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UP.627. *Proposed by Daniel Sitaru, Romania.* Find the surface integral of the second kind:

$$\iint_S 2x\,dy\,dz + y\,dz\,dx + 5z\,dx\,dy$$

where S is the surface of the ellipsoid:

$$E : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. To find the surface integral we will use the Gauss's Divergence Theorem. It relates a surface integral over a closed, oriented surface S to a triple integral over the solid region V enclosed by S. That is,

$$\iint_S (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy) = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$$

Here, our vector field components are $P = 2x, Q = y$ and $R = 5z$. Next, we find the divergence of the vector field $\mathbf{F} = \langle 2x, y, 5z\rangle$. That is,

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(5z)$$

$$\nabla \cdot \mathbf{F} = 2 + 1 + 5 = 8$$

The surface integral now simplifies to

$$8 \iiint_V dV$$

where $\iiint_V dV$ is simply the volume of the ellipsoid V.

Now, we calculate the Volume of the Ellipsoid. The standard equation for an ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

From our given equation, the semi-axes are $a = 2, b = 3, c = 5$. It is well-known that the formula for the volume of an ellipsoid is Volume $= \frac{4}{3}\pi abc$. Substituting the values of a, b, and c yields

$$\text{Volume} = \frac{4}{3}\pi(2)(3)(5) = 40\pi,$$

and the value of the surface integral is

$$\iint_S 2x\,dy\,dz + y\,dz\,dx + 5z\,dx\,dy = 8 \times (40\pi) = 320\pi$$