$ and $\lambda \ge 1$, the coefficient $\lambda - 1$ in (2.4) *is the largest coefficient valid for all admissible positive weights and variables, with the coefficient of the normalized third moment fixed at* 1*.*

Remark 2.2. When $\lambda = 1$, the second inequality in (2.5) is an equality for every admissible data set. Nevertheless, equality in $F_\lambda \ge M_3$ still holds precisely under condition (2.6).

3 A remainder identity and proof of the inequalities

3.1 Homogeneity and normalization

Both sides of (2.4) are homogeneous of degree one in the variables a_i. Since $m > 0$, we may introduce

$$x_i = \frac{a_i}{m}, \qquad \sum_{i=1}^n w_i x_i = 1, \qquad 0 < x_i < 1 + \frac{1}{\lambda}. \tag{3.1}$$

Set

$$Q = \sum_{i=1}^{n} w_i x_i^3, \qquad W = \sum_{i=1}^{n} w_i (x_i - 1)^2.$$

Then

$$F_\lambda = m \sum_{i=1}^{n} w_i \frac{x_i^2}{1 + \lambda - \lambda x_i}, \qquad \frac{M_3^3}{m^2} = mQ, \qquad \frac{V}{m} = mW.$$

Thus (2.4) is equivalent to

$$\sum_{i=1}^{n} w_i \frac{x_i^2}{1 + \lambda - \lambda x_i} \geq Q + (\lambda - 1)W. \tag{3.2}$$

3.2 The algebraic decomposition

Lemma 3.1. *If $\lambda \geq 1$ and $0 < x < 1 + 1/\lambda$, then*

$$\begin{aligned} \frac{x^2}{1 + \lambda - \lambda x} &= x^3 + (\lambda - 1)(x - 1) + (\lambda - 1)(x - 1)^2 \\ &\quad + \frac{x(x - 1)^2(\lambda x + \lambda^2 - 1)}{1 + \lambda - \lambda x}. \end{aligned} \tag{3.3}$$

The last term is nonnegative and vanishes if and only if $x = 1$.

Proof. Put $D = 1 + \lambda - \lambda x > 0$. Since $(x - 1) + (x - 1)^2 = x(x - 1)$, the asserted identity is equivalent to

$$x^2 - Dx^3 - (\lambda - 1)Dx(x - 1) = x(x - 1)^2(\lambda x + \lambda^2 - 1).$$

Indeed, factoring $x(x - 1)$ from the left-hand side leaves

$$\begin{aligned} x(\lambda x - 1) - (\lambda - 1)D &= \lambda x^2 + (\lambda^2 - \lambda - 1)x - (\lambda^2 - 1) \\ &= (x - 1)(\lambda x + \lambda^2 - 1). \end{aligned}$$

This proves (3.3). The sign assertion follows from $x > 0$, $D > 0$, and $\lambda x + \lambda^2 - 1 > 0$. The last factor remains strictly positive when $\lambda = 1$, in which case it equals x. $\square$

3.3 The third-moment bound and equality cases

Apply Lemma 3.1 to each x_i, multiply by w_i, and sum over i. The linear term disappears because

$$\sum_{i=1}^{n} w_i (x_i - 1) = 0.$$

We obtain the exact remainder identity

$$\begin{aligned} &\sum_{i=1}^{n} w_i \frac{x_i^2}{1 + \lambda - \lambda x_i} - Q - (\lambda - 1)W \\ &\quad = \sum_{i=1}^{n} w_i \frac{x_i(x_i - 1)^2(\lambda x_i + \lambda^2 - 1)}{1 + \lambda - \lambda x_i} \geq 0. \end{aligned} \tag{3.4}$$

This establishes (3.2); multiplication by m gives (2.4).

All weights are positive, so the sum on the right-hand side vanishes if and only if every remainder term vanishes. By the lemma, this is equivalent to $x_1 = \cdots = x_n = 1$, or, in the original variables, to (2.6). In particular, (2.4) is strict whenever the variables are not all equal.

3.4 Passage to the power mean of order three

The comparison needed to deduce (2.5) is a special case of the power-mean inequality; see [1]. For completeness, we give a direct proof. The elementary identity

$$x^3 - 1 - 3(x-1) = (x-1)^2(x+2)$$

yields, after taking weighted sums,

$$Q - 1 = \sum_{i=1}^{n} w_i(x_i - 1)^2(x_i + 2) \geq 0. \tag{3.5}$$

Hence $Q \geq 1$ and therefore $Q \geq Q^{1/3}$. It follows that

$$\frac{M_3^3}{m^2} = mQ \geq mQ^{1/3} = M_3.$$

Together with (2.4), this proves (2.5). If the variables are not all equal, the remainder in (3.4) is positive. Thus both $F_\lambda \geq M_3 + (\lambda - 1)V/m$ and $F_\lambda \geq M_3$ are strict. Conversely, equal variables give equality by direct substitution.

4 Optimality of the variance coefficient

Fix $\lambda \geq 1$. Suppose that a real constant C satisfies

$$\sum_{i=1}^{n} w_i \frac{x_i^2}{1 + \lambda - \lambda x_i} \geq \sum_{i=1}^{n} w_i x_i^3 + C \sum_{i=1}^{n} w_i (x_i - 1)^2 \tag{4.1}$$

for every admissible choice satisfying (3.1). We show that $C \leq \lambda - 1$.

4.1 A two-point extremizing family

First take $n = 2$ and choose

$$\begin{gathered} 0 < \varepsilon < \min\left\{\frac{1}{2}, \frac{1}{2\lambda}\right\}, \\ w_1 = \varepsilon, \quad w_2 = 1 - \varepsilon, \qquad x_1 = \varepsilon, \quad x_2 = 1 + \varepsilon. \end{gathered} \tag{4.2}$$

The weighted mean equals one, since

$$w_1 x_1 + w_2 x_2 = \varepsilon^2 + (1-\varepsilon)(1+\varepsilon) = 1.$$

The two denominators are $1 + \lambda - \lambda\varepsilon$ and $1 - \lambda\varepsilon$, both of which are positive. Moreover,

$$W_\varepsilon = \varepsilon(1-\varepsilon)^2 + (1-\varepsilon)\varepsilon^2 = \varepsilon(1-\varepsilon) > 0. \tag{4.3}$$

Define

$$R_\lambda(x) = \frac{\lambda x^2 + (\lambda - 1)x + \lambda^2 - 1}{1 + \lambda - \lambda x}. \tag{4.4}$$

Rearranging (3.3), we have

$$\frac{x^2}{1 + \lambda - \lambda x} - x^3 - (\lambda - 1)(x - 1) = (x-1)^2 R_\lambda(x).$$

The weighted sum of the linear term is again zero. For the family (4.2), division by (4.3) therefore gives

$$T_\varepsilon := \frac{\sum_{i=1}^{2} w_i \dfrac{x_i^2}{1+\lambda-\lambda x_i} - \sum_{i=1}^{2} w_i x_i^3}{\sum_{i=1}^{2} w_i (x_i-1)^2} \tag{4.5}$$
$$= (1-\varepsilon)R_\lambda(\varepsilon) + \varepsilon R_\lambda(1+\varepsilon).$$

The function R_λ extends continuously to $x = 0$ and is continuous in a neighborhood of $x = 1$. Consequently,

$$\lim_{\varepsilon\to 0+} T_\varepsilon = R_\lambda(0) = \frac{\lambda^2-1}{1+\lambda} = \lambda - 1. \tag{4.6}$$

If (4.1) holds for every admissible choice, then $C \le T_\varepsilon$ for each such ε. Taking the limit gives $C \le \lambda - 1$. Since (2.4) proves that $C = \lambda - 1$ is valid, the optimal coefficient is

$$C_{\mathrm{opt}}(\lambda) = \lambda - 1. \tag{4.7}$$

4.2 Scope of the optimality statement

Every member of the extremizing family has strictly positive weights and variables. A zero weight and a zero variable occur only in the limit; no inadmissible data set is used in the inequality itself.

For any prescribed $n > 2$, split the weight $1 - \varepsilon$ into $n - 1$ positive parts and assign the value $1 + \varepsilon$ to each corresponding variable. The mean, the variance, and the quotient (4.5) remain unchanged. Hence the same optimal constant applies for every fixed $n \ge 2$, provided that the weights may vary. This completes the proof of Theorem 2.1.

Remark 4.1. The coefficient of the normalized third moment cannot be increased either. Indeed, if $a_i = t > 0$ for every i, then $F_\lambda = t$, $M_3^3/m^2 = t$, and $V = 0$. A coefficient greater than one would therefore fail at the equality configuration. This observation and (4.7) do not characterize all admissible pairs of coefficients: the latter result fixes the coefficient of the third-moment term at unity.

5 Application to triangle side lengths

Let a, b, c be the side lengths of a nondegenerate triangle, and set

$$S = a + b + c, \qquad \Delta = (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{5.1}$$

The quantity Δ measures the dispersion of the side lengths and vanishes precisely for an equilateral triangle.

Corollary 5.1. *With the notation above,*

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \ge \frac{9(a^3+b^3+c^3)}{S^2} + \frac{\Delta}{S}. \tag{5.2}$$

In particular,

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \ge 3\sqrt[3]{\frac{a^3+b^3+c^3}{3}} + \frac{\Delta}{S}. \tag{5.3}$$

Equality in either inequality holds if and only if $a = b = c$.

Proof. Apply Theorem 2.1 with $n = 3$, $w_1 = w_2 = w_3 = 1/3$, $\lambda = 2$, and $(a_1, a_2, a_3) = (a, b, c)$. Then $m = S/3$, and

$$d_1 = S - 2a = b + c - a, \quad d_2 = S - 2b = c + a - b, \quad d_3 = S - 2c = a + b - c.$$

Thus positivity of the denominators is equivalent to the strict triangle inequalities. Furthermore,

$$\Delta = 3\left[\left(a - \frac{S}{3}\right)^2 + \left(b - \frac{S}{3}\right)^2 + \left(c - \frac{S}{3}\right)^2\right], \qquad V = \frac{\Delta}{9}.$$

Multiplying (2.4) and the first inequality of (2.5) by three yields (5.2) and (5.3), respectively. The equality conditions follow from the theorem. □

For every nonequilateral triangle, $\Delta/S > 0$. Hence (5.3) gives a strictly larger lower bound than (1.1). The correction quantifies the departure of the side lengths from equality while preserving homogeneity of degree one.

Remark 5.2. We do not assert that the coefficient 1 of Δ/S is optimal in (5.2) or (5.3). The extremizing family used above varies the weights, whereas the three weights here are fixed at 1/3. Determining the best coefficient for this fixed-weight problem requires a separate argument.

6 An unweighted multivariable consequence

The triangle refinement is also a special case of the following unweighted family.

Corollary 6.1. *Let $n \geq 2$, let $a_1, \ldots, a_n > 0$, and put $S = \sum_{i=1}^n a_i$. Suppose that*

$$a_i < \frac{S}{n-1} \qquad (i = 1, \ldots, n). \tag{6.1}$$

Then

$$\begin{aligned}\sum_{i=1}^n \frac{a_i^2}{S - (n-1)a_i} &\geq \frac{n^2 \sum_{i=1}^n a_i^3}{S^2} + \frac{n-2}{S} \sum_{1 \leq i < j \leq n} (a_i - a_j)^2 \\ &\geq n\left(\frac{\sum_{i=1}^n a_i^3}{n}\right)^{1/3} + \frac{n-2}{S} \sum_{1 \leq i < j \leq n} (a_i - a_j)^2.\end{aligned} \tag{6.2}$$

Equality in either inequality holds if and only if $a_1 = \cdots = a_n$.

Proof. Choose $w_i = 1/n$ and $\lambda = n - 1$ in Theorem 2.1. Then $m = S/n$, and

$$(1 + \lambda)m - \lambda a_i = S - (n-1)a_i > 0.$$

To express the variance term in pairwise form, use the identity

$$\sum_{1 \leq i < j \leq n} (a_i - a_j)^2 = n \sum_{i=1}^n \left(a_i - \frac{S}{n}\right)^2 = n \sum_{i=1}^n a_i^2 - S^2, \tag{6.3}$$

which follows by expansion. In particular,

$$V = \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (a_i - a_j)^2.$$

Multiplying (2.4) by n gives the first inequality in (6.2). The second follows from the moment comparison established in (3.5). The equality condition in both steps is precisely that all variables are equal. □

For $n = 2$, condition (6.1) holds for every positive pair, and the explicit variance correction vanishes. For $n = 3$, Corollary 5.1 is recovered. When $n > 3$, the condition is stronger than the usual polygon inequalities $a_i < S - a_i$, since $S/(n-1) < S/2$. This stronger restriction is exactly what ensures positivity of the denominators $S - (n-1)a_i$.

References

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[2] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.