

ROMANIAN MATHEMATICAL MAGAZINE

UP.624 Solve for reals:

$$\sqrt[4]{1+x} + \sqrt[4]{1-x} - \sqrt[4]{1-x^2} = 1$$

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$$\text{Denote: } \begin{cases} \sqrt[4]{1+x} = a \\ \sqrt[4]{1-x} = b \end{cases}; \begin{cases} S = a + b \\ P = ab > 0 \end{cases} \Rightarrow \begin{cases} a + b - ab = 1 \\ a^4 + b^4 = 2 \end{cases} \Rightarrow \begin{cases} S = 1 + P \\ (S^2 - 2P)^2 - 2P^2 = 2 \end{cases}$$

$$((1+P)^2 - 2P)^2 - 2P^2 = 2$$

$$(1 + 2P + P^2 - 2P)^2 - 2P^2 = 2$$

$$(P^2 + 1)^2 - 2P^2 = 2 \Rightarrow P^4 + 2P^2 + 1 - 2P^2 = 2$$

$$\Rightarrow P^4 = 1 \Rightarrow P = 1 \quad (\text{because } P > 0)$$

$$P = 1 \Rightarrow S = 2 \Rightarrow \begin{cases} a = 1 \\ b = 1 \end{cases} \Rightarrow \begin{cases} \sqrt[4]{1+x} = 1 \\ \sqrt[4]{1-x} = 1 \end{cases} \Rightarrow x = 0$$