José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

UP.628. *Proposed by Daniel Sitaru, Romania.* Find the general solution for the system of differential equations:

$$\begin{cases} y_1' = y_1 + y_2 + y_3 \\ y_2' = 2y_2 + y_3 \\ y_3' = 3y_3 \end{cases}$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. First, we write the system in matrix form $\mathbf{y}' = A\mathbf{y}$ as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Since A is an upper triangular matrix, its eigenvalues are the elements on the main diagonal:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

The eigenvectors corresponding to each eigenvalue are

- For $\lambda_1 = 1$: Solving $(A - I)\mathbf{v}_1 = \mathbf{0}$, we get the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 2$: Solving $(A - 2I)\mathbf{v}_2 = \mathbf{0}$, we get the eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.
- For $\lambda_3 = 3$: Solving $(A - 3I)\mathbf{v}_3 = \mathbf{0}$, we get the eigenvector $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

Combining the eigenvalues and eigenvectors, the general solution is given by:

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

or in component form

$$\begin{cases} y_1(t) = C_1 e^t + C_2 e^{2t} + 3C_3 e^{3t} \\ y_2(t) = C_2 e^{2t} + 2C_3 e^{3t} \\ y_3(t) = 2C_3 e^{3t} \end{cases}$$

where C_1, C_2, C_3 are arbitrary constants.