ROMANIAN MATHEMATICAL MAGAZINE

UP.622 Find the surface integral of the first kind:

$$\Omega = \int\int_{\Sigma} (x+y+z)\, d\tau$$

$$\sum = \{(x,y,z) \in \mathbb{R}^3 | x = 2u+3v, y = 3u+2v, z = u+v; u,v \in [0,1]\}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

$$x = \varphi(u,v) = 2u+3v; y = \psi(u,v) = 3u+2v, z = \chi(u,v) = u+v;$$

$$\frac{\partial\varphi}{\partial v} = 2; \frac{\partial\psi}{\partial u} = 3; \frac{\lambda\chi}{\lambda u} = 1$$

$$\frac{\lambda\varphi}{\lambda v} = 3; \frac{\partial\psi}{\partial v} = 2; \frac{\partial\chi}{\partial u} = 1$$

$$E = \left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\psi}{\partial u}\right)^2 + \left(\frac{\partial\chi}{\partial u}\right)^2 = 2^2+3^2+1^2 = 14$$

$$G = \left(\frac{\partial\varphi}{\partial v}\right)^2 + \left(\frac{\partial\psi}{\partial v}\right)^2 + \left(\frac{\partial\chi}{\partial v}\right)^2 = 3^2+2^2+1^2 = 14$$

$$F = \left(\frac{\partial\varphi}{\partial u}\right)\left(\frac{\partial\varphi}{\partial v}\right) + \left(\frac{\partial\psi}{\partial u}\right)\left(\frac{\partial\psi}{\partial v}\right) + \left(\frac{\partial\chi}{\partial u}\right)\left(\frac{\partial\chi}{\partial v}\right) = 2\cdot 3 + 3\cdot 2 + 1\cdot 1 = 13$$

$$d\tau = \sqrt{EG-F^2}du\,dv = \sqrt{14\cdot 14 - 13^2}du\,dv = \sqrt{27}dudv$$

$$f(x,y,z) = x+y+z$$

$$f\big(\varphi(u,v), \psi(u,v), \chi(u,v)\big) = 2u+3v+3u+2v+u+v = 6u+6v$$

$$\Omega = \int\int_{\Sigma} (x+y+z)\, d\tau = \int\int_{D} (6u+6v)\sqrt{27}dudv =$$

$$= 18\sqrt{3}\left(\int_0^1 u\,du \cdot \int_0^1 dv + \int_0^1 du \cdot \int_0^1 v\,dv\right) = 18\sqrt{3}\left(\frac{1}{2}\cdot 1 + 1\cdot\frac{1}{2}\right) = 18\sqrt{3}$$

Solution 2 by Jose Luis Diaz Barrero-Spain

We compute this integral carrying out the following steps:

- First, we compute the partial derivatives with respect to u and v: Let the position vector of the surface be $r(u,v) = (2u+3v, 3u+2v, u+v)$. Then,

$$r_u = \frac{\partial r}{\partial u} = (2,3,1)$$

$$r_v = \frac{\partial r}{\partial v} = (3,2,1)$$

- **Second, we compute the cross product $r_u \times r_v$:**

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1i + 1j - 5k = (1, 1, -5)$$

- **Third, we calculate the surface element differential $dS = |r_u \times r_v| du dv$:**

$$|r_u \times r_v| = \sqrt{1^2 + 1^2 + (-5)^2} = \sqrt{1 + 1 + 25} = \sqrt{27} = 3\sqrt{3}$$

Thus, $dS = 3\sqrt{3} du\, dv$.

- **Fourth, we substitute the parametrization into the integrand $x + y + z$ to obtain**

$$x + y + z = (2u + 3v) + (3u + 2v) + (u + v) = 6(u + v).$$

- **Fifth, we evaluate the double integral over $u, v \in [0, 1]$ and we get**

$$\int\int_S (x + y + z)\, dS = \int_0^1 \int_0^1 (6u + 6v) \cdot 3\sqrt{3} du\, dv =$$

$$= 3\sqrt{3} \int_0^1 \int_0^1 6(u + v)\, du\, dv = 18\sqrt{3} \int_0^1 \left[\frac{u^2}{2} + vu\right]_{u=0}^{u=1} dv =$$

$$= 18\sqrt{3} \int_0^1 \left[\frac{u^2}{2} + vu\right]_{u=0}^{u=1} dv = 18\sqrt{3} \int_0^1 \left(\frac{1}{2} + v\right) dv =$$

$$= 18\sqrt{3} \left[\frac{1}{2} v + \frac{v^2}{2}\right]_0^1 = 18\sqrt{3} \left(\frac{1}{2} + \frac{1}{2}\right) = 18\sqrt{3}$$