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UP.629. *Proposed by Daniel Sitaru, Romania.* Find the general solution for the linear second-order differential equation with constant coefficients:

$$y'' - 5y' + 4y = xe^x$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. The associated homogeneous equation is

$$y'' - 5y' + 4y = 0,$$

and its characteristic equation is:

$$r^2 - 5r + 4 = 0 \Rightarrow (r-1)(r-4) = 0$$

This gives roots $r_1 = 1$ and $r_2 = 4$, and the homogeneous solution is

$$y_h(x) = C_1e^x + C_2e^{4x}$$

To find the particular solution we will use the method of undetermined coefficients: The right-hand side is $f(x) = xe^x$. Since $r = 1$ is a single root of the characteristic equation, we will use the trial solution

$$y_p(x) = x(Ax+B)e^x = (Ax^2+Bx)e^x$$

Next, we compute the derivatives of $y_p(x)$:

$$\begin{aligned} y_p' &= (2Ax+B)e^x + (Ax^2+Bx)e^x = [Ax^2+(2A+B)x+B]e^x \\ y_p'' &= (2Ax+2A+B)e^x + [Ax^2+(2A+B)x+B]e^x \\ &= [Ax^2+(4A+B)x+(2A+2B)]e^x \end{aligned}$$

Substituting into the original differential equation yields

$$y'' - 5y' + 4y = xe^x,$$

and factoring out e^x, we get

$$\Big[Ax^2+(4A+B)x+(2A+2B)\Big] - 5\Big[Ax^2+(2A+B)x+B\Big] + 4\Big[Ax^2+Bx\Big] = x$$

Group the terms by powers of x:

- For x^2: $A - 5A + 4A = 0$ (matches correctly)
- For x: $(4A+B) - 5(2A+B) + 4B = -6A = 1 \Rightarrow A = -\frac{1}{6}$
- For the constant term: $(2A+2B) - 5B = 2A - 3B = 0 \Rightarrow 3B = 2A \Rightarrow B = \frac{2}{3}A = -\frac{1}{9}$

Thus, the particular solution is

$$y_p(x) = \left(-\frac{1}{6}x^2 - \frac{1}{9}x\right)e^x,$$

and the general solution is

$$y(x) = y_h(x) + y_p(x) = C_1e^x + C_2e^{4x} - \left(\frac{1}{6}x^2 + \frac{1}{9}x\right)e^x$$