

ROMANIAN MATHEMATICAL MAGAZINE

UP.623 If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ and $\lambda \geq \frac{1}{3}$ then:

$$\sum a^2 b \sum \frac{1}{\sqrt{a^2 + \lambda}} \leq \frac{3}{\sqrt{1 + 3\lambda}}$$

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Solution by proposer

We prove: If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ then:

$$\sum a^2 b \leq \frac{1}{3}(a + b + c)$$

Proof: We have:

$$\begin{aligned} 3 \sum a^2 b &\stackrel{(1)}{\leq} \sum a^2 \sum a = 1 \cdot \sum a = \sum a, \text{ where } (1) \Leftrightarrow 3 \sum a^2 b \leq \sum a^2 \sum a \Leftrightarrow \\ &\Leftrightarrow \sum a(b - c)^2 \geq 0 \end{aligned}$$

Lemma 2: If $a, b, c > 0$, $a^2 + b^2 + c^2 = 1$ and $\lambda \geq \frac{1}{3}$ then

$$\left(\sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 \leq 3 \left(3 - \frac{\lambda(a + b + c)^2}{1 + 3\lambda} \right).$$

Proof: We have

$$\begin{aligned} \left(\sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 &\stackrel{CBS}{\leq} 3 \sum \frac{1}{a^2 + \lambda} = 3 \frac{1}{\lambda} \sum \left(1 - \frac{a^2}{a^2 + \lambda} \right) = \frac{3}{\lambda} \left(3 - \sum \frac{a^2}{a^2 + \lambda} \right) \stackrel{CS}{\leq} \\ &\leq \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{\sum(a^2 + \lambda)} \right) = \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \end{aligned}$$

Let's get back to the main problem. Using the Lemmas, we obtain:

$$\begin{aligned} LHS^2 &= \left(\sum a^2 b \sum \frac{1}{\sqrt{a^2 + \lambda}} \right)^2 \stackrel{Lemma 1+2}{\leq} \frac{1}{9}(a + b + c)^2 \cdot \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \stackrel{(2)}{\leq} \\ &\leq \frac{9}{1 + 3\lambda} = RHS^2, \end{aligned}$$

$$\text{where } (2) \Leftrightarrow \frac{1}{9}(a + b + c)^2 \cdot \frac{3}{\lambda} \left(3 - \frac{(a + b + c)^2}{1 + 3\lambda} \right) \leq \frac{9}{1 + 3\lambda} \Leftrightarrow \frac{1}{9}t \cdot \frac{3}{\lambda} \left(3 - \frac{t}{1 + 3\lambda} \right) \leq \frac{9}{1 + 3\lambda} \Leftrightarrow$$

$$t(3 + 9\lambda - t) \leq 27\lambda \Leftrightarrow t^2 - 3(3\lambda + 1)t + 27\lambda \geq 0 \Leftrightarrow (t - 3)(t - 9\lambda) \geq 0,$$

which follows from $t \leq 3$ and $\lambda \geq \frac{1}{3}$,

$$(\text{see } t = (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 3 \cdot 1 = 3).$$

Equality holds if and only if $a = b = c = \frac{1}{\sqrt{3}}$.